

Morphism preserving some kinds of languages

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Summary

Morphism preserving some kinds of languages are investigated. We study a necessary and sufficient condition for a morphism to preserve the primitivity, the d-primitivity, the square-freeness, the prefix codes, the suffix codes, and the comma-free codes.

Keywords: morphism, primitive word, d-primitive word, prefix code, suffix code, comma-free code

1 Introduction

The notion of primitive words, d-primitive words, square-free words, and codes play an important role in formal language theory, algebraic coding theory, combinatorial theory of words ([2], [3],[5], [6], [7]). Some studies on a problem concerning the behavior of morphisms with respect to primitivity has been done. A sufficient condition for a morphism to preserve the primitivity has been presented in [8], and classification of morphisms from the point of view of their primitivity-preserving properties has been done in [4].

On the other hand, few studies have been done on d-primitive preserving and the preserveness of many kinds of codes, except for prefix codes.

We study a necessary and sufficient condition for a morphism to preserve the primitivity, the d-primitivity, the square-freeness, the prefix codes, the suffix codes, and the comma-free codes.

2 Preliminaries

Let Σ be an alphabet consisting of at least two letters. Σ^* denotes the free monoid generated by Σ , that is, the set of all finite words over Σ , including the empty word ϵ , and $\Sigma^+ = \Sigma^* - \{\epsilon\}$. For w in Σ^* , $|w|$ denotes the length of w . A *language* over Σ is a set $L \subseteq \Sigma^*$.

For a word $u \in \Sigma^+$, if $u = vw$ for some $v, w \in \Sigma^*$, then v (w) is called a *prefix* (*suffix*) of u , denoted by $v \leq_p u$ ($w \leq_s u$, resp.). If $v \leq_p u$ ($w \leq_s u$) and $u \neq v$ ($u \neq w$), then v (w) is called a *proper prefix* (*proper suffix*) of u , denoted by $v <_p u$ ($w <_s u$, resp.). If $u = vxw$ for some $v, w \in \Sigma^*$, x is called an *infix* or a *factor* of u . For a word w , let $Pref(w)(Suff(w))$ be the set of all prefixes (suffixes, resp.) of w .

A nonempty word u is called a *primitive word* if $u = f^n$, for some $f \in \Sigma^+$, and some $n \geq 1$ always implies that $n = 1$. Let Q be the set of all primitive words over Σ .

A nonempty word u is a *non-overlapping word* if $u = vx = yv$ for $x, y \in \Sigma^+$ always implies that $v = \epsilon$. Let $D(1)$ be the set of all non-overlapping words over Σ . A word in $D(1)$ is also called a *d-primitive word* (See [1] and [7]).

For a word $w \in \Sigma^+$, there exist a unique primitive word x and a unique integer $i \geq 1$ such that $w = x^i$. Let $x = \sqrt{i}w$ and call x the *root* of w .

A word $u \in \Sigma^*$ is square-free if $u = vw^2x$ for some $v, w, x \in \Sigma^*$, implies $w = \epsilon$. Let SF be the set of all square-free words.

A language $L \subseteq \Sigma^+$ is a *code* if $x_1x_2\dots x_n = y_1y_2\dots y_m$, $x_i, y_j \in L$ implies $n = m$ and $x_i = y_i$, for $i = 1, \dots, n$.

A language $L \subseteq \Sigma^+$ is a *prefix code* (*suffix code*) if the condition $L \cap L\Sigma^+ = \phi$ ($L \cap \Sigma^+L = \phi$) is true. L is a *bifix code* if L is both a prefix code and also a suffix code. L is an *infix code* if, for all $x, y, u \in \Sigma^*$, $u \in L$ and $xuy \in L$ together imply $x = y = \epsilon$. L is an *intercode* if $L^{m+1} \cap \Sigma^+L^m\Sigma^+ = \phi$ for some $m \geq 1$. The integer m is called the *index* of L . An intercode of index 1 is called a *comma-free code*.

A language L is a *pure code* if it is a code such that, for any $x \in L^*$, $\sqrt{x} \in L^*$

A mapping $h : \Sigma_1^* \rightarrow \Sigma_2^*$ such that $h(xy) = h(x)h(y)$ for all $x, y \in \Sigma_1^*$ is a *morphism* of Σ_1^* into Σ_2^* . A morphism h is *primitive preserving* if $h(x) \in Q$ for all $x \in Q$. A morphism h is *d-primitive preserving* if $h(x) \in D(1)$ for all $x \in D(1)$. A morphism h is *square-free preserving* if $h(x) \in SF$ for all $x \in SF$. A morphism h is *prefix(suffix) preserving* if h preserves the prefix(suffix) codes. A morphism h is *comma-free preserving* if h preserves the comma-free codes.

3 Morphism preserving some kinds of languages

Lemma 1 ([8])

Let $h : \Sigma^* \rightarrow \Sigma^*$ be a morphism. Then h is injective if and only if $h(\Sigma)$ is a code and $|h(\Sigma)| = |\Sigma|$.

□

The "Only if" part of the following proposition is proved in [9]. We give another simpler proof.

Proposition 2 Let $h : \Sigma^* \rightarrow \Sigma^*$ be an injective morphism. Then h is primitive preserving iff $h(\Sigma)$ is a pure code.

Proof.

[If]([8])

[Only if] By Lemma 1, $h(\Sigma)$ is a code. Let $z = h(w)$ for some $w \in \Sigma^*$, and let $w = p^i$ for some $p \in Q$ and an integer i . We have $h(p) \in Q$ since h is primitive preserving. Hence $\sqrt{z} = h(p) \in h(\Sigma^*) = [h(\Sigma)]^*$. Thus $h(\Sigma)$ is pure. $\square$

Proposition 3 *A morphism $h : \{a, b\}^* \rightarrow \{a, b\}^*$ is d -primitive preserving iff $h(u)$ is d -primitive for each d -primitive word u of length ≤ 2 .*

Proof.

[Only if] Trivial.

[If] If $h(u)$ is d -primitive for each d -primitive word u of length ≤ 2 , then $h(\Sigma)$ is a prefix code and $|h(\Sigma)| = |\Sigma|$. Thus h is injective by Lemma 1. Suppose that that a morphism h is not d -primitive preserving.

There exist $u, v \in \Sigma^+$, and $w \in \Sigma^*$ such that $u \in D(1)$, and $h(u) = v w v$. Let $u = \nu_1 \nu_2 \dots \nu_k$ for some $\nu_i \in \Sigma$. If $v = h(\nu_1 \dots \nu_i) = h(\nu_j \dots \nu_k)$ for some i and j , then $j + i - 1 = k$, and $h(\nu_1) = h(\nu_j), h(\nu_2) = h(\nu_{j+1}), \dots, h(\nu_i) = h(\nu_k)$. Since h is injective, $\nu_1 = \nu_j, \nu_2 = \nu_{j+1}, \dots, \nu_i = \nu_k$.

This means that $u \notin D(1)$. We have that $v \notin (h(\Sigma))^*$. We can assume that $v = v' w'$ and $v = v'' y$ for some $w' \in \text{Pref}(h(\Sigma))$ and $y \in h(\Sigma), v', v'' \in \Sigma^*$.

If $|w'| \leq |y|$, then either $h(ab) \notin D(1)$ or $h(ba) \notin D(1)$. Thus we can write $w' = xy$ for some $x \in \Sigma^+$. Without loss of generality, we can assume that $h(a) = xyz$ and $h(b) = y$ for some $z \in \Sigma^+$.

(Case 1) $\nu_{k-1} = a$

(Case 1-1) $z = sx; |x| < |z|, s \in \Sigma^+$.

$h(a) = xyz = xysx$, a contradiction.

(Case 1-2) $x = y'' z; |z| < |x| < |yz|$, with $y = y' y''$, for $y'' \in \Sigma^+$.

$h(a) = y'' z y z, h(b) = y = y' y''$, contradiction.

(Case 1-3) $x = x'' y z; |yz| < |x|, x = x' x'', x'' \in \Sigma^+$.

$h(a) = xyz = x'' y z y z = x' x'' y z$, a contradiction.

(Case 2) $\nu_{k-1} = b$

(Case 2-1) $x = x'' y z y^n$ for some $x', x'' \in \Sigma^+$ with $x = x' x''$, and $n \geq 1$.

$h(a) = xyz = x'' y z y^n y z = x' x'' y z$, contradiction.

(Case 2-2) $x = y''y^n$ for some $y'', y' \in \Sigma^+$ with $y = y'y''$, and $n \geq 1$. We have that $h(a) = xyz = y''y^nyz$ and $h(b) = y'y''$, a contradiction. $\square$

Proposition 4 ([3])

Let $h : \Sigma_1^* \rightarrow \Sigma_2^*$ be a morphism such that (1) $h(u)$ is square-free for each square-free word u of length ≤ 3 . (2) For $a, b \in \Sigma_1$, $a \neq b$, no $h(a)$ is a proper factor of $h(b)$. Then h is a square-free preserving morphism.

$\square$

The previous Proposition gives a sufficient condition for a morphism to be square-free preserving.

Example 1 Define a morphism $h : \{a, b, c\}^* \rightarrow \{a, b, c, d\}^*$ by $h(a) = abcd, h(b) = b, h(c) = c$. Then h is square-free preserving. Both $h(b)$ and $h(c)$ are proper factor of $h(a)$.

Proposition 5 ([8]) $h : \Sigma^* \rightarrow \Sigma^*$ be a morphism such that $|h(\Sigma)| = |\Sigma|$. Then $h(\Sigma)$ is prefix code iff h preserves the prefix codes.

Corollary 6 $h : \Sigma^* \rightarrow \Sigma^*$ be a morphism such that $|h(\Sigma)| = |\Sigma|$. Then $h(\Sigma)$ is bifix code iff h preserves the bifix codes

Proposition 7 Let $h : \Sigma^* \rightarrow \Sigma^*$ be an injective morphism. Then $h(\Sigma)$ is a comma-free code iff h preserves the comma-free codes.

Proof.

[If] Trivial.

[Only if] Suppose h does not preserve the comma-free codes. For some comma-free code L , $h(L)$ is not a comma-free code. There exist $x, y \in \Sigma^+$, $w, u, v \in L$ such that $xh(w)y = h(u)h(v)$. Let $u = a_1 \dots a_n; v = b_1 \dots b_m; w = c_1 \dots c_k$, $a_1, \dots, a_n; b_1, \dots, b_m; c_1, \dots, c_k \in \Sigma$. Let $d_1 = a_1, \dots, d_n = a_n, d_{n+1} = b_1, \dots, d_{n+m} = b_m$. Note that $h(u) = h(a_1) \dots h(a_n); h(v) = h(b_1) \dots h(b_m); h(w) = h(c_1) \dots h(c_k)$.

(Case 1) There exists an integer i such that $h(d_i) = h(c_1), \dots, h(d_{i+k-1}) = h(c_k)$. By injectivity of h , L is not comma-free.

(Case 2) For some $i, j \geq 1$, $h(c_1) = h(d_i), \dots, h(c_j) = h(d_{i+j-1})$, $h(c_{j+1}) \neq h(d_{i+j})$. Then either $h(c_{j+1}) <_p h(d_{i+j})$ or $h(c_{j+1}) <_s h(d_{i+j})$. Thus $h(\Sigma)$ is not comma-free since it is not a prefix code.

(Case 3) For some integer $i \geq 2$, $h(d_1) \dots h(d_{i-1}) <_p x <_p h(d_1) \dots h(d_i)$

(3-1) $h(d_1) \dots h(d_i) <_p xh(c_1)$

If $h(d_1) \dots h(d_{i+1}) <_p xh(c_1)$, then $h(c_1)$ has a proper infix $h(d_{i+1})$. If $xh(c_1) <_p h(d_1) \dots h(d_{i+1})$, then $h(d_i)h(d_{i+1})$ has a proper infix $h(c_1)$. In either case, $h(\Sigma)$ is not comma-free. (3-2) $xh(c_1) <_p h(d_1) \dots h(d_i)$

$h(d_i)$ has a proper infix $h(c_1)$. Thus $h(\Sigma)$ is not an infix code.

(Case 4) $x <_p h(d_1)$

(4-1) $xh(c_1) <_p h(d_1)$

$h(d_1)$ has a proper infix $h(c_1)$. Thus $h(\Sigma)$ is not an infix code.

(4-2) $h(d_1) <_p xh(c_1)$

If $xh(c_1) <_p h(d_1)h(d_2)$, then $h(d_1)h(d_2)$ has a proper infix $h(c_1)$.

If $h(d_1)h(d_2) <_p xh(c_1)$, then $h(c_1)$ has a proper infix $h(d_2)$.

In either case, $h(\Sigma)$ is not comma-free. □

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