

Self-diffusion constants of non-colliding interacting Brownian motions in one spatial dimension

By

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Abstract

We prove that self-diffusion constants of interacting Brownian particles in $\mathbb{R}$ always vanish if the particles do not collide with each other. We represent self-diffusion constants by additive functionals of reversible Markov processes as obtained in [13].

§ 1. Introduction

We consider infinitely many Brownian particles $\mathbf{X} = (X^i)_{i \in \mathbb{Z}}$ moving in $\mathbb{R}^d$ with interaction Ψ and inverse temperature $\beta > 0$. Intuitively, $\mathbf{X}$ is given by the infinite-dimensional stochastic differential equation (ISDE)

$$(1.1) \quad dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \in \mathbb{Z}, j \neq i} \nabla \Psi(X_t^i - X_t^j) dt, \quad (i \in \mathbb{Z})$$

and is called an interacting Brownian motion in infinite dimensions [9, 10, 24, 2, 27, 17, 23]. We set the configuration-valued process $\mathbf{X}$, called unlabeled dynamics, to be

$$(1.2) \quad \mathbf{X}_t = \sum_{i \in \mathbb{Z}} \delta_{X_t^i}.$$

The dynamics in the present paper are quite general and not necessarily given by ISDEs of the form (1.1). We later present the unlabeled dynamics $\mathbf{X}$ using Dirichlet form theory (2.2) where the labeled dynamics $\mathbf{X}$ are an $(\mathbb{R}^d)^{\mathbb{Z}}$ -valued additive functional of $\mathbf{X}$ (2.9).

Received January 31, 2016. Revised May 12, 2016.

2010 Mathematics Subject Classification(s): 60J60, 60J70, 60F17, 82C21, 82C22

Key Words: homogenization, self-diffusion constant, self-diffusion matrix, interacting Brownian motions, Kipnis–Varadhan theory, Dirichlet forms.

Supported in part by a Grant-in-Aid for Scientific Research (KIBAN-A, No. 24244010) of the Japan Society for the Promotion of Science.

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We suppose that the system is in equilibrium in the sense that

$$X_t \underset{\text{law}}{=} X_0 \quad \text{for all } t,$$

and that X is a μ -reversible diffusion, where μ is the distribution of X_0 . Thus μ is the equilibrium state of the unlabeled dynamics X . By definition, μ is a probability measure on the configuration space S over $\mathbb{R}^d$:

$$(1.3) \quad S = \{s = \sum_i \delta_{s_i}; s_i \in \mathbb{R}^d, s(\{|s_i| \leq r\}) < \infty \text{ for all } r \in \mathbb{N}\}.$$

We equip S with the vague topology under which S is a Polish space. Throughout the paper we assume that μ is translation invariant:

$$(1.4) \quad \mu \circ \theta_x^{-1} = \mu \quad \text{for each } x \in \mathbb{R}^d,$$

where $\theta_x: S \rightarrow S$ is the shift $\theta_x(s) = \sum_i \delta_{s_i+x}$ for $s = \sum_i \delta_{s_i}$. By definition, $\theta_{-x} = \theta_x^{-1}$, and θ_x is a homeomorphism for each x .

We tag a particle $X^0 = \{X_t^0\}$, say, and study its asymptotic behavior. In particular, we investigate the diffusive scaling limit. The celebrated Kipnis–Varadhan theory [8] asserts that tagged particles of reversible systems converge to (a constant multiple of) the Brownian motion B :

$$\lim_{\varepsilon \rightarrow 0} \varepsilon X_{\cdot/\varepsilon^2}^0 = \sigma(s)B,$$

where $\sigma(s)$ may depend on the initial configuration s of environment seen from the tagged particle, that is, $s = \sum_{i \neq 0} \delta_{X_0^i - X_0^0}$. The average $\alpha[\mu]$ of $\sigma^t \sigma$ with respect to the reduced Palm measure $\mu_0 = \mu(\cdot - \delta_0 | s(\{0\}) \geq 1)$ conditioned at the origin is called a self-diffusion matrix (see (3.8)).

It is known that $\alpha[\mu]$ is always positive definite if $d \geq 2$ and that the interaction Ψ is of Ruelle's class with hard core having positive volume [14]. It is expected that self-diffusion matrices for point processes with Ruelle's class potentials are always positive definite in multiple dimensions. The only known degenerate example in multiple dimensions is Ginibre interacting Brownian motion, in which an infinite-particle system interacts via the two-dimensional Coulomb potential $\Psi(x) = -\log|x|$ with $\beta = 2$ [20]. We remark that the two-dimensional Coulomb potential is not of Ruelle's class because it is unbounded at infinity.

In one dimension, there are degenerate examples such as hard rods (Harris [6]) and Dyson's model (Spohn [25, 26]), which have scaling orders such that $O(t^{1/4})$ in [6] and $O((\log t)^{1/2})$ in [26], respectively.

The main purpose of this paper is to give a sufficient condition for the degeneracy of the self-diffusion constant in one dimension. We prove that $\alpha[\mu] = 0$ if particles do

not collide with each other (Theorem 2.1). Let

$$\Delta = \{\mathbf{x} = (x_i)_{i \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}; x_i < x_{i+1} \quad \text{for all } i \in \mathbb{Z}\}.$$

Then the non-collision condition (2.6) requires that $\mathbf{X}_t \in \Delta$ for all $0 \leq t < \infty$ for all labeled particles $\mathbf{X} = (X^i)_{i \in \mathbb{Z}}$ that start at $\mathbf{X}_0 \in \Delta$. The set Δ is very tiny compared with the whole space $\mathbb{R}^{\mathbb{Z}}$, which is the state space of independent Brownian motions $\mathbf{B} = (B^i)_{i \in \mathbb{Z}}$ and interacting Brownian motions $\mathbf{X}$ with Ruelle's class potentials but without non-collision condition. Intuitively, such a small state space of interacting one-dimensional Brownian motions with the non-collision condition results in sub-diffusive behavior. Briefly, smallness of the state space implies sub-diffusivity. To some extent, this phenomenon resembles the sub-diffusivity of a simple random walk in the incipient infinite cluster, which is a random domain enjoying a fractal structure at the critical point of Bernoulli percolation.

Another purpose of this paper is to give a statement of the main theorem in a more natural fashion than previously (Theorem 3.7). Traditionally, these problems are stated for tagged particles $\{X_t^0\}$ as functionals of the stationary environment process $\mathbf{Y} = \{\sum_{i \neq 0} \delta_{X_t^i - X_t^0}\}$, with the reduced Palm measure μ_0 as the invariant probability measure [1, 5, 8, 14]. One statement of the result is that, for any $F \in C_b(C([0, \infty); \mathbb{R}^d))$,

$$(1.5) \quad \lim_{\epsilon \rightarrow 0} \mu_0(\{\mathbf{s}; |\mathbb{E}_{\mathbf{s}}^{\mathbf{Y}}[F(\epsilon X_{\cdot/\epsilon^2}^0)] - E[F(\sigma(\mathbf{s})B)]| \geq \kappa\}) = 0 \quad \text{for each } \kappa > 0.$$

Here, the expectation $\mathbb{E}_{\mathbf{s}}^{\mathbf{Y}}$ is with respect to the distribution $\mathbf{P}_{\mathbf{s}}^{\mathbf{Y}}$ of the environment process $\mathbf{Y} = \{\sum_{i \neq 0} \delta_{X_t^i - X_t^0}\}$ starting at $\mathbf{s}$, and μ_0 is the initial distribution of the whole environment process $\mathbf{Y}$. Furthermore, $B = \{B_t\}$ denotes the standard d -dimensional Brownian motion and $E[\cdot]$ is the expectation with respect to B . This reduction of the problem through the idea of “tagged particle and environment seen from the tagged particle” is a key idea in Kipnis-Varadhan theory for tagged particle problems.

We define a label $\mathbf{l}(\mathbf{s}) = (\mathbf{l}_i(\mathbf{s}))_{i \in \mathbb{Z}} = (s_i)_{i \in \mathbb{Z}}$ for μ -a.s. $\mathbf{s} \in \mathbf{S}$, and construct the labeled process $\mathbf{X} = (X^i)_{i \in \mathbb{Z}} = \mathbf{l}_{\text{path}}(\mathbf{X})$ from the unlabeled process $\mathbf{X} = \sum_{i \in \mathbb{Z}} \delta_{X^i}$ (see (2.9)). A refinement of statement (1.5) is that, for each $i \in \mathbb{Z}$,

$$(1.6) \quad \lim_{\epsilon \rightarrow 0} \mu(\{\mathbf{s}; |\mathbb{E}_{\mathbf{s}}[F(\epsilon X_{\cdot/\epsilon^2}^i)] - E[F(\sigma(\mathbf{s})B)]| \geq \kappa\}) = 0 \quad \text{for each } \kappa > 0.$$

Here, $\mathbb{E}_{\mathbf{s}}$ is the expectation with respect to the distribution $\mathbf{P}_{\mathbf{s}}$ of the original μ -reversible diffusion $\mathbf{X} = \sum_{i \in \mathbb{Z}} \delta_{X^i}$ given by (1.2) starting at $\mathbf{s}$.

We thus state the theorem in terms of the scaling limit of each tagged particle X^i of the original unlabeled dynamics $\mathbf{X} = \sum_i \delta_{X^i}$ (Theorem 3.7). To formulate the statement as (1.6), we need to prepare a label $\mathbf{l}$ to choose tagged particles from the unlabeled system.

We remark that the equilibrium state for the unlabeled dynamics is μ . Note that μ_0 is not necessary absolutely continuous with respect to μ . Indeed, the Ginibre point process is an example where μ and μ_0 are singular each other [22]. We nonetheless deduce claim (1.6) from (1.5).

Example 1.1 (Ruelle's class potentials). Typical examples of the interaction Ψ in ISDE (1.1) are Ruelle's class potentials, where the point processes μ are translation invariant canonical Gibbs measures with interaction potential Ψ and inverse temperature β . If $d \geq 2$ and Ψ has a hard core with positive volume, then $\alpha[\mu]$ is positive definite for any $\beta \geq 0$ [14]. If Ψ does not have a hard core, then the positivity of $\alpha[\mu]$ has only been proved for $\Psi \in C_0(\mathbb{R}^d)$ and sufficiently small β [1]. This result is valid for $d \geq 1$. It is plausible that the positivity of $\alpha[\mu]$ holds for all Ruelle's class potentials without any hard core condition or restriction on β if $d \geq 2$. This problem is still open in this framework.

Example 1.2 (Ruelle's class potentials: $d = 1$). Let Ψ be of Ruelle's class. Suppose that $\Psi \in C^3(\mathbb{R}^d \setminus \{0\})$ is non-negative with bounded support and satisfies the non-collision condition (2.5). The point processes μ are translation invariant canonical Gibbs measures with interaction potential Ψ and inverse temperature β . In [26], Spohn showed that $\alpha[\mu]$ vanishes and that the correct scaling is $\varepsilon X_{t/\varepsilon^4}^0$.

Example 1.3 (Hard rods in $\mathbb{R}$). Consider Brownian motions $\mathbf{B} = (B^i)_{i \in \mathbb{Z}}$ in $\mathbb{R}$, and pose the reflecting boundary condition for the set of multiple points $\Gamma = \cup_{i \neq j} \{\mathbf{s} = (s_i); s_i = s_j\}$. Harris proved that $\alpha[\mu] = 0$ and that the correct order is $\varepsilon X_{t/\varepsilon^4}^0$ [6].

Example 1.4 (Dyson's model). Another interesting example is Dyson's model in infinite dimensions [25]:

$$(1.7) \quad dX_t^i = dB_t^i + \lim_{R \rightarrow \infty} \frac{\beta}{2} \sum_{j \neq i, |X_t^j| < R} \frac{1}{X_t^i - X_t^j} dt.$$

The ISDE was solved for $\beta = 1, 2, 4$ in [17, 18] (weak solution) and in [23] (pathwise unique, strong solution). The case $\beta = 1, 2, 4$ fulfill the assumptions in Theorem 2.1. For general $1 \leq \beta < \infty$, Tsai [28] solved (1.7) at the level of non-equilibrium, pathwise unique strong solutions. He did not, however, prove the μ -reversibility of the associated unlabeled diffusions. Hence, these diffusions have not yet been shown to be associated with Dirichlet forms. The problem is thus still open for cases other than $\beta = 1, 2, 4$.

In [26], Spohn proved that $E[|X_t^0 - X_0^0|^2] \sim \text{const.} \log t$ as $t \rightarrow \infty$. This suggests that $X_t \approx (\log t)^{1/2}$ as $t \rightarrow \infty$, which is also an open problem.

The organization of the paper is as follows. In Section 2, we set up the problem and state the main result (Theorem 2.1). In Section 3, we prepare an invariance principle and

state Theorem 3.7. In Section 4, we present a representation of self-diffusion constants. In Section 5, we prove that the self-diffusion constant vanishes under the non-collision condition in one-dimension. In Section 6, we complete the proof of Theorem 2.1.

§ 2. Set up and the main result

In this section, we set up and state the main theorem (Theorem 2.1).

Let $\mathbf{S}$ be the configuration space over $\mathbb{R}^d$ as in (1.3). Let μ be a point process on $\mathbb{R}^d$ supported on a set consisting of infinitely many particles. We assume that μ is translation invariant as in (1.4).

A symmetric and locally integrable function $\rho^n : (\mathbb{R}^d)^n \rightarrow [0, \infty)$ is called the n -point correlation function of a random point field μ on $\mathbb{R}^d$ with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \dots, x_n) dx_1 \cdots dx_n = \int_{\mathbb{R}^d} \prod_{i=1}^m \frac{s(A_i)!}{(s(A_i) - k_i)!} d\mu$$

for any sequence of disjoint bounded measurable sets $A_1, \dots, A_m \in \mathcal{B}(\mathbb{R}^d)$ and some sequence of natural numbers $k_1, \dots, k_m$ satisfying $k_1 + \cdots + k_m = n$. If $s(A_i) - k_i < 0$, according to our interpretation, $s(A_i)!/(s(A_i) - k_i)! = 0$ by convention.

For a function f on $\mathbf{S}$, we denote by $\check{f}$ the symmetric function defined on a subset of $\{\cup_{n=0}^{\infty} (\mathbb{R}^d)^n\} \cup (\mathbb{R}^d)^{\mathbb{Z}}$ such that $\check{f}(s_1, \dots) = f(\mathbf{s})$, where $\mathbf{s} = \sum_i \delta_{s_i} \in \mathbf{S}$. We say that a function f on $\mathbf{S}$ is smooth if $\check{f}$ is smooth, and local if $f = f \circ \pi_r$ for some r , where $\pi_r : \mathbf{S} \rightarrow \mathbf{S}$ such that $\pi_r(\mathbf{s}) = \mathbf{s}(\cdot \cap \{|x| \leq r\})$.

Let $\mathcal{D}_o$ be the set of all smooth, local functions on $\mathbf{S}$. Let $\mathbb{D}$ be the square field on $\mathbf{S}$ such that, for $f, g \in \mathcal{D}_o$,

$$(2.1) \quad \mathbb{D}[f, g](\mathbf{s}) = \frac{1}{2} \left\{ \sum_{i \in \mathbb{Z}} \frac{\partial \check{f}}{\partial s_i} \cdot \frac{\partial \check{g}}{\partial s_i} \right\}(\mathbf{s}).$$

We use the square field $\mathbb{D}$ in (2.1) to define a function $F_{\mathbb{D}}$ from the space of point processes to the space of bilinear forms on the configuration space $\mathbf{S}$. Indeed, we set

$$F_{\mathbb{D}}(\mu) = (\mathcal{E}^\mu, \mathcal{D}_o^\mu),$$

where the bilinear form $(\mathcal{E}^\mu, \mathcal{D}_o^\mu)$ on $L^2(\mathbf{S}, \mu)$ is such that

$$(2.2) \quad \begin{aligned} \mathcal{E}^\mu(f, g) &= \int_{\mathbf{S}} \mathbb{D}[f, g] \mu(ds), \\ \mathcal{D}_o^\mu &= \{f \in \mathcal{D}_o; f \in L^2(\mathbf{S}, \mu), \mathcal{E}^\mu(f, f) < \infty\}. \end{aligned}$$

If $F_{\mathbb{D}}(\mu) = (\mathcal{E}^\mu, \mathcal{D}_o^\mu)$ is closable on $L^2(\mathbf{S}, \mu)$, and its closure $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is a quasi-regular Dirichlet form, then by the general theory of Dirichlet forms there exists a μ -reversible

diffusion associated with the Dirichlet space $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathbf{S}, \mu))$ [11]. Hence we assume that

$$(2.3) \quad (\mathcal{E}^\mu, \mathcal{D}_\circ^\mu) \text{ is closable on } L^2(\mathbf{S}, \mu),$$

and

$$(2.4) \quad \text{the } n\text{-point correlation function } \rho^n \text{ of } \mu \text{ is locally bounded for each } n \in \mathbb{N}.$$

We can deduce from (2.4) that the closure $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ of $(\mathcal{E}^\mu, \mathcal{D}_\circ^\mu)$ is a quasi-regular Dirichlet form [12, 18]. Thus the associated diffusion $(\mathbf{P}, \mathbf{X})$ exists [11], where $\mathbf{P} = \{\mathbf{P}_s\}_{s \in \mathbf{S}}$ is the family of diffusion measures and $\mathbf{X} = \{\mathbf{X}_t\}_{t \in [0, \infty)}$ denotes the canonical process. By construction, $(\mathbf{P}, \mathbf{X})$ is a μ -reversible diffusion.

If μ is a Poisson point process with Lebesgue intensity, then the associated diffusion is an $\mathbf{S}$ -valued Brownian motion $\mathbf{B} = \sum_i \delta_{B^i}$. In some sense, this correspondence is natural. If μ is a $(0, \Psi)$ -quasi-Gibbs measure with upper semi-continuous potential Ψ in the sense of [18], then (2.3) is satisfied [18, 19]. We also remark that a $(0, \Psi)$ -Gibbs measure is a $(0, \Psi)$ -quasi-Gibbs measure by definition. We present the ISDEs associated with these unlabeled diffusions at the end of this section.

We assume that

$$(2.5) \quad \text{Cap}(\mathcal{N}_1) = 0,$$

where $\mathcal{N}_1 = \{s \in \mathbf{S}; s(\{x\}) \geq 2 \text{ for some } x \in \mathbb{R}^d\}$ and Cap is the 1-capacity of the Dirichlet space $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathbf{S}, \mu))$. Assumption (2.5) is equivalent to

$$(2.6) \quad \mathbf{P}_s(X_t^i \neq X_t^j \text{ for all } i \neq j, t \in [0, \infty)) = 0 \quad \text{for q.e. } s,$$

where $\mathbf{X}_t = \sum_i \delta_{X_t^i}$, and q.e. indicates quasi-everywhere (see [11, 3]).

We refer to Inukai [7] for the necessary and sufficient conditions for (2.5) in terms of Ψ . This result is for finite-particle systems, but can be used to obtain a precise sufficient condition for (2.5) because the limiting Dirichlet form is a decreasing limit of finite-particle Dirichlet forms [12]. We also refer to [15] for the non-collision property of unlabeled diffusions associated with determinantal point processes.

We remark that the translation invariance of μ yields the non-explosion of each tagged particle X^i [16, Theorem 2.5]:

$$\mathbf{P}_s(\sup_{t \in [0, T]} |X_t^i| < \infty \text{ for all } i \in \mathbb{Z}, T \in [0, \infty)) = 0 \quad \text{for q.e. } s.$$

Let $\mathbf{u}$ be a function on $(\mathbb{R}^d)^\mathbb{Z}$ such that $\mathbf{u}((s_i)) = \sum_i \delta_{s_i}$ and let

$$(2.7) \quad \mathbf{S}_{s, i} = \{s \in \mathbf{S}; s(\{x\}) \leq 1 \text{ for all } x, s(\mathbb{R}^d) = \infty\}.$$

From (2.5) and the translation invariance of μ we see that

$$(2.8) \quad \mu(\mathbb{S}_{\mathbf{s},\mathbf{l}}) = 1.$$

Let $\mathbf{l} : \mathbb{S}_{\mathbf{s},\mathbf{l}} \rightarrow (\mathbb{R}^d)^{\mathbb{Z}}$ be a measurable map such that $\mathbf{u} \circ \mathbf{l} = \text{id}$. We call $\mathbf{l}$ a label and write $\mathbf{l}$ as $\mathbf{l}(\mathbf{s}) = (\mathbf{l}_i(\mathbf{s}))_{i \in \mathbb{Z}} = (s_i)_{i \in \mathbb{Z}}$, where $\mathbf{s} = \sum_{i \in \mathbb{Z}} \delta_{s_i}$.

Example: Let $\mathbf{l} = (s_i)_{i \in \mathbb{Z}}$ be a label. The label $\mathbf{l}$ is well defined for all $\mathbf{s} = \sum_i \delta_{s_i} \in \mathbb{S}_{\mathbf{s},\mathbf{l}}$ from the following.

(1) When $d = 1$, a typical example of the label $\mathbf{l}$ is as follows: $s_{-1} < s_0 < s_1$ and

$$\cdots < s_{-2} < s_{-1} < 0 < s_1 < s_2 < \cdots .$$

(2) Another example for $d \geq 1$ is:

$$|s_0| < |s_1| < |s_{-1}| < |s_2| < |s_{-2}| < \cdots .$$

We can lift the map $\mathbf{l}$ to $\mathbf{l}_{\text{path}} : C([0, \infty); (\mathbb{R}^d)^{\mathbb{Z}}) \rightarrow C([0, \infty); (\mathbb{R}^d)^{\mathbb{Z}})$ in an obvious fashion. Indeed, once $\mathbf{l}$ is given, the dynamics $\mathbf{X}_t = \sum_i \delta_{X_t^i}$ can keep the initial label for all $t \in [0, \infty)$ because the particles neither collide with each other nor explode. Hence we write

$$(2.9) \quad \mathbf{X}_t = (X_t^i)_{i \in \mathbb{Z}} = \mathbf{l}_{\text{path}}(\mathbf{X})_t.$$

We assume that a label $\mathbf{l} = (\mathbf{l}_i(\mathbf{s}))_{i \in \mathbb{Z}} = (s_i)_{i \in \mathbb{Z}}$ is given and fix this throughout the paper. If $\mathbf{X}_0 = \mathbf{s}$ and $\mathbf{X}$ satisfies (2.9), then $\mathbf{X}_0 = (\mathbf{l}_i(\mathbf{s}))_{i \in \mathbb{Z}}$ by definition. We study the diffusive scaling limit of each tagged particle X^i of the labeled process $\mathbf{X} = (X^i)_{i \in \mathbb{Z}}$.

Let $\mu_x = \mu(\cdot - \delta_x | \mathbf{s}(\{x\}) \geq 1)$ be the reduced Palm measure conditioned at $x \in \mathbb{R}^d$. Let $\mu^{[1]}(dx ds) = \rho^1(x) \mu_x(ds) dx$ be the one-Campbell measure of μ . Note that $\rho^1(x)$ is constant in x because μ is translation invariant by (1.4). Let ∇_x be the nabla in $x \in \mathbb{R}^d$. We regard $\mathbb{D}$ as a square field on $C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$ in an obvious fashion. Let

$$\begin{aligned} \mathcal{E}^{[1]}(f, g) &= \int_{\mathbb{R}^d \times \mathbb{S}} \left\{ \frac{1}{2} \nabla_x f \cdot \nabla_x g + \mathbb{D}[f, g] \right\} \mu^{[1]}(dx ds), \\ \mathcal{D}_\circ^{[1]} &= \{f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ; f \in L^2(\mathbb{S}, \mu^{[1]}), \mathcal{E}^{[1]}(f, f) < \infty\}. \end{aligned}$$

We assume the following:

$$(2.10) \quad (\mathcal{E}^{[1]}, \mathcal{D}_\circ^{[1]}) \text{ is closable on } L^2(\mathbb{R}^d \times \mathbb{S}, \mu^{[1]}).$$

Recall that from (2.3) and (2.4) we obtain a μ -reversible diffusion $(\mathbf{P}, \mathbf{X})$. Using the label $\mathbf{l}$, we can write $\mathbf{X}_t = \sum_{i \in \mathbb{Z}} \delta_{X_t^i}$. We thus obtain the labeled process $\mathbf{X} = (X^i)_{i \in \mathbb{Z}}$. Our main theorem is the following:

Theorem 2.1. *Assume (1.4), (2.3)–(2.5), and (2.10), and assume that $d = 1$. Then, for each $i \in \mathbb{Z}$, we obtain*

$$\lim_{\epsilon \rightarrow 0} \epsilon X_{\cdot/\epsilon^2}^i = 0 \quad \text{weakly in } C([0, \infty); \mathbb{R}) \text{ under } \mathbb{P}_s \text{ in } \mu\text{-probability.}$$

That is, for any $F \in C_b(C([0, \infty); \mathbb{R}))$ and for each $\kappa > 0$,

$$\lim_{\epsilon \rightarrow 0} \mu(\{s; |\mathbb{E}_s[F(\epsilon X_{\cdot/\epsilon^2}^i)] - F(0)| \geq \kappa\}) = 0,$$

where 0 in $F(0)$ denotes the constant path with value 0.

The Dirichlet forms describing $(X^0, \sum_{i \neq 0} \delta_{X^i - X^0})$ and $\sum_{i \neq 0} \delta_{X^i - X^0}$ will be given in Section 3. Assumption (2.10) is necessary for this.

We emphasize that our framework does not require any ISDE. Indeed, only a Dirichlet form constructing the unlabeled diffusion is sufficient.

If the point process μ satisfies the geometric condition below, then the unlabeled diffusion given by the Dirichlet form is a solution of the ISDE [17]. Suppose that μ has a logarithmic derivative $\mathbf{d}^\mu = \mathbf{d}^\mu(s, s)$ in the sense of [17]. Then the labeled dynamics $\mathbf{X}$ are described by the infinite-dimensional stochastic differential equation

$$dX_t^i = dB_t^i + \frac{1}{2} \mathbf{d}^\mu(X_t^i, \mathbf{X}_t^{\diamond, i}) dt,$$

where, for $\mathbf{X} = (X_t^i)_i$, we set

$$\mathbf{X}_t^{\diamond, i} = \sum_{j \in \mathbb{Z}, j \neq i} \delta_{X_t^j}.$$

If μ is a canonical Gibbs measure with inverse temperature β and potential Ψ , then

$$\mathbf{d}^\mu(x, s) = -\beta \sum_i \nabla_x \Psi(x, s_i) \quad (s = \sum_i \delta_{s_i}).$$

If μ is a Ginibre point process or a Sine $_\beta$ point process, then $\mathbf{d}^\mu$ is given by

$$\mathbf{d}^\mu(x, s) = \beta \lim_{R \rightarrow \infty} \sum_{|s_i| < R} \frac{x - s_i}{|x - s_i|^2}.$$

This justifies the intuition such that the interaction potentials of these point processes are logarithmic function $\Psi(x, y) = -\log|x - y|$.

We remark that our framework [13] is very general, and contains many examples beyond Gibbs measures and point processes with pairwise interactions. For example, if μ is a distribution of the zero points of planar Gaussian analytic functions (GAF), then its logarithmic derivative would not be given by a two-body potential Ψ . We can still

apply our result to this model. We plan to study this problem in a forth coming paper. We refer to [4] for discussion of GAFs.

§ 3. Invariance principle and self-diffusion matrix

In this section, we quote a general theorem on an invariance principle for additive functionals of reversible Markov processes from [13], and present a refinement corresponding to (1.6).

Throughout this section, we assume (1.4), (2.3)–(2.5), and (2.10). That is, we make the assumptions in Theorem 2.1 except $d = 1$. We suppress this in the statements of the lemmas in this section.

Let (P, X) be the diffusion given by the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(S, \mu)$ as in Section 2. Let $\mathfrak{l}$ be a label, and write $\mathbf{X} = \sum_{i \in \mathbb{Z}} \delta_{X^i}$. Let $\mathbf{X} = (X^i)_{i \in \mathbb{Z}}$ be the associated labeled dynamics.

A standard device for the tagged particle problem for interacting Brownian motions is to introduce processes of the environment seen from the tagged particles [1, 5, 8, 21, 13, 14]. Following this, we define a change of coordinates for $\mathbf{X}$ as follows. Let $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, and set

$$(3.1) \quad X = X^0, \quad Y^i = X^i - X^0, \quad (i \in \mathbb{Z}^*).$$

Then X denotes the tagged particle and $\mathbf{Y} = (Y^i)_{i \in \mathbb{Z}^*}$ is the (labeled) environment seen from the tagged particle. Let $\mathbf{Y} = \{Y_t\}$ be the unlabeled process associated with $\mathbf{Y} = (Y^i)_{i \in \mathbb{Z}^*}$:

$$(3.2) \quad Y_t = \sum_{i \in \mathbb{Z}^*} \delta_{Y_t^i} = \sum_{i \in \mathbb{Z}^*} \delta_{X_t^i - X_t^0}.$$

Then $\mathbf{Y}$ is the process representing the (unlabeled) environment seen from the tagged particle $X = X^0$. We call $\mathbf{Y}$ the environment process. We also call the pair $(X, \mathbf{Y})$ the tagged particle and environment process.

Remark. If $\mathbf{X}$ is given by (1.1), then from (3.1) we see that $(X, \mathbf{Y})$ is given by

$$\begin{aligned} dX_t &= dB_t^0 - \frac{\beta}{2} \sum_{j \in \mathbb{Z}^*} \nabla \Psi(Y_t^j) dt, \\ dY_t^i &= \sqrt{2} d\tilde{B}^i - \frac{\beta}{2} \nabla \Psi(Y_t^i) dt + \frac{\beta}{2} \sum_{j \in \mathbb{Z}^*} \nabla \Psi(Y_t^j) dt - \frac{\beta}{2} \sum_{j \in \mathbb{Z}^*} \nabla \Psi(Y_t^i - Y_t^j) dt, \end{aligned}$$

where $\{\tilde{B}^i\}_{i \in \mathbb{Z}^*}$ are d -dimensional Brownian motions given by

$$\tilde{B}_t^i = \frac{1}{\sqrt{2}} (B_t^i - B_t^0).$$

We remark that $\{\tilde{B}^i\}_{i \in \mathbb{Z}^*}$ are not independent but only identically distributed random variables equivalent to standard Brownian motion.

Using (3.1) and (3.2), we have constructed dynamics $\mathbf{Y}$, $(X, \mathbf{Y})$, $\mathbf{Y}$, and $(X, \mathbf{Y})$ from $\mathbf{X}$. We now specify the Dirichlet forms associated with $\mathbf{Y}$ and $(X, \mathbf{Y})$.

We remark that, although $\mathbf{Y}$ and $(X, \mathbf{Y})$ are also diffusions with state space $(\mathbb{R}^d)^{\mathbb{Z}}$ and $\mathbb{R}^d \times (\mathbb{R}^d)^{\mathbb{Z}}$, respectively, there exist no associated Dirichlet spaces because of the lack of suitable invariant measures. For example, if μ is a Gibbs measure with interaction Ψ , then such measures $\tilde{\mu}_0$ and $dx \times \tilde{\mu}_0$ for $\mathbf{Y}$ and $(X, \mathbf{Y})$ are loosely given by

$$\tilde{\mu}_0 = \frac{1}{Z} \exp\left\{-\beta \left(\sum_{i < j, i, j \in \mathbb{Z}} \Psi(y_i - y_j) - \sum_{k \in \mathbb{Z}} \Psi(0 - y_k) \right)\right\} \prod_{l \in \mathbb{Z}} dy_l.$$

This cannot be justified because of the presence of the infinite product of Lebesgue measures $\prod_{l \in \mathbb{Z}} dy_l$. In contrast, $\mathbf{Y}$ and $(X, \mathbf{Y})$ are diffusions with invariant measures μ_0 and $dx \times \mu_0$, respectively. As a result, they have associated Dirichlet spaces. This fact is key to analysis in the Dirichlet form version of Kipnis–Varadhan theory [13].

Once the Dirichlet forms describing the processes $\mathbf{Y}$ and $(X, \mathbf{Y})$ have been established, we can dispense with ISDE (1.1), which yields the generality of our result. In fact, the process $\mathbf{X}$ in Theorem 2.1 is not necessary given by ISDE (1.1). Thus our framework is much more general than the classical one in [1] and [5].

Let $D^{\text{sft}} = (D_1^{\text{sft}}, \dots, D_d^{\text{sft}})$, where $D_k^{\text{sft}}: \mathcal{D}_\circ \rightarrow \mathcal{D}_\circ$ is such that

$$D_k^{\text{sft}} f(\mathbf{s}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{f(\theta_{\epsilon \mathbf{e}_k}(\mathbf{s})) - f(\mathbf{s})\},$$

and $\mathbf{e}_k$ is the k th unit vector in $\mathbb{R}^d$. We set

$$D^{\text{sft}}[f, g] = \frac{1}{2} D^{\text{sft}} f \cdot D^{\text{sft}} g.$$

Let $\mathbb{D}$ be defined as in (2.1). Let $\mathbb{D}_\mathbf{Y}$ be the square field on $\mathcal{D}_\circ$ such that

$$(3.3) \quad \mathbb{D}_\mathbf{Y}[f, g] = D^{\text{sft}}[f, g] + \mathbb{D}[f, g].$$

Let $(\mathcal{E}_\mathbf{Y}, \mathcal{D}_{\mathbf{Y}\circ})$ be the bilinear form such that

$$\begin{aligned} \mathcal{E}_\mathbf{Y}(f, g) &= \int_S \mathbb{D}_\mathbf{Y}[f, g] d\mu_0, \\ \mathcal{D}_{\mathbf{Y}\circ} &= \{f \in L^2(S, \mu_0) \cap \mathcal{D}_\circ; \mathcal{E}_\mathbf{Y}(f, f) < \infty\}. \end{aligned}$$

The next lemma is a special case of a result for translation invariant diffusions on S in [16], and gives a Dirichlet form for $\mathbf{Y}$.

Lemma 3.1 ([16, Th. 2.6 (1), Th. 2.7 (2.33)]).

- (1) $(\mathcal{E}_Y, \mathcal{D}_{Y\circ})$ is closable on $L^2(S, \mu_0)$.
- (2) Y in (3.2) is a diffusion associated with $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(S, \mu_0)$, where $(\mathcal{E}_Y, \mathcal{D}_Y)$ is the closure of $(\mathcal{E}_Y, \mathcal{D}_{Y\circ})$ on $L^2(S, \mu_0)$.

We next specify the Dirichlet space associated with the coupled process (X, Y) . We naturally regard ∇_x and D^{sft} as operators on $C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$. For $f, g \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$, we set

$$\begin{aligned} (\nabla_x - D^{\text{sft}})[f, g] &= \frac{1}{2}(\nabla_x - D^{\text{sft}})f \cdot (\nabla_x - D^{\text{sft}})g, \\ \mathbb{D}_{XY}[f, g] &= (\nabla_x - D^{\text{sft}})[f, g] + \mathbb{D}[f, g], \\ \mathcal{E}_{XY}(f, g) &= \int_{\mathbb{R}^d \times S} \mathbb{D}_{XY}[f, g] dx d\mu_0. \end{aligned}$$

Applying a general result in [16] to translation invariant diffusions on S , we obtain:

Lemma 3.2 ([16], Th. 2.6). $(\mathcal{E}_{XY}, C_0(\mathbb{R}^d) \otimes \mathcal{D}_\circ)$ is closable on $L^2(\mathbb{R} \times S, dx d\mu_0)$.

We denote by P_s^Y the distribution of the diffusion $Y = \{Y_t\}$ starting at s given by the Dirichlet form $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(S, \mu_0)$. We denote by $P_{(x,s)}^{XY}$ the distribution of the diffusion $(X, Y) = \{(X_t, Y_t)\}$ starting at (x, s) given by the Dirichlet form $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R} \times S, dx d\mu_0)$. By the general theory of Dirichlet forms [3], P_s^Y and $P_{(x,s)}^{XY}$ are unique up to quasi-everywhere starting points. The next two lemmas show the existence of suitable versions of these diffusion measures.

The next lemma explains the relationship between Y and (X, Y) and recalls the identities involving P_s^Y and $P_{(x,s)}^{XY}$. We set

$$X - X_0 = \{X_t - X_0\}_{t \in [0, \infty)}.$$

Lemma 3.3 ([13, Lem. 2.3]). The diffusions P_s^Y and $P_{(x,s)}^{XY}$ satisfy the following:

$$\begin{aligned} P_s^Y &= P_{(x,s)}^{XY}(Y \in \cdot) \quad \text{for each } x \in \mathbb{R}^d, \\ P_{(0,s)}^{XY}((X - X_0, Y) \in \cdot) &= P_{(x,s)}^{XY}((X - X_0, Y) \in \cdot) \quad \text{for each } x \in \mathbb{R}^d. \end{aligned}$$

We next clarify the relationship between the original diffusion X and the diffusion (X, Y) . Let $S_{s,i}$ be defined as in (2.7), and let

$$S_x = \{s \in S_{s,i}; s(\{x\}) = 1\}.$$

For $s \in S_x$ and a label $l(s) = (s_i)_{i \in \mathbb{Z}}$, we set $i(s, x) \in \mathbb{Z}$ such that

$$(3.4) \quad s_{i(s,x)} = x.$$

Let $P = \{P_s\}_{s \in S}$ be the distribution of the original unlabeled diffusion X given by the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(S, \mu)$ as before. Because μ is translation invariant, there is a version of P_s such that

$$P_s \circ \theta_x^{-1} = P_{\theta_x(s)} \quad \text{for all } x \in \mathbb{R}^d \text{ and } s \in S.$$

Lemma 3.4 ([16, Th. 2.7 (2.32)]). Let $i(s, x)$ be defined as above. Then, for each $x \in \mathbb{R}^d$ and $s \in S_x$, (a version of) $P_{(x,s)}^{XY}$ satisfies

$$(3.5) \quad P_s(X^{i(x,s)} \in \cdot) = P_{(x, \theta_{-x}(s - \delta_x))}^{XY}(X \in \cdot).$$

We use an invariance principle obtained in [13]. Applying [13, Th. 1, Lem. 5.5] to P^Y and P^{XY} , we obtain:

Lemma 3.5. There exists a non-negative definite matrix-valued function $\hat{a}$ such that, for each x ,

$$(3.6) \quad \lim_{\varepsilon \rightarrow 0} (\varepsilon X_{\cdot/\varepsilon^2} - \varepsilon X_{0/\varepsilon^2}) = \sqrt{\hat{a}(s)}B \quad \text{in law in } C([0, \infty); \mathbb{R}^d)$$

under $P_{(x,s)}^{XY}$ in μ_0 -probability. That is, for any $F \in C_b(C([0, \infty); \mathbb{R}^d))$ and each $\kappa > 0$,

$$(3.7) \quad \lim_{\varepsilon \rightarrow 0} \mu_0(\{s; |E_{(x,s)}^{XY}[F(\varepsilon X_{\cdot/\varepsilon^2} - \varepsilon X_{0/\varepsilon^2})] - E[F(\sqrt{\hat{a}(s)}B)]| \geq \kappa\}) = 0.$$

From Lemma 3.5, we introduce the self-diffusion matrix $\alpha[\mu]$ given by

$$(3.8) \quad \alpha[\mu] = \int_S \hat{a}(s) \mu_0(ds).$$

Lemma 3.6. Let $x \in \mathbb{R}^d$ and set $\hat{\mu}_x = \mu(\cdot | s(\{x\}) = 1)$. Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon X_{\cdot/\varepsilon^2}^{i(x,s)} = \sqrt{\hat{a}(s)}B \quad \text{in law in } C([0, \infty); \mathbb{R}^d)$$

under P_s in $\hat{\mu}_x$ -probability. That is, for any $F \in C_b(C([0, \infty); \mathbb{R}^d))$ and each $\kappa > 0$,

$$(3.9) \quad \lim_{\varepsilon \rightarrow 0} \hat{\mu}_x(\{s; |E_s[F(\varepsilon X_{\cdot/\varepsilon^2}^{i(x,s)})] - E[F(\sqrt{\hat{a}(s)}B)]| \geq \kappa\}) = 0.$$

Proof. Note that $\varepsilon X_{0/\varepsilon^2}^{i(x,s)} = x$ for P_s -a.s. and $\varepsilon X_{0/\varepsilon^2} = x$ for $P_{(x, \theta_{-x}(s - \delta_x))}^{XY}$ -a.s.. From this and (3.5) in Lemma 3.4, we see that

$$(3.10) \quad P_s \circ (\varepsilon X_{\cdot/\varepsilon^2}^{i(x,s)} - \varepsilon X_{0/\varepsilon^2}^{i(x,s)})^{-1} = P_{(x, \theta_{-x}(s - \delta_x))}^{XY} \circ (\varepsilon X_{\cdot/\varepsilon^2} - \varepsilon X_{0/\varepsilon^2})^{-1}.$$

We easily see that

$$(3.11) \quad \hat{\mu}_x \circ \{\theta_{-x}(s - \delta_x)\}^{-1} = \mu_0.$$

Using (3.10) and (3.11) and applying (3.7) in Lemma 3.5, we obtain, for each x ,

$$\begin{aligned}
 (3.12) \quad & \lim_{\varepsilon \rightarrow 0} \hat{\mu}_x(\{\mathbf{s}; |E_{\mathbf{s}}[F(\varepsilon X_{\cdot/\varepsilon^2}^{i(x,\mathbf{s})})] - E[F(\sqrt{\hat{a}(\mathbf{s})}B)]| \geq \kappa\}) \\
 &= \lim_{\varepsilon \rightarrow 0} \hat{\mu}_x(\{\mathbf{s}; |E_{\mathbf{s}}[F(\varepsilon X_{\cdot/\varepsilon^2}^{i(x,\mathbf{s})} - \varepsilon X_{0/\varepsilon^2}^{i(x,\mathbf{s})})] - E[F(\sqrt{\hat{a}(\mathbf{s})}B)]| \geq \kappa\}) \\
 &= \lim_{\varepsilon \rightarrow 0} \hat{\mu}_x(\{\mathbf{s}; |E_{(x,\theta-x(\mathbf{s}-\delta_x))}^{XY}[F(\varepsilon X_{\cdot/\varepsilon^2} - \varepsilon X_{0/\varepsilon^2})] - E[F(\sqrt{\hat{a}(\mathbf{s})}B)]| \geq \kappa\}) \\
 &= \lim_{\varepsilon \rightarrow 0} \mu_0(\{\mathbf{s}; |E_{(x,\mathbf{s})}^{XY}[F(\varepsilon X_{\cdot/\varepsilon^2} - \varepsilon X_{0/\varepsilon^2})] - E[F(\sqrt{\hat{a}(\mathbf{s})}B)]| \geq \kappa\}) \\
 &= 0.
 \end{aligned}$$

We immediately deduce (3.9) from (3.12). $\square$

Theorem 3.7. Assume (1.4), (2.3)–(2.5), and (2.10). Then, for each $i \in \mathbb{Z}$,

$$(3.13) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon X_{\cdot/\varepsilon^2}^i = \sqrt{\hat{a}(\mathbf{s})}B \quad \text{weakly in } C([0, \infty); \mathbb{R}^d)$$

under $P_{\mathbf{s}}$ in μ -probability. That is, for any $F \in C_b(C([0, \infty); \mathbb{R}^d))$,

$$(3.14) \quad \lim_{\varepsilon \rightarrow 0} \mu(\{\mathbf{s}; |E_{\mathbf{s}}(F(\varepsilon X_{\cdot/\varepsilon^2}^i)) - F(\sqrt{\hat{a}(\mathbf{s})}B)| \geq \kappa\}) = 0 \quad \text{for each } \kappa > 0.$$

Proof. Let $i(\mathbf{s}, x)$ be as in (3.4). Set $\hat{\mu}_x = \mu(\cdot | \mathbf{s}(\{x\}) = 1)$ as in Lemma 3.6. Recall that $\mu(\mathbf{S}_{\mathbf{s},i}) = 1$ by (2.8). Let $\mathbf{l}(\mathbf{s}) = (\mathbf{l}_i(\mathbf{s}))_{i \in \mathbb{Z}}$ be the label as before.

Without loss of generality, we can and do assume $i = 0$ in (3.13). By a straightforward calculation, we have

$$\begin{aligned}
 (3.15) \quad & \mu(X^0 \in \cdot) = \int_{\mathbb{R}^d} \mu(X^0 \in \cdot | \mathbf{l}_0(\mathbf{s}) = x) \mu \circ \mathbf{l}_0^{-1}(dx) \\
 &= \int_{\mathbb{R}^d} \mu(X^0 \in \cdot | \mathbf{l}_0(\mathbf{s}) = x, \mathbf{s}(\{x\}) = 1) \mu \circ \mathbf{l}_0^{-1}(dx) \\
 &= \int_{\mathbb{R}^d} \hat{\mu}_x(\{X^{i(\mathbf{s},x)} \in \cdot\} \cap \{i(\mathbf{s},x) = 0\} | \mathbf{l}_0(\mathbf{s}) = x) \mu \circ \mathbf{l}_0^{-1}(dx) \\
 &= \int_{\mathbb{R}^d} \hat{\mu}_x(X^{i(\mathbf{s},x)} \in \cdot | \mathbf{l}_0(\mathbf{s}) = x) \mu \circ \mathbf{l}_0^{-1}(dx).
 \end{aligned}$$

Let $F \in C_b(C([0, \infty); \mathbb{R}^d))$, and set

$$\begin{aligned}
 (3.16) \quad & G_{\varepsilon}^1(\mathbf{s}) = E_{\mathbf{s}}[F(\varepsilon X_{\cdot/\varepsilon^2}^0)] - E[F(\sqrt{\hat{a}(\mathbf{s})}B)], \\
 & G_{\varepsilon}^2(\mathbf{s}) = E_{\mathbf{s}}[F(\varepsilon X_{\cdot/\varepsilon^2}^{i(\mathbf{s},x)})] - E[F(\sqrt{\hat{a}(\mathbf{s})}B)].
 \end{aligned}$$

For each $\kappa > 0$, we see from (3.15) that

$$(3.17) \quad \mu(\{\mathbf{s}; |G_{\varepsilon}^1(\mathbf{s})| \geq \kappa\}) = \int_{\mathbb{R}^d} \hat{\mu}_x(\{\mathbf{s}; |G_{\varepsilon}^2(\mathbf{s})| \geq \kappa\} | \mathbf{l}_0(\mathbf{s}) = x) \mu \circ \mathbf{l}_0^{-1}(dx).$$

Let $S_R = \{x; |x| \leq R\}$. For any $v > 0$, take $R = R(v)$ such that

$$\mu \circ \mathfrak{l}_0^{-1}(S_R^c) \leq v.$$

Then

$$(3.18) \quad \int_{S_R^c} \hat{\mu}_x(\{\mathbf{s}; |G_\varepsilon^2(\mathbf{s})| \geq \kappa\} | \mathfrak{l}_0(\mathbf{s}) = x) \mu \circ \mathfrak{l}_0^{-1}(dx) \leq v.$$

It is not difficult to see that, for each $i, R \in \mathbb{N}$,

$$\int_{S_R} \frac{1}{\hat{\mu}_x(\mathfrak{l}_i(\mathbf{s}) = x)} \mu \circ \mathfrak{l}_i^{-1}(dx) = \int_{S_R} \rho^1 dx < \infty.$$

Using this and Lemma 3.6, we apply the bounded convergence theorem to obtain

$$(3.19) \quad \begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int_{S_R} \hat{\mu}_x(\{\mathbf{s}; |G_\varepsilon^2(\mathbf{s})| \geq \kappa\} | \mathfrak{l}_0(\mathbf{s}) = x) \mu \circ \mathfrak{l}_0^{-1}(dx) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \int_{S_R} \hat{\mu}_x(\{\mathbf{s}; |G_\varepsilon^2(\mathbf{s})| \geq \kappa\}) \frac{1}{\hat{\mu}_x(\mathfrak{l}_0(\mathbf{s}) = x)} \mu \circ \mathfrak{l}_0^{-1}(dx) \\ & = \int_{S_R} \limsup_{\varepsilon \rightarrow 0} \hat{\mu}_x(\{\mathbf{s}; |G_\varepsilon^2(\mathbf{s})| \geq \kappa\}) \frac{1}{\hat{\mu}_x(\mathfrak{l}_0(\mathbf{s}) = x)} \mu \circ \mathfrak{l}_0^{-1}(dx) \\ & = 0. \end{aligned}$$

Putting (3.16)–(3.19) together we have

$$(3.20) \quad \limsup_{\varepsilon \rightarrow 0} \mu(\{\mathbf{s}; |G_\varepsilon^1(\mathbf{s})| \geq \kappa\}) \leq v.$$

Because v is arbitrary, we see that the left-hand side of (3.20) equals zero. Together with (3.16), this yields (3.14). We have thus completed the proof. $\square$

Remark. (1) When using Kipnis–Varadhan theory, one has to assume the existence of the mean forward velocity $\varphi \in L^2(\mu_0)$ of tagged particles and that

$$\left| \int_{\mathbb{S}} \varphi f d\mu_0 \right| \leq C \mathcal{E}_Y(f, f)^{1/2} \quad \text{for all } f \in \mathcal{D}_Y$$

for some constant C . Roughly speaking, the existence of the mean forward velocity is captured by the condition

$$\lim_{t \rightarrow 0} \frac{1}{t} E_{\mathbf{s}}[X_t^0 - X_0^0] =: \varphi(\mathbf{s}) \quad \text{in } L^2(\mathbb{S}, \mu_0),$$

the average of which must vanish with respect to the reduced Palm measure μ_0 . It is often difficult to check these conditions for interacting Brownian motions with singular or long range potentials. Indeed, one essentially uses the fact that the dynamics are given by a strong solution of an ISDE (1.1).

In [13], we developed a Dirichlet form version of Kipnis–Varadhan theory that allows us to verify these conditions and generalize the results themselves. In fact, only the existence of the coupled Dirichlet form $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ on $L^2(\mathbb{R} \times S, dxd\mu_0)$ was necessary as a substitute for the mean forward velocity condition above. This follows from (1.4), (2.3)–(2.5), and (2.10) as we see in Lemma 3.2.

(2) In [13], Dirichlet forms are assumed to satisfy the strong sector condition, which is a generalization of reversibility. It is easy to see that Theorem 3.7 holds under the strong sector condition.

§ 4. Representation of the self-diffusion constant $\alpha[\mu]$

Before quoting a representation theorem for the self-diffusion matrix $\alpha[\mu]$ from [13], we introduce the quotient Dirichlet form associated with $(\mathcal{E}_Y, \mathcal{D}_Y)$. Although the ideas in this section are valid for $d \geq 1$, we restrict our discussion to $d = 1$ for simplicity. We refer the reader to [13] for the general case.

Recall the decomposition $\mathbb{D}_Y = D^{\text{sft}} + \mathbb{D}$ in (3.3), and choose bilinear forms on $\mathcal{D}_{Y\circ}$ such that

$$\mathcal{E}_Y^1(f, g) = \int_S D^{\text{sft}}[f, g] d\mu_0, \quad \mathcal{E}_Y^2(f, g) = \int_S \mathbb{D}[f, g] d\mu_0.$$

We can naturally extend the domain $\mathcal{D}_{Y\circ}$ of the operator $D^{\text{sft}}: \mathcal{D}_{Y\circ} \rightarrow L^2(S, \mu_0)$ to $\mathcal{D}_Y$, where these bilinear forms are represented by the Dirichlet form $\mathcal{E}_Y$.

$$(4.1) \quad \mathcal{E}_Y = \mathcal{E}_Y^1 + \mathcal{E}_Y^2.$$

Let $\tilde{\mathcal{D}}_Y^2 = \mathcal{D}_Y / \mathcal{E}_Y^2$ be the quotient space of $\mathcal{D}_Y$ with the equivalence relation $\sim_{\mathcal{E}_Y^2}$ given by $\mathcal{E}_Y^2$, that is, $f \sim_{\mathcal{E}_Y^2} g$ if and only if $\mathcal{E}_Y^2(f - g, f - g) = 0$. Let $\tilde{\mathcal{D}}_{Y\circ}$ be a vector space

$$\tilde{\mathcal{D}}_{Y\circ} = \{f = (D^{\text{sft}}f, f / \sim_{\mathcal{E}_Y^2}) \in L^2(S, \mu_0) \times \tilde{\mathcal{D}}_Y^2; f \in \mathcal{D}_Y\}$$

with inner product

$$\tilde{\mathcal{E}}_Y(f, g) = (D^{\text{sft}}f, D^{\text{sft}}g)_{L^2(S, \mu_0)} + \mathcal{E}_Y^2(f, g).$$

Let $\tilde{\mathcal{D}}_Y$ be the completion of $\tilde{\mathcal{D}}_{Y\circ}$ with inner product $\tilde{\mathcal{E}}_Y$ as above. Note that $\tilde{\mathcal{E}}_Y(f, g) = \mathcal{E}_Y(f, g)$. Hence $(\tilde{\mathcal{D}}_Y, \tilde{\mathcal{E}}_Y)$ gives a representation of the quotient Hilbert space of $\mathcal{D}_Y$ with inner product $\mathcal{E}_Y$.

Let $(\tilde{\mathcal{E}}_Y^1, \tilde{\mathcal{D}}_Y)$ and $(\tilde{\mathcal{E}}_Y^2, \tilde{\mathcal{D}}_Y)$ be the quotient bilinear forms of $(\mathcal{E}_Y^1, \mathcal{D}_Y)$ and $(\mathcal{E}_Y^2, \mathcal{D}_Y)$ defined in the same manner as $(\tilde{\mathcal{E}}_Y, \tilde{\mathcal{D}}_Y)$. By definition, the domain of these bilinear forms is $\tilde{\mathcal{D}}_Y$, and (4.1) yields the representation

$$\tilde{\mathcal{E}}_Y = \tilde{\mathcal{E}}_Y^1 + \tilde{\mathcal{E}}_Y^2.$$

For $\mathbf{f} = (f_1, f_2)$ and $\mathbf{g} = (g_1, g_2) \in \tilde{\mathcal{D}}_Y$, we have $\tilde{\mathcal{E}}_Y(\mathbf{f}, \mathbf{g}) = \tilde{\mathcal{E}}_Y^1(f_1, g_1) + \tilde{\mathcal{E}}_Y^2(f_2, g_2)$.

For $(f_1, f_2), (g_1, g_2) \in L^2(S, \mu_0) \times \tilde{\mathcal{D}}_Y^2$ we extend the domain of $\tilde{\mathcal{E}}_Y$ so that

$$\tilde{\mathcal{E}}_Y((f_1, f_2), (g_1, g_2)) = (f_1, g_1)_{L^2(S, \mu_0)} + \tilde{\mathcal{E}}_Y^2(f_2, g_2).$$

We quote the following lemma from [13]:

Lemma 4.1. There exists a unique solution $\chi \in \tilde{\mathcal{D}}_Y$ of the equation

$$(4.2) \quad \tilde{\mathcal{E}}_Y(\chi, \mathbf{g}) = \tilde{\mathcal{E}}_Y((1, 0), \mathbf{g}) \quad \text{for all } \mathbf{g} \in \tilde{\mathcal{D}}_Y.$$

Proof. Because $F(\mathbf{g}) = \tilde{\mathcal{E}}_Y((1, 0), \mathbf{g})$ can be regarded as a bounded linear functional of the Hilbert space $\tilde{\mathcal{D}}_Y$ with inner product $\tilde{\mathcal{E}}_Y$, Lemma 4.1 is obvious from the Riesz theorem. $\square$

Lemma 4.2. Let $\chi = (\chi_1, \chi_2) \in \tilde{\mathcal{D}}_Y$ be the unique solution of equation (4.2). Then the self-diffusion constant $\alpha[\mu]$ is given by

$$(4.3) \quad \alpha[\mu] = \tilde{\mathcal{E}}_Y^1(1 - \chi_1, 1 - \chi_1) + \tilde{\mathcal{E}}_Y^2(\chi_2, \chi_2).$$

In particular, if $(1, 0) \in \tilde{\mathcal{D}}_Y$, then $\alpha[\mu] = 0$.

Proof. Applying [13, Theorems 1,2] to X , we deduce that X has the scaling limit in (3.6) with the self-diffusion constant $\alpha[\mu]$ given by (4.3). This completes the proof of the first claim. If $(1, 0) \in \tilde{\mathcal{D}}_Y$, then $\chi = (1, 0)$. Hence the second claim follows from the first. $\square$

In the rest of this section, we explain the back-ground of the representation formula (4.3). To prove the convergence of $\varepsilon X_{t/\varepsilon^2}$, we use the technique of corrector [8], that is, we use a function χ_ε , called corrector, for which

$$\varepsilon X_{t/\varepsilon^2} - \chi_\varepsilon(\theta_{\varepsilon X_{t/\varepsilon^2}}(\mathbf{s})) = \text{a continuous local martingale} + o(\varepsilon),$$

and for which

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sup_{|x| < R} E[|\chi_\varepsilon(\theta_{\varepsilon X_{t/\varepsilon^2}}(\mathbf{s}))|^2] &= 0 \quad \text{for any } R \in \mathbb{N}, \\ \lim_{\varepsilon \rightarrow 0} \nabla_x \chi_\varepsilon(\theta_x(\mathbf{s}))|_{x=0} &= \lim_{\varepsilon \rightarrow 0} D^{\text{sft}} \chi_\varepsilon(\mathbf{s}) = \psi(\mathbf{s}) \quad \text{in } L^2(S, \mu_0). \end{aligned}$$

Then we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon X_{t/\varepsilon^2} = \lim_{\varepsilon \rightarrow 0} \{\varepsilon X_{t/\varepsilon^2} - \chi_\varepsilon(\theta_{\varepsilon X_{t/\varepsilon^2}}(\mathbf{s}))\} = M_t.$$

Very roughly, using Fukushima decomposition (the Dirichlet form version of the Itô-Tanaka formula), we see that the quadratic variation of the limit martingale M is given by

$$(4.4) \quad \langle M \rangle_t = \mathcal{E}_Y(x - \chi, x - \chi)t,$$

where χ is a formal limit $\chi = \lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(\mathbf{s})$. In practice, χ_ε diverge as $\varepsilon \rightarrow 0$ and x is not in the domain of the Dirichlet space $(\mathcal{E}_Y, \mathcal{D}_Y)$; nevertheless, we can still justify (4.4) from formula (4.3) by introducing the quotient Dirichlet form $(\tilde{\mathcal{E}}_Y, \tilde{\mathcal{D}}_Y)$ and regarding ψ as an element of $\tilde{\mathcal{D}}_Y$. In fact, $\mathcal{E}_Y(x - \chi, x - \chi)$ can be replaced by $\tilde{\mathcal{E}}_Y^1(1 - \chi_1, 1 - \chi_1) + \tilde{\mathcal{E}}_Y^2(\chi_2, \chi_2)$, where χ corresponds to (χ_1, χ_2) .

To apply Fukushima decomposition to the function $x - \chi_\varepsilon(\theta_x(\mathbf{s}))$, we use the coupled Dirichlet form $(\mathcal{E}_{XY}, \mathcal{D}_{XY})$ because $x - \chi_\varepsilon(\theta_x(\mathbf{s}))$ belongs to $\mathcal{D}_{XY}$ locally. We remark that $x - \chi_\varepsilon(\theta_x(\mathbf{s}))$ cannot be in the domain of $(\mathcal{E}_Y, \mathcal{D}_Y)$ even locally.

§ 5. Vanishing self-diffusion constant in one dimension

In this section, we prove that $\alpha[\mu] = 0$ for $d = 1$ and μ satisfying (2.5).

Theorem 5.1. *Suppose that $d = 1$, and assume (1.4), (2.3)–(2.5), and (2.10). Then the self-diffusion constant $\alpha[\mu]$ vanishes.*

Lemma 5.2. Let $\mathcal{N}_2 = \{\mathbf{s}; \mathbf{s}(\{0\}) \geq 1\}$, and let Cap_Y be the capacity of the Dirichlet space $(\mathcal{E}_Y, \mathcal{D}_Y)$ on $L^2(\mathbf{S}, \mu_0)$. Then we obtain

$$(5.1) \quad \text{Cap}_Y(\mathcal{N}_1 \cup \mathcal{N}_2) = 0.$$

Proof. Recall that $\text{Cap}(\mathcal{N}_1) = 0$ by assumption, which implies the non-collision property of $\mathbf{X}$. Because $\mathbf{Y} = \sum_{i \in \mathbb{Z}^*} \delta_{Y_t^i} = \sum_{i \in \mathbb{Z}^*} \delta_{X_t^i - X_t^0}$ is given by (3.1) and (3.2), $\mathbf{Y}$ inherits the non-collision property from $\mathbf{X}$. Hence we deduce $\text{Cap}_Y(\mathcal{N}_1) = \text{Cap}_Y(\mathcal{N}_2) = 0$ from $\text{Cap}(\mathcal{N}_1) = 0$. This implies (5.1). $\square$

Let φ_N be the function on $\mathbf{S}$ such that

$$\varphi_N(\mathbf{s}) = \frac{1}{N} \{s_1 + \cdots + s_N\},$$

where we write $\mathbf{s} = \sum_{i \in \mathbb{Z}^*} \delta_{s_i}$ in such a way that $s_{-1} < 0 < s_1 < s_2 < s_3, \dots$. We note that φ_N is neither continuous on $\mathcal{N}_1$ nor smooth on $\mathcal{N}_1 \cup \mathcal{N}_2$; nevertheless, φ_N is an element of the domain of the Dirichlet form. Indeed, Lemma 5.2 implies the following.

Lemma 5.3. For each $N \in \mathbb{N}$,

$$(5.2) \quad \varphi_N \in \mathcal{D}_Y.$$

Furthermore, $\{\varphi_N\}_{N \in \mathbb{N}}$ is an $\mathcal{E}_Y$ -Cauchy sequence.

Proof. By Lemma 5.2 we have $\text{Cap}_Y(\mathcal{N}_1 \cup \mathcal{N}_2) = 0$. The points of discontinuity of φ are included in $\mathcal{N}_1 \cup \mathcal{N}_2$. Away from the points of discontinuity, φ satisfies

$$\mathbb{D}_Y[\varphi_N, \varphi_N] \leq \frac{1}{2} \left\{ 1 + \frac{1}{N} \right\}.$$

Combining these, we see that

$$\mathcal{E}_Y(\varphi_N, \varphi_N) \leq \frac{1}{2} \left\{ 1 + \frac{1}{N} \right\}.$$

Hence we obtain claim (5.2).

A straightforward calculation shows that $D^{\text{sft}}[\varphi_M - \varphi_N, \varphi_M - \varphi_N] = 0$. Recall that $\mathbb{D}_Y = D^{\text{sft}} + \mathbb{D}$ by (3.3). Then,

$$\begin{aligned} \mathbb{D}_Y[\varphi_M - \varphi_N, \varphi_M - \varphi_N] &= \mathbb{D}[\varphi_M - \varphi_N, \varphi_M - \varphi_N] \\ &\leq 2 \left\{ \mathbb{D}[\varphi_M, \varphi_M] + \mathbb{D}[\varphi_N, \varphi_N] \right\} = \left\{ \frac{1}{M} + \frac{1}{N} \right\}. \end{aligned}$$

Thus we have

$$\lim_{M, N \rightarrow \infty} \mathcal{E}_Y(\varphi_M - \varphi_N, \varphi_M - \varphi_N) = 0.$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. Let $\tilde{\varphi}_N$ be the element of $\tilde{\mathcal{D}}_Y$ whose representative is φ_N . Then

$$(5.3) \quad \lim_{N \rightarrow \infty} \tilde{\varphi}_N = (1, 0) \quad \text{in } \tilde{\mathcal{E}}_Y.$$

In particular, $(1, 0) \in \tilde{\mathcal{D}}_Y$.

Proof. By Lemma 5.3, $\{\varphi_N\}_{N \in \mathbb{N}}$ is an $\mathcal{E}_Y$ -Cauchy sequence. Hence we easily deduce that $\{\tilde{\varphi}_N\}_N$ is a Cauchy sequence in the quotient Dirichlet space $(\tilde{\mathcal{E}}_Y, \tilde{\mathcal{D}}_Y)$.

By a direct calculation we see that, for all N and μ_0 -a.s. $\mathbf{s}$,

$$D^{\text{sft}}\varphi_N(\mathbf{s}) = 1, \quad \mathbb{D}[\varphi_N, \varphi_N](\mathbf{s}) = 1/2N.$$

Hence we have $\tilde{\mathcal{E}}_Y^2(\varphi_N, \varphi_N) = 1/2N$. Combining these we obtain (5.3). □

Proof of Theorem 5.1 From Lemma 5.4, we see that $(1, 0) \in \tilde{\mathcal{D}}_Y$. Hence we obtain $\alpha[\mu] = 0$ from Lemma 4.2. □

§ 6. Proof of Theorem 2.1

Theorem 2.1 follows immediately from Theorem 3.7 and Theorem 5.1.

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