

Remarks on the relation between the
Lee-Yang circle theorem and the correlation inequalities

by

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Abstract

We investigate the relation between the Lee-Yang circle theorem and the correlation inequalities. These results are general and independent of models. General properties of the partition functions which belongs to the Lee-Yang class are given.

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1. Introduction

Recently several authors have investigated the Euclidean boson quantum field models (the so-called $P(\phi)_d$ -models) as a classical statistical mechanics [1],[2] . In these articles we see that the Lee-Yang circle theorem and the correlation inequalities do play a central role in the studying. On the other hand, Griffiths et al conjectured that a set of correlation inequalities will determine the forms of the interactions [3],[4] . From the view points of these applications and the conjectures, it is an interesting problem to decide the partition functions which satisfy the Lee-Yang circle theorem or the desired correlation inequalities.

Adding to these problems, Newman recently proved that the Lee-Yang circle theorem leads to some correlation inequalities [5]. Therefore it is also an interesting problem to discuss the relation between the Lee-Yang circle theorem and the correlation inequalities. Finally since the properties of the partition functions which satisfy the Lee-Yang circle theorem seem to be open, we investigate the general properties of them.

We organize the paper as follows:

In section 2, we define classes of the partition functions $\mathcal{L}_e, \mathcal{L}, \mathcal{D}, \mathcal{N}$, and summarize the relevant correlation inequalities without proof. In section 3, we investigate the

Griffiths first (G-I) and the second (G-II) inequalities and discuss the relation between these inequalities and the Lee-Yang circle theorem. In sections 4 and 5, we investigate the Griffiths-Hurst-Sherman inequality (GHS-inequality) and the Lebowitz inequality. In section 6, general properties of the partition functions which belong to the Lee-Yang class are given.

2. Classes $\mathcal{L}, \mathcal{D}, \mathcal{L}_e$

We summarize notations and definitions used in the following [6] :

D ; unit disk = $\{ z \in \mathbb{C} ; |z| \leq 1 \}$

∂D ; boundary of $D = \{ z \in \mathbb{C} ; |z| = 1 \}$

$\mathcal{L}_e^{(n)}$ or $\mathcal{L}_e$; polynomials of n -variables $z_1, \dots, z_n$ which are linear with respect to each z_i , and satisfy
 $P(z_1^{-1}, \dots, z_n^{-1}) = P(z_1, \dots, z_n) \prod z_i^{-1}$
 with $P(0, 0, \dots, 0) = 1$.

For the sake of the brevity, we restrict ourselves to the case where all the coefficients are real, thus $P \in \mathcal{L}_e$ is typically given by

$$P = (1 + z_1 z_2 \dots z_n) + \sum_i \beta_i^{(1)} (z_i + z_1 z_2 \dots \hat{z}_i \dots z_n) + \sum_{ij} \beta_{ij}^{(2)} (z_i z_j + z_1 z_2 \dots \hat{z}_i \dots \hat{z}_j \dots z_n) + \dots \quad (2-1)$$

with $\beta_{i_1 i_2 \dots i_l}^{(l)} \in (-\infty, \infty)$.

Here $\hat{z}_i$ (or $\hat{i}$) means that the variable z_i should be omitted.

$\mathcal{L}^{(n)}$ or $\mathcal{L}$; the Lee-Yang class $\mathcal{L}_e$. We say that $P \in \mathcal{L}_e$ belongs to $\mathcal{L}$ provided that any root of $P=0$ satisfies $z_i(z_j; j \neq i) \in D^c$ provided $z_j \in D$ ($j \neq i$) and $z_k \in D^0$ for some k ($k \neq i$).

$\mathcal{D}$; Set of $P \in \mathcal{L}_e$ such that all the roots of $P(z, \dots, z)=0$ lies on ∂D . Obviously $\mathcal{D} \supset \mathcal{L}$.

These definitions are general and independent of models. In order to define class $\mathcal{L}$, we use the Ising model of spin 1/2 where there are only ferromagnetic pair interactions:

$$H_\Lambda = - \sum_{i < j} J_{ij} (s_i s_j - 1)/2 - \sum h_i (s_i + 1)/2 \quad (2-2)$$

where $0 \leq J_{ij} \leq \infty$ and s_i ($i \in \Lambda$) is a random variable at the lattice site $i \in \Lambda$ which takes the values ± 1 . Let P be the relevant partition function;

$$P = \sum_{\{s_i = \pm 1\}} \exp(-H_\Lambda) \quad (2-3)$$

with $z_i = \exp(h_i)$.

Therefore P is given by the coefficients

$$\beta_{i_1 i_2 \dots i_\ell}^{(\ell)} = \prod_{i=i_1}^{i_\ell} \prod_{j \in \Lambda; j \neq i_1 i_2 \dots i_\ell} \gamma_{ij} \quad (2-4)$$

with $\gamma_{ij} = \exp(-J_{ij})$.

Then obviously $0 \leq \gamma_{ij} \leq 1$, however we extend this as $-1 \leq \gamma_{ij} \leq 1$, and denote the resultant set by $\mathcal{D}$.

For $P \in \mathcal{L}_e$, we identify P with its coefficients $\{\beta_{i_1 i_2 \dots i_\ell}^{(\ell)}\} \in \mathbb{R}^d$ ($d = 2^{n-1} - 1$), and consider the sets of functions $\mathcal{L}_e$ and $\mathcal{L}$ as the set of the coefficients. In this sense, we denote the

convex hulls of $\mathcal{L}, \mathcal{D}, \mathcal{D}$ by $\hat{\mathcal{L}}, \hat{\mathcal{D}}$ and $\hat{\mathcal{D}}$ respectively, and the closures of $\mathcal{L}$ by $\bar{\mathcal{L}}$.

Finally in order to study the correlation inequalities, we sometimes restrict ourselves to the subsets where all the coefficients are real non negative. We denote these by $\mathcal{L}_e^+, \mathcal{L}^+$ and $\mathcal{D}^+$ respectively.

Now we define the so-called ursell functions:

for $P \in \mathcal{L}_e^+$, we define

$$\left. \begin{aligned} u^{(2)}(i_1, \dots, i_\ell) &= \left(\prod_{i=1}^{\ell} z_i \frac{\partial}{\partial z_i} \right) \log P \quad \ell \geq 2 \\ u^{(1)}(i) &= z_i \frac{\partial}{\partial z_i} \log P - 1/2 \end{aligned} \right\} \quad (2-5)$$

As is well known $\mathcal{D} \subset \bar{\mathcal{L}}$ ($\mathcal{D}^+ \subset \bar{\mathcal{L}}^+$), and for $P \in \mathcal{D}^+$, we see [3], [4], [7]

[8], [9] :

Griffiths first inequality ; $u^{(1)}(i) \geq 0$ for $z_j \geq 1, j \in \Lambda$,

Griffiths second inequality ; $u^{(2)}(i, j) \geq 0$ for $z_j \geq 1, j \in \Lambda$,

GHS-inequality ; $u^{(3)}(i_1, i_2, i_3) \leq 0$ for $z_j \geq 1, j \in \Lambda$,

Lebowitz inequality ; $u^{(4)}(i_1, i_2, i_3, i_4) \leq 0$ for $z_j = 1, j \in \Lambda$,

Sylvester inequality ; $u^{(6)}(i_1, \dots, i_6) \geq 0$ for $z_j = 1, j \in \Lambda$,

where $\Lambda = \{1, 2, \dots, n\}$.

Following inequalities are conjectured by Newman for $P \in \mathcal{D}^+$ [5], [3]:

$$(-1)^{\ell-1} u^{(2\ell)}(i_1, \dots, i_{2\ell}) \geq 0 \text{ for } z_j = 1, j \in \Lambda.$$

C_i ; The set of the partition functions $P \in \mathcal{L}_e^+$ which satisfy the expected inequality for the i 'th ursell function.

3. $\mathcal{L}^+$ and $u^{(1)}, u^{(2)}$

Lemma 1. Let $P \in \mathcal{L}^+$, then $u^{(1)}(i) \geq 0$ provided $z_j \geq 1, j \in \Lambda$.

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proof. Let P be given by

$$P = B(z_1, \dots, z_{n-1}) + A(z_1, \dots, z_{n-1})z_n$$

where A, B are linear functions of $z_1, \dots, z_{n-1}$ with positive coefficients. $P \in \mathcal{L}$ implies

$$|B/A| \leq 1 \text{ provided } |z_i| \geq 1 \text{ } i=1, 2, \dots, n-1.$$

On the other hand,

$$u^{(1)}(n) = z_n (Az_n - B) / 2P.$$

Lemma 2. For $P \in \mathcal{L}_e$, followings hold:

- (i) If $u^{(1)}(i) \geq 0$ provided $z_j \geq 1, j \in \Lambda$, then
 $u^{(2)}(i_1, i_2) \big|_{z=1} \geq 0$.
- (ii) If $u^{(2)}(i_1, i_2) \geq 0$ provided $z_j \geq 1, j \in \Lambda$, then
 $u^{(1)}(i_1) \geq 0$ provided $z_j \geq 1, j \in \Lambda$

proof. (i) Let all z except z_{i_2} be equal to 1. Since $P \in \mathcal{L}_e$, $u^{(1)}(i_1) = (z_{i_2} - 1)f(z_{i_2})$ where the G-1 inequality ensures $f(z_{i_2}) \geq 0$ provided $z_{i_2} \geq 1$. Thus $u^{(2)}(i_1, i_2) \big|_{z=1} = f(1) \geq 0$.

(ii) Since $P \in \mathcal{L}_e$, $u^{(1)}(i_1) \big|_{z=1} = 0$. ▮

However, unfortunately $P \in \mathcal{L}^+$ does not necessarily imply the second Griffiths inequality with positive external fields, i.e., $P \in \mathcal{L}^+$ does not imply

$$u^{(2)}(i_1, i_2) \geq 0 \text{ with } z_j \geq 1, j \in \Lambda.$$

An explicit counterexample is given in the next section.

Finally for $P \in \mathcal{L}^+$, we can show the correlation inequalities which correspond to $\langle s_1 s_2 \dots s_\ell \rangle \geq 0$ provided $h_i \geq 0, i \in \Lambda$. This is the G-1 inequality in usual sense.

Theorem 1. Let $P(z_1, \dots, z_n) \in \mathcal{L}^+$ then

$$\prod_{i \in S} (z_i \partial / \partial z_i) [P(z_1, \dots, z_n) (\prod_{i=1}^n z_i)^{-1/2}] \geq 0 \quad (3-1)$$

provided $z_i \geq 1, i \in \Lambda$ where $S \subset \Lambda$ denotes the set of indices.

proof. It is sufficient to consider the case that all the indices are different. Let P be given by

$$P = \sum_{\{i_1, i_2, \dots, i_l\} \subset S} a_{i_1, i_2, \dots, i_l} z_{i_1} z_{i_2} \dots z_{i_l}$$

where $\{a_{i_1, i_2, \dots, i_l}\}$ are linear functions of $z_j, j \in \Lambda \setminus S$ with positive coefficients. Then

$$\prod_{i \in S} (z_i \partial / \partial z_i) [P \prod_{i=1}^n z_i^{-1/2}] = [2^{|S|} \prod_{i=1}^n z_i^{1/2}]^{-1} Q,$$

where

$$\begin{aligned} Q &= 2^{|S|} z^S P_S - 2^{|S|-1} \sum_{i \in S} z^{S \setminus i} P_{S \setminus i} + 2^{|S|-2} \sum_{i, j \in S} z^{S \setminus \{i, j\}} P_{S \setminus \{i, j\}} \\ &\quad + \dots + (-1)^{|S|} P \\ &= \sum_{\{i_1, i_2, \dots, i_l\} \subset S} (-1)^{|S| - |I|} a_I z^I \end{aligned} \quad (3-2)$$

with

$$a_I = a_{i_1, i_2, \dots, i_l}; \quad z^I = \prod_{i \in I} z_i; \quad P_I = \prod_{i \in I} \partial / \partial z_i P$$

Let $z_j, j \in \Lambda \setminus S$ be fixed and ≥ 1 , thus we study the necessary and sufficient condition that ensures $Q \geq 0$ provided that $z_i \geq 1, i \in S$.

Following Lemma 4, which will be proved later, this is

$$(1) \quad a_{1, 2, \dots, l} \geq 0$$

$$(2) \quad a_{1, 2, \dots, l} - a_{1, 2, \dots, \hat{i}, l} \geq 0 \quad i \in S,$$

$$(3) \quad a_{1, 2, \dots, l} - a_{1, 2, \dots, \hat{i}, l} - a_{1, 2, \dots, \hat{j}, l} + a_{1, 2, \dots, \hat{i}, \hat{j}, l} \geq 0, i, j \in S,$$

.....

$$\begin{aligned} (l+1) \quad & a_{1, 2, \dots, l} - \sum_{i \in S} a_{1, 2, \dots, \hat{i}, l} + \sum_{i, j \in S} a_{1, 2, \dots, \hat{i}, \hat{j}, l} \\ & + \dots + (-1)^l a_{\hat{1}, \hat{2}, \dots, \hat{l}} \geq 0. \end{aligned}$$

Here without loss of generality, we put $S = \{1, 2, 3, \dots, l\} \subset \Lambda$.

These conditions are equivalent to

$$\begin{aligned} & (-1)^{|S \setminus I|} P_I(z_j = -1; j \in S \setminus I) \\ &= (-1)^{|S \setminus I|} \left(\prod_{i \in I} \partial / \partial z_i \right) P|_{z_j = -1; j \in S \setminus I} \geq 0 \end{aligned}$$

for any subset $I \subset S$. Since $P \in \mathcal{L}^+$, all the roots of $P_I(z_j = z; j \in S \setminus I) = 0$ lie in the unit disk D . Now we investigate the sign of $P_I(-1)$.

$$\begin{aligned} P_I(z) &= a_S z^{|S \setminus I|} + \dots + a_I = \sum_{J: I \subset J \subset S} a_J z^{|J \setminus I|} \\ &= a_S \prod_{i=1}^r [(z - \omega_i)(z - \bar{\omega}_i)] \prod_{j=1}^{|S \setminus I| - 2r} (z - \zeta_j) \end{aligned}$$

where $\{\omega_i, \bar{\omega}_i \mid |\omega_i| \leq 1\}$ are the complex roots, and $\{\zeta_j \mid |\zeta_j| \leq 1\}$ are the real roots of $P_I = 0$. Since $a_S > 0$, $(1 - \omega_i)(1 - \bar{\omega}_i) > 0$ and $\text{sgn} \prod_{j=1}^{|S \setminus I| - 2r} (-1 - \zeta_j) = \text{sgn} (-1)^{|S \setminus I|}$, we see

$$(-1)^{|S \setminus I|} P_I(-1) \geq 0.$$

This completes the proof. ///

Finally we would like ^{to} point out that if $P \in \mathcal{L}$, $u^{(1)}(i)$ or $\langle s_1 s_2 \dots s_\ell \rangle \propto \prod_{i \in S} z_i \partial / \partial z_i [P \prod_{j \in S} z_j^{-1/2}]$ also satisfy the definition of $\mathcal{L}$ except the evenness condition $P(z_1^{-1}, \dots, z_n^{-1}) = P(z_1, \dots, z_n^{-1}) \prod_{i=1}^n z_i^{-1}$. This is obvious because if $P(z_1, \dots, z_n) \in \mathcal{L}$, $P(\exp(i\theta_1)z_1, \dots, \exp(i\theta_n)z_n)$ with $\theta_i \in \mathbb{R}$ again satisfies the definition of $\mathcal{L}$ except the evenness condition, and these correlation functions are essentially given by (3-2). However, this is not true for the higher order ursell functions. In fact if it were true, the higher order ursell functions would have definite signs in $\{z_i \geq 1; i \in \Lambda\}$.

4. $\mathcal{L}^+$ and $u^{(3)}, u^{(4)}$

In the cases of $n=1,2$, $\bar{\mathcal{L}} = \mathcal{D}$ ($\bar{\mathcal{L}}^+ = \mathcal{D}^+$), and $C_i = \mathcal{D}^+ = \mathcal{L}^+$, ($i=1,2,3,4$). In the case of $n=3$, we will easily see that $\bar{\mathcal{L}} = \hat{\mathcal{D}}$ ($\bar{\mathcal{L}}^+ = \hat{\mathcal{D}}^+$), and $P \in \mathcal{L}^+$ does not imply the desired inequality.

Lemma 3. Let $P \in \mathcal{L}_e$, and be given by

$$P = 1 + z_1 z_2 z_3 + \sum_{i=1}^3 \beta_i (z_i + z_1 \bar{z}_i z_3) . \quad (4-1)$$

Then $P \in \mathcal{L}$ if and only if

$$|1 + \beta_i| > |\beta_j + \beta_k| . \quad (4-2)$$

proof. It is necessary and sufficient that

$(z_3)^{-1} = -[\beta_3 + \beta_1 z_1 + \beta_2 z_2 + z_1 z_2] / [1 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_1 z_2] \in D^0$ provided $z_1, z_2 \in D$ and some of them $\in D^0$. Remark that the Shilov boundary of the polydisk $D \otimes D \otimes \dots \otimes D$ is $\partial D \otimes \dots \otimes \partial D$. Since $z_3 \in \partial D$ provided $(z_1, z_2) \in \partial D \times \partial D$, the problem reduces to obtain a condition which is equivalent to

$$P(z_1, z_2, z_3=0) = 1 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_1 z_2 \neq 0$$

$$\text{provided } (z_1, z_2) \in D \otimes D .$$

Therefore

$$|z_2|^{-1} = |(\beta_2 + \beta_3 z_1) / (1 + \beta_1 z_1)| < 1 \text{ provided } z_1 \in D .$$

This completes the proof. $\square$

Now we investigate the correlation inequalities for $P \in \mathcal{L}$ given in the previous lemma 3. When all the arguments are different with each other, we have

$$u^{(3)}(1,2,3) = z_1 z_2 z_3 \Pi \partial / \partial z_i \log P = z_1 z_2 z_3 f_3 / P^3$$

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where

$$f_3 = p^2 p_{1,2,3} - \sum p_i p_{j,k} + p_1 p_2 p_3$$

$$= s_0(1 - z_1 z_2 z_3) + s_1(z_1 - z_2 z_3) + s_2(z_2 - z_3 z_1) + s_3(z_3 - z_1 z_2)$$

with

$$s_0 = 1 - \beta_1^2 - \beta_2^2 - \beta_3^2 + 2\beta_1 \beta_2 \beta_3,$$

$$s_1 = \beta_1(1 - \beta_1^2 + \beta_2^2 + \beta_3^2) - 2\beta_2 \beta_3,$$

$$s_2 = \beta_2(1 + \beta_1^2 - \beta_2^2 + \beta_3^2) - 2\beta_3 \beta_1,$$

$$s_3 = \beta_3(1 + \beta_1^2 + \beta_2^2 - \beta_3^2) - 2\beta_1 \beta_2.$$

(4-3)

Lemma 4. Let

$$f = a_0 z_1 z_2 \dots z_n + \sum a_i z_1 z_2 \dots \hat{z}_i \dots z_n + \sum a_{ij} z_1 z_2 \dots \hat{z}_i \dots \hat{z}_j \dots z_n \\ + \dots + a_{1,2,\dots,n}.$$

Then the necessary and sufficient condition that $f \geq 0$ provided $z_i \geq 1$ ($i \in \Lambda$) is

$$a_0 \geq 0$$

$$a_0 + a_i \geq 0 \quad i \in \Lambda$$

$$a_0 + a_i + a_j + a_{i,j} \geq 0 \quad i, j \in \Lambda$$

$$\dots \dots \dots$$

$$a_0 + \sum a_i + \sum a_{i,j} + \dots + a_{1,2,\dots,n} \geq 0$$

proof. Remark that f is a linear function with respect to each variable. Therefore the necessary and sufficient condition that $f \geq 0$ provided $z_i \geq 1$ is

$$f(z_1, z_2, \dots, z_{n-1}, 1) \geq 0$$

$$\partial / \partial z_n f(z_1, \dots, z_n) = \partial / \partial z_n f(z_1, \dots, z_n) |_{z_n=1} \geq 0$$

provided $z_i \geq 1, i \in \Lambda$. This discussion leads to the following condition:

$$\text{for any } i \in \Lambda, \left(\prod_{i \in \Lambda} \partial / \partial z_i \right) f |_{z=1} \geq 0.$$



Theorem 2. For $P \in \mathcal{L}^+$, we see:

- (i) $u^{(3)}(1,2,3) \leq 0$ provided $z_1, z_2, z_3 \geq 1$ and $z_4 = z_5 = \dots = z_n = 1$.
- (ii) $u^{(2)}(1,2)$ is not necessarily positive provided $z_i \geq 1$,
however, $u^{(2)}(1,2) \geq 0$ provided $z_1, z_2 \geq 1$ and $z_3 = z_4 = \dots = z_n = 1$.
- (iii) For $P \in \mathcal{L}_e^+$, let $\tilde{P} = P|_{z_4 = \dots = z_n = 1}$, then for $\tilde{P}$ we see
that if $u^{(1)}(i) \geq 0$ provided $z_j \geq 1$ ($j=1,2,3$), $\tilde{P} \in \mathcal{L}^+$.

proof. (i) It is sufficient to consider P given by (4-1).

Thus following lemma 4 and (4-3), we must prove

$$\begin{aligned} s_0 + s_i + s_j - s_k &\geq 0, & (i,j,k) &= (1,2,3) \\ s_0 + s_i &\geq 0, & (i &= 1,2,3) \\ s_0 &\geq 0 \end{aligned}$$

provided $(\beta_1, \beta_2, \beta_3) \in \mathcal{L}^+$. It is a straightforward but tedious calculation.

(ii) We present an example. For P given by (4-1),

$$u^{(2)}(1,2) = z_1 z_2 [(\beta_3 - \beta_1 \beta_2)(1 + z_3^2) + (1 - \beta_1^2 - \beta_2^2 + \beta_3^2)z_3] / P^2.$$

Thus obviously $u^{(2)} \geq 0$ provided $P \in \mathcal{L}^+$ and $z_3 = 1$ (this includes general cases), however, consider the point $(1/3, 1/3, 0) \in \mathcal{L}^+$. At the point $\beta_3 - \beta_1 \beta_2 = -1/9$, then if z_3 is large enough, we see $u^{(2)} < 0$.

(iii) Following lemma 4, the necessary sufficient condition that $u^{(1)}(i) \geq 0$ provided $z_j \geq 1$ ($j=1,2,3$) is

$$1 + \beta_i \geq \beta_j + \beta_k \quad (i,j,k) = (1,2,3) \quad \text{///}$$

Finally we investigate the fourth ursell function [7],[8]:

$$u^{(4)}(1,2,3,4) = \prod_{i=1}^4 (z_i \partial / \partial z_i) \log P.$$

As is well known, even if $P \in \mathcal{D}^+$, $u^{(4)}$ does not necessarily

negative when $z_i \geq 1$, but negative provided $z_i = 1$ (the so-called Lebowitz Inequality). Contrary to the case of $u^{(2)}, u^{(4)}$ is not necessarily negative even if $P \in \mathcal{L}^+$ and $z_i = 1$ ($i \in \Lambda$). In fact let

$$P(z_1, z_2, z_3, z_4) = P \mid z_5 = \dots = z_n = 1$$

$$= \text{const.} [(1+z_1 \dots z_4) + \sum \beta_i (z_i + z_1 \dots \hat{z}_i \dots z_4) + \sum \beta_{ij} (z_i z_j + z_1 \dots \hat{z}_i \dots \hat{z}_j \dots z_4)] \quad (4-4)$$

Thus

$$u^{(4)}(1, 2, 3, 4) = z_1 z_2 z_3 z_4 [P^3 P_{1,2,3,4} - P^2 \sum P_{ij} P_{kl} - P^2 \sum P_{ij} P_{kl} + 2P \sum P_{ij} P_{kl} - 6P_1 P_2 P_3 P_4] / P^4 \quad (4-5)$$

and

$$u^{(4)}(1, 2, 3, 4) \mid z=1 = \text{positive const.} \{ (\sum \beta_i)^2 / 2 - 2 \sum \beta_i \beta_j - 1/2 \} + \{ (\sum \beta_{ij})^2 / 2 - \sum \beta_{ij}^2 + \sum \beta_{ij} \} \} \quad (4-6)$$

A point $(\beta_i = 0, \beta_{ij} = 1/3) \in \mathbb{R}^7$ is $\in \mathcal{L}^+$, but at the point $[\] = 2/3 > 0$. Thus we see that $P \in \mathcal{L}^+$ implies neither $u^{(4)} \leq 0$ with zero external fields nor $u^{(3)} \leq 0$ with positive external fields (see the next section).

5. Some remarks on the correlation inequalities.

Now we see that the partition functions which belong to $\mathcal{L}^+$ do not necessarily satisfy the correlation inequalities expected from the results seen in $P \in \mathcal{D}^+$. The reason is obvious, in fact $P \in \mathcal{L}^+$ is a property which is derived from

the behavior of P on $D \otimes D \otimes \dots \otimes D \subset C^n$, and on the other hand correlation inequalities crucially depend on the behavior on $[1, \infty)^n \subset R^n$. Our examples suggest

$$D^+ \subset \bigcap_i \bigcup_i^{C_i} C_i \subset \bar{\mathcal{L}}^+ \quad (5-1)$$

However, Newman showed [5]

$$(-1)^{l-1} (z \partial / \partial z)^{2l} \log P(z, z, \dots, z) \Big|_{z=1} \geq 0 \quad (5-2)$$

provided $P \in \mathcal{D}$.

From our standing point of view, these are special urself functions.

Our analysis implies that $\mathcal{L}^+$ -class is too wide to satisfy all the correlation inequalities. Finally we show that the even'th correlation inequalities with zero external fields follow from the odd'th correlation inequalities with positive external fields (see also the note-added in [9]).

Theorem 3. For $P \in \mathcal{L}_e$, if $u^{(3)}(i, j, k) \leq 0$ with positive external fields holds, $u^{(4)}(i, j, k, l)_{z=1} \leq 0$.

proof.

$$u^{(4)}(i, j, k, l) = z_1 \partial / \partial z_1 u^{(3)}(i, j, k) \Big|_{z=1}$$

Since we can put all z except z_1 equal to 1 in $u^{(3)}$ and $P \in \mathcal{L}_e$, then we have

$$u^{(3)}(i, j, k) = (1 - z_1) f(z_1) \quad (5-3)$$

where the GHS inequality ensures $f(z_1) \geq 0$ provided $z_1 \geq 1$.

This completes the proof. 

Remark. As is well known, the higher order urself functions do not satisfy the expected inequality for $z_i \geq 1$.

even for $P \in \mathcal{D}^+$. However, if they satisfy the conjectured inequalities (including odd'th ursell functions) for $1 \leq z_i < 1+\epsilon$ with $\epsilon > 0$, $i \in \Lambda$, we see that $(-1)^{\ell-1} u^{(2\ell)}|_{z=1} \geq 0$ can be derived from $(-1)^{\ell-1} u^{(2\ell-1)} \geq 0$ with $1 \leq z_i < 1+\epsilon$. If this is true, $u^{(2\ell)}$ and $u^{(2\ell-1)}$ should be considered as a pair. See also the dicussions in lemma 2, and by the same discussions, we see that the converse is true.

Corollary 1. For $P \in \mathcal{L}^+$, $u^{(4)}(i,j,k,1)|_{z=1} \leq 0$ provided that at least two of $(i,j,k,1)$ are equal.
proof. (i) Two arguments are equal.

Following Theorem 2 and Theorem 3, it is obvious.

(ii) Three arguments are equal.

Without loss of generality, let $(i,j,k,1) = (1,2,2,2)$. Thus

$u^{(4)}|_{z=1} = (PP_{1,2} - P_1 P_2)(P^2 + 6P_2^2 - 6PP_2) / P^4|_{z=1}$. This is negative since $P \in \mathcal{L}^+$. Finally if all the arguments are equal, the problem reduces to $P = \text{const.}(1+z)$. $\square$

6. Structure of $\mathcal{L}$ and $\mathcal{D}$.

Before studying the topological structure of $\mathcal{L}$, we would like to point out that a product can be defined on $\mathcal{L}$ [6], [10], [11]. We call this product the Asano product.

Theorem 4. Let $\{\alpha_{i_1, \dots, i_\ell}^{(\ell)}\} \in \mathcal{L}^{(n)}$, $\{\beta_{i_1, \dots, i_\ell}^{(\ell)}\} \in \mathcal{L}^{(n)}$
then $\{\alpha_{i_1, \dots, i_\ell}^{(\ell)} \beta_{i_1, \dots, i_\ell}^{(\ell)}\} \in \mathcal{L}^{(n)}$

This is a very well known theorem, and we do not repeat the proof. Details are shown in [6],[10],[11]. Therefore $\mathcal{L}$ has a semi-group structure by this product. Remark that $\mathcal{D}$ is also closed under the product. We denote this product by $\{\alpha\beta\}$ or $A[P_\alpha P_\beta]$.

The Lee-Yang class $\mathcal{L}$ is a much complicated set in the space of the d -coefficients ($d=2^{n-1}-1$). Let P be given by (2-1), then we identify P with $\{\beta_1^{(1)}, \dots, \beta_{1,2}^{(2)}, \dots\} \in \mathbb{R}^d$.

Lemma 5. If $P \in \mathcal{L}$, then $-1 \leq \beta_i^{(\ell)}, \dots \leq 1$.

proof. If $P \in \mathcal{L}$, then $A[P^N] = A[PA[\dots A[PP]\dots]] \in \mathcal{L}$. $A[P^N]$ is given by $\{\beta_i^{(\ell)N}\}$. Since all the coefficients of $\mathcal{L}$ must be bounded, $|\beta_i^{(\ell)}| \leq 1$. This completes the proof. $\square$

Theorem 4.

(i) $\mathcal{L}$ is an open, arcwise-connected set.

(ii) $\mathcal{L}$ is homeomorphic to d -dimensional opendisk $D^{(d)}$.

proof. (i) The openness of $\mathcal{L}$ follows from the definition.

Let $P_t \in \mathcal{D}$ be given by putting all $\gamma_{ij} = t \in [0,1]$.

Therefore $P_1 \in \bar{\mathcal{L}}$ and $P_t \in \mathcal{L}$ for $t \in [0,1)$. P_t is a continuous line connecting $\Pi^n(1+z_i) \in \bar{\mathcal{L}}$ and $1 + \Pi^n z_i \in \mathcal{L}$, and lies in $\mathcal{L}$. For any $P \in \mathcal{L}$, $\tilde{P}_t = A[P_t P] \in \mathcal{L}$ is continuous with respect to $t \in [0,1]$, and $\tilde{P}_1 = P \in \mathcal{L}$, $\tilde{P}_0 = 1 + \Pi z_i \in \mathcal{L}$. Thus $\mathcal{L}$ is arcwise connected.

(ii) From the above discussions, we see, by operating $A[P_t \cdot \cdot]$, that any sub set of $\mathcal{L}$ can be continuously contracted to the origin. This completes the proof. $\square$

Remark. Even if the coefficients are complex, these statements can be extended by suitable redefinitions [5].

The main theorem in this section is:

Theorem 5.

- (i) $\mathcal{D} \subset \bar{\mathcal{L}} \subset \hat{\mathcal{D}}$,
- (ii) $\mathcal{D}^+ \subset \bar{\mathcal{L}}^+ \subset \hat{\mathcal{D}}^+$,
- (iii) $\hat{\mathcal{L}} = \hat{\mathcal{D}}$; $\hat{\mathcal{D}}^+ = \hat{\mathcal{L}}^+$.

proof. (i) $\mathcal{D} \subset \bar{\mathcal{L}}$, $\mathcal{D}^+ \subset \bar{\mathcal{L}}^+$ are well known. Consider the following (d+1) functions:

$$\prod^n (1 \pm z_i)$$

where the number of (-) sign is even, and which ensures that these functions belong to $\mathcal{L}_e$. We denote these functions by P_i ($i=1, \dots, d+1$), and remark that $P_i \in \mathcal{D}$ ($P_i \in \bar{\mathcal{L}}$) and these are all linearly independent. Then $P = \sum \alpha_i P_i$ with $\alpha_i \geq 0$, $\sum \alpha_i = 1$ becomes a d-dimensional convex cell in the d-dimensional space of the coefficients. Thus, denoting this convex cell by $\hat{\mathcal{D}}'$, we show $\partial \hat{\mathcal{D}}' \notin \mathcal{L}$. If once it is proved, (i) follows from theorem 4 and the fact $\mathcal{D} \subset \bar{\mathcal{L}}$. Each of $\partial \hat{\mathcal{D}}'$ is a (d-1) dimensional convex cell. We rewrite P as

$$\begin{aligned} P &= P|_{z_n=0} + z_n \partial/\partial z_n P \\ &= B(z_1, \dots, z_{n-1}) + A(z_1, \dots, z_{n-1}) z_n \end{aligned}$$

Since $P \in \mathcal{L}$ is equivalent to

$$|A/B| < 1$$

provided $z_i \in D$ and some $z_j \in D^0$, and $P \in \mathcal{L}_e$ implies $|A/B| = 1$ provided all $z_j \in \partial D$, it is necessary and sufficient that $B \neq 0$ provided $(z_1, z_2, \dots, z_{n-1}) \in \partial D^{n-1}$.

B is given by $\sum_{i=1}^n \alpha_i P_i$ $| z_n=0$, and consider the point $(z_1, z_2, \dots, z_{n-1}) = (\pm 1, \pm 1, \dots, \pm 1)$. There are 2^{n-1} points. For the given point, the function which does not vanish at the point is one of the following two possible functions:

$$\prod_{i=1}^{n-1} (1 \pm z_i) (1 + z_n)$$

$$\prod_{i=1}^{n-1} (1 \pm z_i) (1 - z_n)$$

and only one of these functions belongs to $\mathcal{L}_e$. Thus d -functions of $\{P_i\}$ vanish at the point. Therefore $\mathcal{L} \neq \partial \hat{\mathcal{D}}$, and (i) follows.

(ii) Each hypersurface of $\partial \hat{\mathcal{L}}$ is given by

$$P = \sum_{i=1}^d \alpha_i P_i \quad \alpha_i \geq 0, \quad \sum_{i=1}^d \alpha_i = 1.$$

Since we restrict ourselves to $\mathcal{L}^+$, one of P_i ($i=1, 2, \dots, d$) is $\prod_{i=1}^n (1 + z_i)$. These hypersurfaces intersect the positive part of the coordinate-axis at the points

$$(1 + \prod_{i \in I} z_i) (1 + \prod_{j \in I^c} z_j) \in \mathcal{D}^+$$

$$I \neq \emptyset, \Lambda.$$

The convex hull of these points together with $(1 + \prod_{i=1}^n z_i)$ and $\prod_{i=1}^n (1 + z_i)$ includes $\mathcal{L}^+$.

(iii) This is obvious from the above discussions. 

Remark

(i) Set $\mathcal{L}$ is much complicated, and $\partial \hat{\mathcal{L}}$ is constructed by algebraic manifolds. $\mathcal{L}$ seems to be a "concave set". To confirm the conjecture, consider the function $P = (1 - \lambda)(1 + \prod_{i=1}^n z_i) + \lambda \prod_{i=1}^n (1 + z_i)$ with $0 \leq \lambda \leq 1, n \geq 3$. $P \in \mathcal{L}^+$ if and only if $0 \leq \lambda < [1 + (2 \cos \pi / (n-1))^{n-1}]^{-1}$.

(ii) One may define the vertices of $\hat{\mathcal{L}}, \mathcal{D}$. In order to define the vertices, however, we must use the terminologies

of algebraic geometry. Since $\bar{\mathcal{L}}$ and $\mathcal{D}$ are semi-analytic sets, we can define the vertices as the zero dimensional singularities of $\partial\bar{\mathcal{L}}, \partial\mathcal{D}$. This is usually done through the stratification of the singularity. We conjecture

$$(i) \text{Ver } \bar{\mathcal{L}} = \text{Ver } \mathcal{D} ,$$

$$(ii) \text{Ver } \bar{\mathcal{L}}^+ = \text{Ver } \mathcal{D}^+ .$$

These can be confirmed for $n=1,2,3,4$.

Finally we comment on the some interesting properties of $\mathcal{D}$. Let $P(z, \dots, z) = (1+z^n) + a_1(z+z^{n-1}) + a_2(z^2+z^{n-2}) + \dots \in \mathcal{D}$. Recently Millard et al have obtained a generalization of the Ruelle's lemma [6], [12]:

Theorem 6. Let A and B be closed circular regions not containing the origin. If $f = \sum_{i=0}^n b_i z^i$ vanishes only in $A \subset \mathbb{C}$, and $g = \sum_{i=0}^n c_i z^i$ vanishes only in $B \subset \mathbb{C}$, then $A[fg] = \sum_{i=0}^n \frac{1}{n} c_i^{-1} b_i c_i z^i$ vanishes only in $AB = \{ z \in \mathbb{C}; z = -z_1 z_2, z_1 \in A, z_2 \in B \}$.

Therefore, using the same techniques in Theorem 4, we have:

Theorem 7.

(i) $\mathcal{D}$ is closed, arcwise-connected set and all of the homotopies of $\mathcal{D}$ vanish.

(ii) Let $\mathcal{F}_S = \{ z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + 1; (a_{n-1}, \dots, a_1) \in \mathbb{R}^{n-1} \}$ be functions whose roots are all in an open region SCC which is invariant under the rotation around the origin. Then $\mathcal{F}_S$ is homeomorphic to $(n-1)$ -dimensional open disk D^{n-1} .

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