

Initial boundary value problem for nonlinear Schrödinger equations

By

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Abstract

We survey our recent results on inhomogeneous boundary value problem for the nonlinear Schrödinger equation on the half-space [9],[10] and nonlinear Neumann boundary value problem for nonlinear Schrödinger equations on the half-line [11].

§ 1. Introduction

We survey our recent results on inhomogeneous boundary value problem for the nonlinear Schrödinger equation on the half-space [9], [10] and nonlinear Neumann boundary value problem for nonlinear Schrödinger equations on the half-line [11]. Inhomogeneous boundary value problem is represented as

$$(1.1) \quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u, t > 0, x = (x', x_n) \in \mathbb{R}_+^n, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+^n, \\ Bu(t, x', 0) = h(t, x'), t > 0, x' \in \mathbb{R}^{n-1}, \end{cases}$$

where $\mathbb{R}_+^n = \{x = (x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\}$ denotes the upper half space, $\lambda \neq 0$ is a complex constant, $\mathbb{R}_+ = (0, \infty)$, $B = \partial_{x_n}$ (Neumann boundary value problem) or $B = I$ (Dirichlet boundary value problem), p is the order of nonlinearity satisfying

$$(1.2) \quad 1 + \frac{4}{n+2} < p < \begin{cases} 1 + \frac{4}{n-2}, n \geq 3 \\ \infty, n = 2 \end{cases},$$

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$u_0 = u_0(x', x_n)$ and $h(t, x')$ are given initial and boundary data. Nonlinear Neumann boundary value problem is defined by

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = f(t, x), t > 0, x \in \mathbb{R}_+, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+, \\ -\partial_x u(t, 0) = g(u)(t, 0), t > 0, \end{cases}$$

where f is a given data and

$$g(t) = g(u)(t, 0) = \mu |u|^{q-1} u(t, 0) \text{ or } \mu |u(t, 0)|^q, \mu \in \mathbb{C},$$

$q > 1$ is the order of nonlinearity.

Nonlinear Schrödinger equation

$$(1.3) \quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u, t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), x \in \mathbb{R}^n \end{cases}$$

was studied by many authors extensively. Let $U(t)$ be the free Schrödinger evolution group on $\mathbb{R}^n$ defined by

$$U(t)\phi = c_n t^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2it}} \phi(y) dy, \phi \in L^2(\mathbb{R}^n).$$

Then have the integral equation associated with (1.3)

$$(1.4) \quad u(t) = U(t)u_0 - i\lambda \int_0^t U(t-\tau) |u|^{p-1} u(\tau) d\tau.$$

There are several critical exponents for (1.4). Equation (1.4) is called scattering critical for $p = 1 + \frac{2}{n}$ (in the case of nonlinear heat equation, $p = 1 + \frac{2}{n}$ is the Fujita exponent), $L^2(\mathbb{R}^n)$ critical for $p = 1 + \frac{4}{n}$, energy critical for $p = 1 + \frac{4}{n-2}$, Strauss critical if p is a positive root of $\frac{n}{2} \frac{p-1}{p+1} p = 1$, namely $p = \frac{n+2+\sqrt{n^2+12n+4}}{2n}$. Note that exponent $1 + \frac{4}{n+2}$ is in the between scattering critical exponent and Strauss critical exponent. Cazenave and Weissler [3] showed that (1.4) has a small global solution $u \in C([0, \infty); H^1(\mathbb{R}^n) \cap H^{0,1}(\mathbb{R}^n))$ when $\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_0\|_{H^{0,1}(\mathbb{R}^n)}$ is small and the order of nonlinearity satisfies (1.2), where

$$H^{0,s}(\mathbb{R}^n) = \left\{ v; \|v\|_{H^{0,s}(\mathbb{R}^n)} = \left\| \left(1 + |x|^2\right)^{\frac{s}{2}} v \right\|_{L^2(\mathbb{R}^n)} < \infty \right\}, s \geq 0$$

and $H^1(\mathbb{R}^n)$ is the energy space

$$H^1(\mathbb{R}^n) = \left\{ v; \|v\|_{H^1(\mathbb{R}^n)} = \|v\|_{L^2(\mathbb{R}^n)} + \|\nabla v\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

We let the exponent $p = 1 + \frac{4}{n+2s}$ be $H^1(\mathbb{R}^n) \cap H^{0,s}(\mathbb{R}^n)$ critical exponent.

We put

$$\begin{cases} p_F = 1 + \frac{2}{n}, \text{ Fujita exponent,} \\ p_{CW} = 1 + \frac{4}{n+2}, \text{ Cazenave-Weissler exponent,} \\ p_S = \frac{n+2+\sqrt{n^2+12n+4}}{2n}, \text{ Strauss exponent,} \\ p_{L^2} = 1 + \frac{4}{n}, \text{ Mass critical exponent,} \\ p_E = 1 + \frac{4}{n-2}, \text{ Energy critical exponent.} \end{cases}$$

We restrict our attention to the case

$$p_{L^2} \leq p < p_E,$$

then in [14], it was shown that (1.4) has a small global solution u satisfying the a-priori estimate

$$\|u\|_{L^q(\mathbb{R}_t; W^{1,r}(\mathbb{R}^n))} + \|u\|_{L^\infty(\mathbb{R}_t; H^1(\mathbb{R}^n))} \leq C \|u_0\|_{H^1(\mathbb{R}^n)}.$$

if the data are small in $H^1(\mathbb{R}^n)$, where

$$(1.5) \quad \frac{1}{r} = \frac{1}{p+1}, \frac{1}{q} = \frac{n}{4} \frac{p-1}{p+1}$$

and

$$W^{1,r}(\mathbb{R}^n) = \left\{ v; \|v\|_{W^{1,r}(\mathbb{R}^n)} = \|v\|_{L^r(\mathbb{R}^n)} + \|\nabla v\|_{L^r(\mathbb{R}^n)} < \infty \right\}.$$

The choice of an admissible pair (q, r) of the Strichartz estimate satisfying (1.5) was used by Strauss [14].

§ 2. Inhomogeneous Neumann boundary value problem

In [9] asymptotic behavior in time of small solutions for $p_{CW} < p < p_{L^2}$ was studied. In order to state our results, we use the weighted Sobolev space based on

$$H^{0,1}(\mathbb{R}_+^n) = \left\{ v; \|v\|_{H^{0,1}(\mathbb{R}_+^n)} = \|v\|_{L^2(\mathbb{R}_+^n)} + \|xv\|_{L^2(\mathbb{R}_+^n)} < \infty \right\}.$$

We consider the integral equation associated with (1.1) which is written as

$$(2.1) \quad u(t) = U_N(t) u_0 + z(t) - i\lambda \int_0^t U_N(t-\tau) |u|^{p-1} u(\tau) d\tau$$

when $B = \partial_{x_n}$, where

$$\begin{aligned} U_N(t) &= U_{nN}(t) \tilde{U}(t), \\ z(t, x) &= -i(2i\pi)^{-1/2} \int_0^t \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) h(t-\tau, x') d\tau, \\ U_{nN}(t) \phi &= c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n-y_n)^2}{2it}} + e^{-\frac{(x_n+y_n)^2}{2it}} \right) \phi(y_n) dy_n \end{aligned}$$

and

$$\tilde{U}(t)\phi = c_{n-1}t^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'|^2}{2it}} \phi(y') dy', \quad c_n = (2i\pi)^{-n/2}.$$

If the data (u_0, h) are smooth, then solutions of (2.1) satisfies (1.1).

There are many results concerns with nonlinear Schrödinger equations on the half-line or the half-space with inhomogeneous boundary value conditions. In [1], global in time of small solutions to the boundary value problem for

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u, t > 0, x = (x', x_n) \in \mathbb{R}_+^n, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+^n, \\ Bu(t, x', 0) = h(t, x'), t > 0, x' \in \mathbb{R}^{n-1} \end{cases}$$

was shown when $p_{L^2} \leq p < p_E$, $B = I$ (Dirichlet problem),

$$(u_0, h) \in H^1(\mathbb{R}_+^n) \times L^2\left(\mathbb{R}_+; H^{3/2}(\mathbb{R}^{n-1})\right) \cap W^{3/4,2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})),$$

and (u_0, h) is small in the norm

$$\|u_0\|_{H^1(\mathbb{R}_+^n)} + \|h\|_{L^2(\mathbb{R}_+; H^{3/2}(\mathbb{R}^{n-1})) \cap W^{3/4,2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))}.$$

The estimate of solutions u such that

$$\begin{aligned} & \|u\|_{L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^{n-1}))} \\ & \leq C \left(\|u_0\|_{H^1(\mathbb{R}_+^n)} + \|h\|_{L^2(\mathbb{R}_+; H^{3/2}(\mathbb{R}^{n-1})) \cap W^{3/4,2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} \right), \end{aligned}$$

was obtained, where the fractional derivatives in space and time are defined by

$$H^{3/2}(\mathbb{R}^{n-1}) = \left\{ v \in L^2(\mathbb{R}^{n-1}); \left\| \langle \xi \rangle^{3/2} \mathcal{F}v \right\|_{L^2(\mathbb{R}_\xi^{n-1})} < \infty \right\}$$

and

$$\begin{aligned} & W^{3/4,2}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})) \\ & = \left\{ v \in L^2(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})); \left\| \langle \delta \rangle^{3/4} \mathcal{F}_t v \right\|_{L^2(\mathbb{R}_\delta; L^2(\mathbb{R}^{n-1}))} < \infty \right\}. \end{aligned}$$

Though nonlinear Neumann problem (1.1) is not discussed explicitly, Strichartz estimates obtained in [1] are valid for

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = f, t > 0, x = (x', x_n) \in \mathbb{R}_+^n, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+^n, \\ \partial_{x_n} u(t, x', 0) = h(t, x'), t > 0, x' \in \mathbb{R}^{n-1} \end{cases}$$

and can be applicable to (1.1) for $p_{L^2} \leq p < p_E$ when $h(0, x') = 0$. Therefore the results in [1] are considered as the extensions from the results of [14] to the boundary value problem.

In papers, [4], [7], [8] we studied (1.1) with a critical power nonlinearity $p = 1 + \frac{2}{n}$ on $\mathbb{R}_+^n$ for $n = 1, 2$. In [5], inhomogeneous Dirichlet boundary value problem was considered. In these papers sufficient conditions to show asymptotic behavior of solutions and the optimal time decay estimates of solutions in L^∞ space were presented. As far as we know, there are no results of time decay of solutions on the inhomogeneous (Neumann or Dirichlet) boundary value problem of nonlinear Schrödinger equations on the upper half space in general space dimensions $n \geq 3$.

Notations and main results. We use the following notations : For the spatial variable $x = (x', x_n)$, $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$ and $j = 1, 2, \dots, n$, let

$$\begin{aligned} \partial_{x_j} &= \frac{\partial}{\partial x_j}, \partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \tilde{\partial}^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_{n-1}}^{\alpha_{n-1}}, \\ \nabla &= (\partial_{x_1}, \dots, \partial_{x_n}), \tilde{\nabla} = (\partial_{x_1}, \dots, \partial_{x_{n-1}}), \Delta = \sum_{j=1}^n \partial_{x_j}^2, \tilde{\Delta} = \sum_{j=1}^{n-1} \partial_{x_j}^2, \\ J_{x_j} &= x_j + it\partial_{x_j}, J = (J_{x_1}, \dots, J_{x_n}), \tilde{J} = (J_{x_1}, \dots, J_{x_{n-1}}) \\ J^\alpha &= J_{x_1}^{\alpha_1} \dots J_{x_n}^{\alpha_n}, |\alpha| = \sum_{j=1}^n \alpha_j, \tilde{J}^\alpha = J_{x_1}^{\alpha_1} \dots J_{x_{n-1}}^{\alpha_{n-1}}, |\alpha| = \sum_{j=1}^{n-1} \alpha_j, \end{aligned}$$

We define the Schrödinger evolution operator associated with the homogeneous Neumann boundary value problem .

For $\phi \in L^2(\mathbb{R}_+^n)$, let the evolution group for the linear Schrödinger equation to the homogeneous Neumann boundary condition be

$$(2.2) \quad U_N(t) = U_{nN}(t)\tilde{U}(t),$$

where

$$(2.3) \quad U_{nN}(t)\phi = c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n - y_n)^2}{2it}} + e^{-\frac{(x_n + y_n)^2}{2it}} \right) \phi(y_n) dy_n$$

and

$$(2.4) \quad \tilde{U}(t)\phi = c_{n-1} t^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x' - y'|^2}{2it}} \phi(y') dy', \quad c_n = (2i\pi)^{-n/2}.$$

It is clear that the evolution operator U_N is a unitary operator from $L^2(\mathbb{R}_+^n)$ to itself. We also recall the Schrödinger evolution operator associated with the homogeneous Dirichlet boundary value problem by

$$U_D(t)\phi = U_{nD}(t)\tilde{U}(t)\phi,$$

where

$$U_{nD}(t)\phi = c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n - y_n)^2}{2it}} - e^{-\frac{(x_n + y_n)^2}{2it}} \right) \phi(y_n) dy_n.$$

Weighted Sobolev spaces $H^{m,s}(\mathbb{R})$ are defined as follows : For $m, s \in \mathbb{R}$, let

$$H^{m,s}(\mathbb{R}) = \left\{ v \in L^2(\mathbb{R}) ; \|v\|_{H^{m,s}(\mathbb{R})} = \sum_{|\alpha| \leq m, |\beta| \leq s} \|\partial^\alpha x^\beta v\|_{L^2(\mathbb{R})} < \infty \right\},$$

$$W^{m,p}(\mathbb{D}) = \left\{ v \in L^p(\mathbb{D}) ; \|v\|_{W^{m,p}(\mathbb{D})} = \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{L^p(\mathbb{D})} < \infty \right\},$$

where $\mathbb{D} = \mathbb{R}_+^n$ or $\mathbb{R}^{n-1}$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. For simplicity, we write $H^m(\mathbb{D}) = H^{m,0}(\mathbb{D})$. For time derivative, we denote

$$W_l^{s,p}(\mathbb{R}_+; X) = \left\{ v \in L^p(\mathbb{R}_+; X) ; \|v\|_{W_l^{s,p}(\mathbb{R}_+; X)} = \sum_{j=0}^s \left\| \langle \cdot \rangle^l \partial_t^j v \right\|_{L^p(\mathbb{R}_+; X)} < \infty \right\},$$

where X is a Banach space and $1 \leq p \leq \infty$. We call a pair of exponents (q, r) as an admissible pair for Strichartz estimates if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, 2 \leq q, r \leq \infty, (q, r, n) \neq (2, \infty, 2).$$

To state our main results, we introduce the function space

$$X_{q,r} = \left\{ v \in C(\mathbb{R}_+; L^2(\mathbb{R}_+^n)) ; \|v\|_{X_{q,r}} < \infty \right\},$$

$$\begin{aligned} \|v\|_{X_{q,r}} &= \sum_{|\alpha| \leq 1} \left(\|J^\alpha v\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}_+^n))} + \|J^\alpha v\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{R}_+^n))} \right) \\ &+ \|v\|_{L^q(\mathbb{R}_+; W^{1,r}(\mathbb{R}_+^n))} + \|v\|_{L^\infty(\mathbb{R}_+; H^1(\mathbb{R}_+^n))} \end{aligned}$$

and

$$\begin{aligned} Z_\gamma &= W_\gamma^{0,\infty}(\mathbb{R}_+; H^2(\mathbb{R}^{n-1})) \cap W_{\gamma-2}^{0,\infty}(\mathbb{R}_+; H^{0,2}(\mathbb{R}^{n-1})) \\ &\cap W_\gamma^{1,\infty}(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})). \end{aligned}$$

For simplicity, we write $L^q L^r = L^q(\mathbb{R}_+; L^r(\mathbb{R}_+^n))$, $L^q W^{1,r} = L^q(\mathbb{R}_+; W^{1,r}(\mathbb{R}_+^n))$. We use notation such that

$$\|h\|_{X_1 \cap X_2} = \|h\|_{X_1} + \|h\|_{X_2}.$$

Now we give our result from [9].

Theorem 2.1. *We assume that $n \geq 3$*

$$p_{CW} < p < p_E$$

and (q, r) is an admissible pair defined by

$$(2.5) \quad \frac{n}{r} = \frac{n}{2} - \frac{2}{q}, \frac{2}{q} = \frac{n-2}{4} (p-1),$$

Suppose that the initial and the boundary data satisfy

$$u_0 \in Y, h \in Z_\gamma$$

with $h(0, x') = 0$ and $\gamma > 2 + 3/4$, where

$$Y = H^1(\mathbb{R}_+^n) \cap H^{0,1}(\mathbb{R}_+^n),$$

Furthermore we assume that for some $\varepsilon > 0$

$$\|u_0\|_Y \leq \varepsilon, \|h\|_{Z_\gamma} \leq \varepsilon^p.$$

Then there exists an $\varepsilon > 0$ such that the integral equation (2.1) has a unique global solution $u \in X_{q,r}$ satisfying

$$(2.6) \quad \|u\|_{X_{q,r}} \leq \sqrt{\varepsilon}.$$

Furthermore we have the estimates of z defined in (2.1) such that

$$(2.7) \quad \sum_{|\alpha| \leq 2} \|J^\alpha z\|_{L^\infty L^2} + \|z\|_{L^\infty H^2} \leq C\varepsilon^{p/2}.$$

We note that the additional technical assumption $h(0, x') = 0$ is used to get the estimate of $(x_n + it\partial_{x_n})^2 z$ which is needed to prove asymptotic behavior of solutions, where z is defined in the integral equation (2.1).

An admissible pair (q, r) of Strichartz estimates defined in (2.5) is the same one which was used in [3].

• Strategy of the proof of the theorem. We have the integral equation (2.1) by using the Fourier cosine transform formally and for (2.1), we have the following result.

Proposition 2.2. ([6]). *Let*

$$u_0 \in L^2(\mathbb{R}_+^n), f \in L_{loc}^1(\mathbb{R}_+; L^2(\mathbb{R}_+^n))$$

and

$$h \in H_{loc}^1(\mathbb{R}_+; L^2(\mathbb{R}^{n-1})) \cap L_{loc}^2(\mathbb{R}_+; H^2(\mathbb{R}^{n-1}))$$

and set

$$z(t, x) = -ic_1 \int_0^t \tau^{-1/2} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) h(t - \tau, x') d\tau,$$

where $\tilde{U}(\tau)$ is given in (2.4). Then

$$u(t) = U_N(t) u_0 + z(t) - i \int_0^t U_N(t - \tau) f(\tau) d\tau$$

is well defined in $C([0, T]; L^2(\mathbb{R}_+^n))$ and if $u_0 \in H^2(\mathbb{R}_+^n)$, then u solves the initial boundary value problem

$$(2.8) \quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = f, t > 0, x \in \mathbb{R}_+^n, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+^n, \\ \partial_{x_n} u(t, x', 0) = h(t, x'), t > 0, x' \in \mathbb{R}^{n-1} \end{cases}$$

in $C([0, T]; L^2(\mathbb{R}_+^n))$.

Since the initial and 0-Neumann boundary value problem for (2.8) ($h = 0$) can be solved by applying (2.2) to the Duhamel formula, we merely show that the inhomogeneous term $z(t)$ solves the inhomogeneous Schrödinger equation (2.8) with $u_0 = 0$ and $f = 0$, formally,

$$(2.9) \quad \begin{cases} i\partial_t z + \frac{1}{2}\Delta z = 0, t > 0, x \in \mathbb{R}_+^n, \\ z(0, x) = 0, x \in \mathbb{R}_+^n, \\ \partial_{x_n} z(t, x', 0) = h(t, x'), t > 0, x' \in \mathbb{R}^{n-1}. \end{cases}$$

We put $y = x_n \tau^{-\frac{1}{2}}$ in (1.3), then

$$(2.10) \quad \begin{aligned} \partial_{x_n} z(t, x) &= c_1 \int_0^t x_n \tau^{-\frac{3}{2}} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) h(t - \tau, x') d\tau \\ &= -2c_1 \int_{-\infty}^{\frac{x_n}{\sqrt{t}}} e^{-\frac{y^2}{2i}} \tilde{U}\left(\frac{x_n^2}{y^2}\right) h\left(t - \frac{x_n^2}{y^2}, x'\right) dy \end{aligned}$$

and

$$(2.11) \quad \begin{aligned} &\partial_{x_n}^2 z(t, x) \\ &= -2c_1 \int_{-\infty}^{\frac{x_n}{\sqrt{t}}} e^{-\frac{y^2}{2i}} \frac{2x_n}{y^2} \partial_{\frac{x_n^2}{y^2}} \left(\tilde{U}\left(\frac{x_n^2}{y^2}\right) h\left(t - \frac{x_n^2}{y^2}, x'\right) \right) dy \\ &\quad - 2c_1 t^{-1/2} e^{\frac{ix_n^2}{2t}} \tilde{U}(t) h(0, x'). \end{aligned}$$

The first term of the right hand side of the above is equal to

$$\begin{aligned} &-2c_1 \int_0^{\frac{x_n}{\sqrt{t}}} e^{-\frac{y^2}{2i}} \frac{2\tau}{x_n} \partial_{\frac{x_n^2}{y^2}} \left(\tilde{U}\left(\frac{x_n^2}{y^2}\right) h\left(t - \frac{x_n^2}{y^2}, x'\right) \right) d\frac{x_n}{\sqrt{\tau}} \\ &= 2c_1 \int_0^t e^{-\frac{x_n^2}{2i\tau}} \tau^{-1/2} \partial_\tau \left(\tilde{U}(\tau) h(t - \tau, x') \right) d\tau \\ &= 2c_1 \int_0^t e^{-\frac{x_n^2}{2i\tau}} \tau^{-1/2} \tilde{U}(\tau) \left(\frac{i}{2} \tilde{\Delta} - \partial_t \right) h(t - \tau, x') d\tau. \end{aligned}$$

Hence it follows

$$\begin{aligned}
& \partial_{x_n}^2 z(t, x) \\
&= 2c_1 \int_0^t e^{-\frac{x_n^2}{2i\tau}} \tau^{-1/2} \tilde{U}(\tau) \left(\frac{i}{2} \tilde{\Delta} - \partial_t \right) h(t - \tau, x') d\tau \\
&\quad - 2c_1 t^{-1/2} e^{-\frac{x_n^2}{2it}} \tilde{U}(t) h(0, x') \\
&= \tilde{\Delta} \left(c_1 \int_0^t e^{-\frac{x_n^2}{2i\tau}} \tau^{-1/2} \tilde{U}(\tau) h(t - \tau, x') d\tau \right) \\
&\quad - 2\partial_t \left(c_1 \int_0^t e^{-\frac{x_n^2}{2i\tau}} \tau^{-1/2} \tilde{U}(\tau) h(t - \tau, x') d\tau \right) \\
(2.12) \quad &+ 2c_1 t^{-1/2} e^{-\frac{x_n^2}{2it}} \tilde{U}(t) h(0, x') - 2c_1 t^{-1/2} e^{-\frac{x_n^2}{2it}} \tilde{U}(t) h(0, x').
\end{aligned}$$

By (1.3) and (2.12) we obtain

$$(2.13) \quad i\partial_t z + \frac{1}{2} \left(\tilde{\Delta} + \partial_{x_n}^2 \right) z = 0.$$

On the other hand from (2.10), we substitute $x_n = 0$ and

$$(2.14) \quad \partial_{x_n} z(t, x', 0) = 2c_1 \int_0^\infty e^{-\frac{y^2}{2i}} h(t, x') dy = h(t, x'),$$

where the y -integration is understood as the oscillatory integral. From (2.13) and (2.14), z solves the problem (2.9) if the data $h(t, x')$ satisfy the assumptions.

Our method is based on the energy method and the generator of the Galilei transform $J_{x_j} = x_j + it\partial_{x_j}$ which commutes with the Schrödinger evolution operator. We apply the operator J_{x_j} to the integral equation (2.1) and use Strichartz estimates to evolution operators of homogeneous problems. In order to show the theorem we use the commutation relation between the operator J_{x_n} and the Schrödinger evolution group for the homogeneous Neumann boundary problem which yields the evolution group of the Schrödinger equation with the homogeneous Dirichlet boundary problem. Furthermore the application of the energy method yields desired estimates of solutions z to (2.9).

§ 3. Inhomogeneous Dirichlet boundary value problem

We consider the integral equation associated with the homogeneous Dirichlet boundary value problem which is defined by

$$(3.1) \quad u(t) = U_D(t) u_0 + z(t) - i\lambda \int_0^t U_D(t - \tau) |u|^{p-1} u(\tau) d\tau,$$

where

$$z(t, x) = c_1 \int_0^t x_n \tau^{-\frac{3}{2}} e^{-\frac{x_n^2}{2i\tau}} \tilde{U}(\tau) h(t - \tau) d\tau,$$

$U_D(t) = U_{nD}(t)\tilde{U}(t)$ with

$$U_{nD}(t)\phi = c_1 t^{-1/2} \int_0^\infty \left(e^{-\frac{(x_n - y_n)^2}{2it}} - e^{-\frac{(x_n + y_n)^2}{2it}} \right) \phi(y_n) dy_n,$$

$$\tilde{U}(t)\phi = c_{n-1} t^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x' - y'|^2}{2it}} \phi(y') dy',$$

and $c_n = (2i\pi)^{-n/2}$. We find that z is the solution of

$$\begin{cases} i\partial_t z + \frac{1}{2}\Delta z = 0, (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n, \\ z(0, x) = 0, x \in \mathbb{R}_+^n, \\ z(t, x', 0) = h(t, x'), (t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}. \end{cases}$$

As far as we know, there are no results of time decay of solutions to (1.1) in general space dimensions $n \geq 3$. We are interested in higher space dimensions.

§ 3.1. Main result of [10].

We now state the main result in [10].

Theorem 3.1. *We assume that*

$$p_{CW} < p < p_E, n \geq 3,$$

$$(u_0, h) \in Y \times Z_\gamma = H^1(\mathbb{R}_+^n) \cap H^{0,1}(\mathbb{R}_+^n) \times Z_\gamma, \gamma > 3$$

with

$$0 < \|u_0\|_Y \leq \varepsilon, 0 < \|h\|_{Z_\gamma} \leq \rho$$

and the compatibility condition $h(0, x') = u_0(x', 0)$ holds, where

$$Z_\gamma = \cap_{k=0}^2 W_\gamma^{k,\infty}(\mathbb{R}_+; H^{4-2k}(\mathbb{R}^{n-1})) \\ \cap_{l=0}^1 W_{\gamma-1}^{l,\infty}(\mathbb{R}_+; H^{2-2l,1}(\mathbb{R}^{n-1})) \cap L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^{n-1})).$$

Furthermore we assume that $h(t) \in C^1(\mathbb{R}^{n-1})$ and

$$|\partial_{x_j} h(t, x')| \leq C |h(t, x')|$$

for all $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$ if $p < 2$. Then there exist $\rho, \varepsilon > 0$ such that the integral equation (3.1) has a unique global solution $u \in X_{q,r}$ where (q, r) is an admissible pair of the Strichartz estimate such that

$$\frac{n}{r} + \frac{2}{q} = \frac{n}{2}, \frac{2}{q} = \frac{p-1}{4} (n-2), n \geq 3.$$

Moreover time decay estimates of solutions

$$\|u(t)\|_{L^\sigma(\mathbb{R}_+^n)} \leq C \langle t \rangle^{-\frac{n}{2}(1-\frac{2}{\sigma})}$$

are valid for $2 \leq \sigma \leq 2 + \frac{4}{n-2}$.

3.1.1. Comments of the theorem of inhomogeneous Dirichlet boundary value problem.

In [9], we have used the commutation relation

$$J_{x_n}(t) U_N(t - \tau) = U_D(t - \tau) J_{x_n}(\tau).$$

Then by using the Strichartz estimates, we get the desired results. However in the Dirichlet case we do not have

$$J_{x_n}(t) U_D(t - \tau) \phi(\tau) = U_N(t - \tau) J_{x_n}(\tau) \phi(\tau)$$

if we do not assume $\phi(\tau, x', 0) = 0$. Indeed, we have the commutation relation

$$(3.2) \quad \begin{aligned} J_{x_n}(t) U_D(t - \tau) \phi(\tau, x) &= U_N(t - \tau) J_{x_n}(\tau) \phi(\tau, x) \\ &+ \frac{2ic_1}{(t - \tau)^{1/2}} e^{-\frac{x_n^2}{2i(t - \tau)}} \tilde{U}(t - \tau) \tau \phi(\tau, x', 0). \end{aligned}$$

The identity is obtained by a direct calculation

$$\begin{aligned} &J_{x_n}(t) U_{nD}(t - \tau) g \\ &= \frac{c_1}{(t - \tau)^{1/2}} J_{x_n}(t - \tau) \int_0^\infty \left(e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) g(\tau, y_n) dy_n \\ &- \frac{c_1}{(t - \tau)^{1/2}} \int_0^\infty \left((i\tau \partial_{x_n}) \left(e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) \right) g(\tau, y_n) dy_n \\ &= \frac{c_1}{(t - \tau)^{1/2}} \int_0^\infty \left(e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} + e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) J_{x_n}(\tau) g(\tau, y_n) dy_n \\ &+ \frac{2c_1}{(t - \tau)^{1/2}} i\tau e^{-\frac{x_n^2}{2i(t - \tau)}} g(\tau, 0), \end{aligned}$$

since

$$\begin{aligned} &J_{x_n}(t - \tau) \left(e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) g(\tau, y_n) \\ &= x_n \left(e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} - e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) g(\tau, y_n) \\ &+ \left(-(x_n - y_n) e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} + (x_n + y_n) e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) g(\tau, y_n) \\ &= \left(e^{-\frac{|x_n - y_n|^2}{2i(t - \tau)}} + e^{-\frac{|x_n + y_n|^2}{2i(t - \tau)}} \right) y_n g(\tau, y_n). \end{aligned}$$

Hence we have

$$\begin{aligned}
 & J_{x_n}(t) \int_0^t U_D(t-\tau) \phi(\tau, x) d\tau \\
 &= \int_0^t U_N(t-\tau) J_{x_n}(\tau) \phi(\tau, x) d\tau \\
 (3.3) \quad &+ 2ic_1 \int_0^t \frac{1}{(t-\tau)^{1/2}} e^{-\frac{x_n^2}{2i(t-\tau)}} \tilde{U}(t-\tau) \tau \phi(\tau, x', 0) d\tau.
 \end{aligned}$$

It seems that it is difficult to control the second term of the right hand side of (3.3).

In order to avoid the commutation relation between $J_{x_n}(t)$ and $U_D(t)$, we use a new integral form and prove a-priori estimate of

$$\left\| J_{x_n}(t) \int_0^t U_D(t-\tau) \phi(\tau, x) d\tau \right\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}_+^n))}$$

directly. More precisely, we use the equation of $I\phi = \int_0^t U_D(t-\tau) \phi(\tau, x) d\tau$ which is written as

$$i\partial_t I\phi + \frac{1}{2}\Delta I\phi = ie^{-x_n}\phi,$$

and the commutation relation between $J_{x_n}(t)$ and $i\partial_t + \frac{1}{2}\Delta$ to get the desired estimates.

§ 4. Nonlinear Neumann boundary value problem

We consider the initial-boundary value problem for nonlinear Schrödinger equations on the positive half-line

$$(4.1) \quad \begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = f(t, x), t > 0, x \in \mathbb{R}_+, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+, \\ -\partial_x u(t, 0) = g(u)(t, 0), t > 0, \end{cases}$$

where

$$f \in L^1\left(\mathbb{R}_t; \dot{H}^s(\mathbb{R}_+)\right), -\frac{1}{2} < s < \frac{1}{2}$$

$$g(t) = g(u)(t, 0) = \mu |u|^{q-1} u(t, 0) \text{ or } \mu |u(t, 0)|^q, \mu \in \mathbb{C}, q > 1,$$

$\dot{H}^s(\mathbb{R}_x)$ is defined as

$$\dot{H}^s(\mathbb{R}_x) = \left\{ u : \|u\|_{\dot{H}^s(\mathbb{R}_x)}^2 = \int_{\mathbb{R}} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty \right\}, -\frac{1}{2} < s < \frac{1}{2}.$$

Our purpose in this section is to present a global existence in time of small solutions to the integral equation associated with (4.1) when $q > 2$ and a blow-up of solutions when $1 < q \leq 2$.

The integral equation associated with (4.1) is written as

$$(4.2) \quad \begin{aligned} u(t, x) = & U_N(t) u_0 + \frac{1}{\sqrt{2\pi}} e^{\frac{\pi}{4}i} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{x^2}{2i(t-\tau)}} g(\tau) d\tau \\ & - e^{\frac{\pi}{2}i} \int_0^t U_N(t-\tau) f(\tau) d\tau, \end{aligned}$$

where

$$U_N(t) u_0 = \frac{1}{\sqrt{2t\pi}} e^{-\frac{\pi}{4}i} \int_0^\infty \left(e^{-\frac{(x-y)^2}{2it}} + e^{-\frac{(x+y)^2}{2it}} \right) u_0(y) dy.$$

We extend $u_0(x)$, $f(t, x)$ to $\mathbb{R}$ as even functions with respect to x variable and denote the extensions by the same letters $u_0(x)$, $f(t, x)$, respectively, then we have the problem

$$(4.3) \quad \begin{aligned} u(t, x) = & U(t) u_0 + \frac{1}{\sqrt{2\pi}} e^{\frac{\pi}{4}i} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{x^2}{2i(t-\tau)}} g(\tau) d\tau \\ & - e^{\frac{\pi}{2}i} \int_0^t U(t-\tau) f(\tau) d\tau, \end{aligned}$$

where

$$U(t) u_0 = \frac{1}{\sqrt{2it\pi}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2it}} u_0(y) dy$$

is the free Schrödinger group. In what follows, we consider the Cauchy problem (4.3) which is associated with (4.1). In order to state our result, we introduce the function space. We define the homogeneous Besov norm by

$$\begin{aligned} & \|v\|_{\dot{B}_{\infty, \infty}^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} \\ & = \|v\|_{\dot{B}_{\infty, \infty, x}^\alpha L_t^\beta} = \sup_{\lambda > 0} \lambda^{-\alpha} \sup_{|\eta| \leq \lambda} \left\| \|v(\cdot + \eta) - v(\cdot)\|_{L^\beta(\mathbb{R}_t)} \right\|_{L^\infty(\mathbb{R}_x)}, \alpha > 0 \end{aligned}$$

We introduce the space of solutions as

$$X_{\alpha, r} = C^\alpha \left(\mathbb{R}_x; L^{\frac{2r}{2+\alpha r}}(\mathbb{R}_t) \right) \cap C_b(\mathbb{R}_x; L^r(\mathbb{R}_t)) \cap C_b \left(\mathbb{R}_t; \dot{H}^{-\left(\frac{2}{r}-\frac{1}{2}\right)}(\mathbb{R}_x) \right),$$

$$0 < \alpha < 1, r \geq 2,$$

where $C^\alpha \left(\mathbb{R}_x; L^{\frac{2r}{2+\alpha r}}(\mathbb{R}_t) \right)$ implies the set of the Hölder continuous of order α in $\mathbb{R}_x$ in the norm $L^{\frac{2r}{2+\alpha r}}(\mathbb{R}_t)$. Namely if $v \in C^\alpha \left(\mathbb{R}_x; L^{\frac{2r}{2+\alpha r}}(\mathbb{R}_t) \right)$, then

$$\lim_{\delta \rightarrow 0} \|v(x+\delta) - v(x)\|_{L^{\frac{2r}{2+\alpha r}}(\mathbb{R}_t)} |\delta|^{-\alpha} = 0,$$

for any $x \in \mathbb{R}_x$. For simplicity, we write

$$C^\alpha \left(\mathbb{R}_x; L^{\frac{2r}{2+\alpha r}}(\mathbb{R}_t) \right) = C_x^\alpha L_t^{\frac{2r}{2+\alpha r}},$$

$$C_b(\mathbb{R}_x; L^r(\mathbb{R}_t)) = C \cap L^\infty(\mathbb{R}_x; L^r(\mathbb{R}_t)) = C_{b,x} L_t^r$$

and

$$C_b\left(\mathbb{R}_t; \dot{H}^{-(\frac{2}{r}-\frac{1}{2})}(\mathbb{R}_x)\right) = C \cap L^\infty\left(\mathbb{R}_t; \dot{H}^{-(\frac{2}{r}-\frac{1}{2})}(\mathbb{R}_x)\right) = C_{b,t} \dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}.$$

Then

$$X_{\alpha,r} = C_x^\alpha L_t^{\frac{2r}{2+\alpha r}} \cap C_{b,x} L_t^r \cap C_{b,t} \dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}.$$

The norms of $C_x^\alpha L_t^{\frac{2r}{2+\alpha r}}$, $C_{b,x} L_t^r$ and $C_{b,t} \dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}$ are defined by

$$\begin{aligned} \|v\|_{C_x^\alpha L_t^{\frac{2r}{2+\alpha r}}} &= \|v\|_{B_{\infty,\infty,x}^\alpha L_t^{\frac{2r}{2+\alpha r}}} = \left\| \|v\|_{L_t^{\frac{2r}{2+\alpha r}}} \right\|_{B_{\infty,\infty,x}^\alpha}, \\ \|v\|_{C_{b,x} L_t^r} &= \|v\|_{L_x^\infty L_t^r} = \left\| \|v\|_{L_t^r} \right\|_{L_x^\infty}, \\ \|v\|_{C_{b,t} \dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}} &= \|v\|_{L_t^\infty \dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}} = \left\| \|v\|_{\dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}} \right\|_{L_t^\infty}. \end{aligned}$$

Therefore we define the norm of $X_{\alpha,r}$ as

$$\|v\|_{X_{\alpha,r}} = \|v\|_{L_x^\infty L_t^r} + \|v\|_{B_{\infty,\infty,x}^\alpha L_t^{\frac{2r}{2+\alpha r}}} + \|v\|_{L_t^\infty \dot{H}_x^{-(\frac{2}{r}-\frac{1}{2})}}.$$

We now state our main results obtained in [11].

Theorem 4.1. *We assume $u_0 \in \dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}$, $f \in L_t^1 \dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}$, $2 < q < \infty$, $0 < \alpha \leq \frac{q-2}{q-1}$. Then there exists an $\varepsilon > 0$ such that (4.3) has a unique global solution*

$$u \in X_{\alpha,2(q-1)}$$

for any (u_0, f) satisfying

$$\|u_0\|_{\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}} + \|f\|_{L_t^1 \dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}} \leq \varepsilon.$$

We remark here that the solution constructed in the theorem is well defined at $x = 0$ and the solution on the positive line satisfies (4.2). Our result says a global existence result in a scale invariant space for $f = 0$. We put

$$u_\lambda(t, x) = \lambda^{-\frac{1}{2(q-1)}} u\left(\frac{t}{\lambda}, \frac{x}{\sqrt{\lambda}}\right),$$

then if u is a solution of

$$u(t, x) = U(t) u_0 + \frac{1}{\sqrt{2\pi}} e^{\frac{\pi}{4}i} \int_0^t \frac{1}{\sqrt{t-\tau}} e^{-\frac{x^2}{2i(t-\tau)}} g(u(\tau, 0)) d\tau$$

u_λ is also a solution. We have

$$\|u_\lambda(0, \cdot)\|_{\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}} = \|u_0\|_{\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}}$$

and

$$\|u_\lambda(0, \cdot)\|_{L_x^{q-1}} = \|u_0\|_{L_x^{q-1}}$$

which implies that $\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}$ and L_x^{q-1} are invariant spaces for $2 \leq q \leq \infty$. We have by the Hardy-Littlewood-Sobolev inequality

$$\begin{aligned} \|\phi\|_{\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}} &\leq C \|\phi\|_{L_x^{q-1}}, \text{ for } 2 < q \leq 3, \\ \|\phi\|_{L_x^{q-1}} &\leq C \|\phi\|_{\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}} \text{ for } 3 \leq q < \infty \end{aligned}$$

provided that the right hand sides are finite. Therefore $q = 2$ and $q = \infty$ are exceptional points. Local or global existence results in a invariant space for $q = 2$ are not known.

Next result says nonexistence of global in time of small solutions for $1 < q \leq 2$. We assume that u is the solution of (4.3) with $g(t) = \mu |u(t, 0)|^q$ in the space

$$C([0, T]; H_x^s), s > \frac{1}{2}.$$

In [2], it was shown that local in time of solutions of (4.3) exists in $C([0, T]; H_x^s)$, $\frac{1}{2} < s < \frac{3}{2}$. We define the maximal existence time of the solution

$$T^* = \sup_T \left\{ T : \sup_{[0, T)} \|u(t)\|_{L^\infty(\mathbb{R})} < \infty \right\}.$$

In [11], we have

Theorem 4.2. *We assume $1 < q \leq 2$, $\frac{1}{\sqrt{2\pi}} \operatorname{Re}(e^{\frac{\pi}{4}i} \mu) = \beta > 0$,*

$$(4.4) \quad \operatorname{Im} \int_0^t (U(t-\tau)f)(\tau, 0) d\tau \leq 0,$$

and there exists a $\rho > 0$ such that

$$(4.5) \quad \operatorname{Re}(U(t)u_0)(t, 0) \geq \frac{\rho}{\sqrt{1+t}}, t \geq 0.$$

Then $T^ < \infty$. More precisely,*

$$T^* \leq \begin{cases} \left(\frac{2-q}{2(q-1)\beta\rho^{q-1}} + 1 \right)^{\frac{2}{2-q}} - 1 & \text{for } 1 < q < 2, \\ e^{\frac{1}{\beta\rho}} & \text{for } q = 2. \end{cases}$$

Remark 4.3. For any $\varepsilon > 0$, we define the initial function as

$$u_0(x) = \varepsilon e^{-\frac{\sigma x^2}{2i}}, \sigma = 1 + i$$

which satisfies the condition (4.5). If we define $f(t, x)$ as

$$f(t, x) = i e^{-\frac{\sigma x^2}{2i}} h(t), h(t) \geq 0.$$

Then we find (4.4) holds. Indeed by a direct calculation we have

$$\begin{aligned} (U(t) u_0)(t, 0) &= \frac{1}{\sqrt{2it\pi}} \int e^{-\frac{y^2}{2it}} u_0(y) dy = \frac{1}{\sqrt{2it\pi}} \varepsilon \int e^{-\frac{y^2}{2i}(\frac{\sigma t+1}{t})} dy \\ &= \frac{1}{\sqrt{2it\pi}} \varepsilon \sqrt{\frac{2it\pi}{\sigma t+1}} = \frac{\varepsilon}{\sqrt{1+\sigma t}} \end{aligned}$$

and

$$\sqrt{1+\sigma t} = \sqrt{(1+t)+it} = \left((1+t)^2 + t^2\right)^{\frac{1}{4}} e^{i\frac{1}{2}\theta}, \tan \theta = \frac{t}{1+t}.$$

Hence

$$(U(t) u_0)(t, 0) = \frac{\varepsilon}{\left((1+t)^2 + t^2\right)^{\frac{1}{4}}} \left(\cos \frac{1}{2}\theta - i \sin \frac{1}{2}\theta\right), 0 \leq \theta < \frac{\pi}{4}$$

which implies

$$\begin{aligned} &\operatorname{Re}(U(t) u_0)(t, 0) \\ &= \frac{\varepsilon}{\left((1+t)^2 + t^2\right)^{\frac{1}{4}}} \cos \frac{1}{2}\theta \geq \frac{\varepsilon}{2^{\frac{1}{4}}\sqrt{1+t}} \frac{1}{2^{\frac{1}{2}}} \geq \frac{\varepsilon}{2} \frac{1}{\sqrt{1+t}}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} &- \operatorname{Im} \int_0^t U(t-\tau) f(\tau) d\tau \\ &= \operatorname{Re} \int_0^t \frac{1}{\sqrt{1+\sigma(t-\tau)}} h(\tau) d\tau \geq \frac{1}{2} \frac{1}{\sqrt{1+t}} \int_0^t h(\tau) d\tau \geq 0 \end{aligned}$$

from which we have (4.4). If we take ρ such that

$$\frac{\varepsilon}{2} \geq \rho > 0$$

then we have (4.5).

Remark 4.4. The lower bound of an existence time of solutions is not shown. Therefore our result does not say the sharp estimate of the life span of solutions.

In the case of $3 \leq q < \infty$, we can consider the problem (4.3) with

$$f(t, x) = \gamma |u|^{2(q-1)} u(t, x) \text{ or } \gamma |u|^{2q-1}(t, x).$$

In this case $\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}$ is also an invariant space. In [12], we have

Theorem 4.5. *We assume $u_0 \in \dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}$, $3 \leq q < \infty$, $0 < \alpha \leq \frac{q-2}{q-1}$. Then there exists an $\varepsilon > 0$ such that (4.3) with*

$$f(t, x) = \gamma |u|^{2(q-1)} u(t, x) \text{ or } \gamma |u|^{2q-1}(t, x), \gamma \in \mathbb{C}$$

has a unique global solution

$$u \in X_{\alpha, 2(q-1)}$$

for any (u_0, f) satisfying

$$\|u_0\|_{\dot{H}_x^{-(\frac{1}{q-1}-\frac{1}{2})}} \leq \varepsilon.$$

Remark 4.6. *For $2 < q < 3$, (4.3) with*

$$f(t, x) = \gamma |u|^{2(q-1)} u(t, x) \text{ or } \gamma |u|^{2q-1}(t, x),$$

and $g(t, 0)$ is a given function satisfying some regularity and decay conditions was studied by [9]. It seems that the boundary conditions used in [9] do not work for the nonlinear problem.

There are few results on nonlinear Neumann boundary problem

$$(4.6) \quad \begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^{p-1} u, t > 0, x \in \mathbb{R}_+, \\ u(0, x) = u_0(x), x \in \mathbb{R}_+, \\ -\partial_x u(t, 0) = \mu |u|^{q-1} u(t, 0), t > 0. \end{cases}$$

In [2], it was shown that if $u_0 \in H^s(\mathbb{R}_+)$, $\frac{1}{2} < s < \frac{7}{2}$, $s \neq \frac{3}{2}$, $p, q \geq 2$, where $H^s(\mathbb{R}_+)$ is defined as the set of restrictions to $\mathbb{R}_+$ of functions in $H^s(\mathbb{R})$, with the restriction norm

$$\|v\|_{H^s(\mathbb{R}_+)} = \inf \left\{ \|\phi\|_{H^s(\mathbb{R})} : \phi(x)|_{x \in \mathbb{R}_+} = v \right\},$$

then (4.2) has a unique local solution $u \in C([0, T]; H^s(\mathbb{R}_+))$ for some $T > 0$ with the zeroth order compatibility condition $-\partial_x u_0(0) = \mu |u_0(0)|^{q-1} u_0(0)$ for $\frac{3}{2} < s < \frac{7}{2}$. Blow up of solutions for large data was discussed in [13] in the case of $\lambda > 0$ (defocusing), $\mu > 0$ (focusing). In [12], we showed if $u_0 \in L^2(\mathbb{R}_+)$, $1 < p \leq 5$, $1 < q \leq 3$, then (4.2) has a unique local solution $u \in C([0, T]; L^2(\mathbb{R}_+))$ for some $T > 0$. In the case of $p = 5$, $q = 3$, there exists a global small solutions for small $L^2(\mathbb{R}_+)$ data.

DATA AVAILABILITY. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest

The author declares that he has no conflict of interest.

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