

A transformation formula for Maass-type Eisenstein series of two variables

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Abstract

Let $s = (s_1, s_2)$ be complex variables, and $z = (z_1, z_2)$ complex parameters with $(z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, where $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) denotes the upper (resp. lower) half-plane. The main object of this report is the double Eisenstein series (of two variables) $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$ defined by (2.1) below, which includes (as a particular case) the non-holomorphic Eisenstein series $\zeta_{\mathbb{Z}^2}(s; z)$ (defined by (1.1)) attached to $SL(2, \mathbb{Z})$.

We first show a Fourier-type series expansion for a two variable extension of the bilateral Hurwitz zeta-function (Theorem 1), which further allows us to obtain a similar type of series expansion for $\zeta_{\mathbb{Z}^2}(s; z)$ (Theorem 2) by means of Mellin-Barnes type integrals. This eventually leads us to establish complete asymptotic expansions for $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$ in the descending order of $z = z_1 - z_2$ as $z \rightarrow \infty$ through the (upper-half) sector $0 < \arg z < \pi$ (Theorem 3). It can be shown from Theorem 3 certain functional properties of $\zeta_{\mathbb{Z}^2}(s; z)$ (Corollaries 1–2), as well as several closed form evaluations for specific values of $\zeta_{\mathbb{Z}^2}(s; z)$ at some integer lattice arguments (Corollary 3).

1 Introduction

Let $s = \sigma + it$ be a complex variable and let $\mathcal{H}^+ = \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi\}$ and $\mathcal{H}^- = \{z \in \mathbb{C} \mid -\pi < \arg(z) < 0\}$ be the complex half-planes. For an arbitrary even integer k and the complex parameter $z \in \mathcal{H}^+$, the non-holomorphic Eisenstein series $E_k(s; z)$ of weight k attached to $SL(2, \mathbb{Z})$ is defined by the meromorphic continuation of the series

$$E_k(s, z) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ \text{GCD}(c,d)=1}} (cz + d)^{-k} |cz + d|^{-2s} \quad (1.1)$$

to the whole s -plane (see [10, Chap.4, Sect.3]). It is readily seen when $k = 0$ that the relation

$$E_0(s; z) = \zeta_{\mathbb{Z}^2}(s; z)/2\zeta(2s) \quad (1.2)$$

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holds with the Riemann zeta-function $\zeta(s)$, and the Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s; z)$ defined by

$$\zeta_{\mathbb{Z}^2}(s; z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |m + nz|^{-2s} \quad (\text{Res} > 1), \quad (1.3)$$

which can be continued to a meromorphic function over the whole s -plane (cf. [1][2]).

The principal aim of this report is to introduce a class of double Eisenstein series (2.1) below, which includes (as a particular case) the Epstein zeta-function (1.3) and so the non-holomorphic (or real analytic) Eisenstein series (1.1), and further to present a Fourier-type series expansion for the double Eisenstein series (of two variables) $\zeta_{\mathbb{Z}^2}(s; z)$ defined by (2.1) below (Theorem 2). This eventually leads us to obtain complete asymptotic expansions of $\zeta_{\mathbb{Z}^2}(s; z)$ in the descending order of $z = z_1 - z_2$ as $z \rightarrow \infty$ through the (upper-half) sector $0 < \arg z < \pi$ (Theorem 3). Certain functional properties of $\zeta_{\mathbb{Z}^2}(s; z)$ (Corollaries 1–2), as well as several closed form evaluations for specific values of $\zeta_{\mathbb{Z}^2}(s; z)$ at some integer lattice arguments will be presented (Corollary 3).

We give in what follows a brief overview of several results relevant to the present direction of research. As to *holomorphic* Eisenstein series, complete asymptotic expansions (with respect to the parameter z) were obtained by Matsumoto in [11, Corollary 1], while Noda [12] studied an asymptotic formula for the *non-holomorphic* Eisenstein series $E_0(s; z)$ as $t \rightarrow +\infty$ on the critical line $\sigma = 1/2$. Katsurada [4] derived complete asymptotic expansions for (1.3) in the descending order of $y = \text{Im } z$ as $y \rightarrow +\infty$, where key rôles in the proofs were played by Mellin-Barnes type integrals. This result further allows him to yield for $\zeta_{\mathbb{Z}^2}(s; z)$ a new proof of its functional equation and its Kronecker limit formula when $s \rightarrow 1$, as well as its closed form evaluations of certain specific values at integer arguments. The main formula in [4, Theorem 1] is readily switched to an asymptotic expansion of $E_0(s; z)$ as $y \rightarrow +\infty$ by the relation (1.2). It is in fact possible to transfer from $E_0(s; z)$ to $E_k(s; z)$ by using Maass' weight change operators (see [10, Chap.4, (12), (13)]); this leads the authors to establish in [5, Theorem 1] complete asymptotic expansions for (1.1) as $y \rightarrow +\infty$ with any even weight k . The main formula in [5, Theorem 1] yields various consequences similar to those in the case of (1.2).

The classical Lipschitz formula (see (2.4) below) was recently extended in [6][7] into a form of two variables, where the pair of parameters (z_1, z_2) belongs to either $(\mathcal{H}^+)^2$ or $(\mathcal{H}^-)^2$. As an application, they further derived a transformation formula for a class of double Eisenstein series, which can be regarded as a two variable analogue of the Fourier series expansion of the *holomorphic* Eisenstein series attached to $SL(2, \mathbb{Z})$.

The class of double Eisenstein series such as in (2.1) can be regarded as one of those of double *bilateral* Dirichlet series. Double and further multiple Dirichlet series have been the subject of recent extensive research, where its major portion covers multiple *unilateral* Dirichlet series. We mention here several results relevant to (2.1). The functional equations of some Euler-type double Eisenstein series were recently given in [8], while Pasles and Pribitkin [13] have shown a kind of triple and quadruple analogues of the Lipschitz formula, which give generalizations of the Maass-type formula. They also applied their results to study a class of generalized non-analytic automorphic forms.

2 Statement of results

Let $s = (s_1, s_2)$ be complex variables, and $z = (z_1, z_2)$ be complex parameters with $z = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$. Throughout this report, the notation $\langle s \rangle = s_1 + s_2$ will be used. We define the double Eisenstein series of Maass-type by

$$\widetilde{\zeta}_{\mathbb{Z}^2}(s; z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} (m + nz_1)^{-s_1} (m + nz_2)^{-s_2}. \quad (2.1)$$

Here the argument of $(m + nz)$ for $z \in \mathcal{H}^+ \cup \mathcal{H}^-$ is chosen so as $|\arg(m + nz)| \leq \pi$. More precisely, for negative integer n in $(m + nz)$, we let $\arg(n) = -\pi$ for $z \in \mathcal{H}^+$ and $\arg(n) = \pi$ for $z \in \mathcal{H}^-$. Similarly, for negative integer m in $(m + nz)$ when $n = 0$, we define $\arg(m) = -\pi$ for $z \in \mathcal{H}^+$ and $\arg(m) = \pi$ for $z \in \mathcal{H}^-$. The argument of $m \in \mathbb{Z}$ in $(m + w)$ for $w \in \mathcal{H}^+ \cup \mathcal{H}^-$ is defined as follows:

$$\arg(m) = \lim_{\delta \rightarrow +0} \arg(m + \delta w) = \begin{cases} 0 & \text{if } m > 0, \\ \pi & \text{if } m < 0 \text{ and } w \in \mathcal{H}^+, \\ -\pi & \text{if } m < 0 \text{ and } w \in \mathcal{H}^-. \end{cases} \quad (2.2)$$

The complex power is defined by $w^s = \exp\{(\sigma + it)(\log |w| + i \arg(w))\}$. The right-hand side of (2.1) converges absolutely and locally uniformly for $\text{Re}\langle s \rangle > 2$.

As usual, $\Gamma(s)$ and $\zeta(s)$ denote the gamma and the Riemann zeta function respectively. We write $\sigma_s(l) = \sum_{0 < d|l} d^s$ and use the notation $e(z) = e^{2\pi iz}$. Let $U(\alpha, \gamma; Z)$ be the confluent hypergeometric function of the second kind, defined by

$$U(\alpha, \gamma; Z) = \frac{1}{\Gamma(s_1)} \int_0^\infty e^{-Zu} u^{\alpha-1} (1+u)^{\gamma-\alpha-1} du$$

for $\text{Re}(\alpha) > 0$ and $|\arg(Z)| < \pi/2$ (see [3, 6.5.(2)]). Further, we let $\zeta_{\mathbb{Z}}(s; z)$ be the bilateral Hurwitz zeta-function defined by

$$\zeta_{\mathbb{Z}}(s; z) = \sum_{m=-\infty}^{\infty} (m + z)^{-s} \quad (\text{Re } s > 1), \quad (2.3)$$

for $z \in \mathcal{H}^+$ or $z \in \mathcal{H}^-$. Then the Lipschitz formula (cf. [6, Proposition 2])

$$\zeta_{\mathbb{Z}}(s; z) = \begin{cases} \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{l=1}^{\infty} l^{s-1} e(lz) & \text{if } z \in \mathcal{H}^+, \\ \frac{(2\pi i)^s}{\Gamma(s)} \sum_{l=1}^{\infty} l^{s-1} e(-lz) & \text{if } z \in \mathcal{H}^-, \end{cases} \quad (2.4)$$

holds. Here the l -sum on the right side converges absolutely for all complex s , and hence (2.4) provides the holomorphic continuation of $\zeta_{\mathbb{Z}}(s; z)$ to the whole s -plane. This includes the classical Lipschitz formula, which asserts that

$$\sum_{m=-\infty}^{\infty} (z + m)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} \exp(2\pi i n z) \quad (z \in \mathcal{H}^+),$$

for any integer $k \geq 2$ (cf. [9]). We next define the bilateral Hurwitz zeta-function of two variables, $\tilde{\zeta}_{\mathbb{Z}}(s; z)$ for complex variables $s = (s_1, s_2)$ and complex parameters $z = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, by

$$\tilde{\zeta}_{\mathbb{Z}}(s; z) = \sum_{m=-\infty}^{\infty} (m + z_1)^{-s_1} (m + z_2)^{-s_2} \quad (\operatorname{Re}\langle s \rangle > 1). \quad (2.5)$$

In this paper, we first extend the Lipschitz formula (2.4) into a form of two variables.

Theorem 1. *Let $z = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, and define*

$$a_l(s; z) = \begin{cases} \Gamma(s_2) e(lz_1) U(s_2, \langle s \rangle; 2\pi i l(z_2 - z_1)) & \text{if } l > 0, \\ \Gamma(s_1) e(lz_2) U(s_1, \langle s \rangle; 2\pi i |l|(z_2 - z_1)) & \text{if } l < 0. \end{cases} \quad (2.6)$$

Then the formula

$$\begin{aligned} \tilde{\zeta}_{\mathbb{Z}}(s; z) &= 2\pi i^{2s_2-1} \frac{\Gamma(\langle s \rangle - 1)}{\Gamma(s_1)\Gamma(s_2)} (z_1 - z_2)^{1-\langle s \rangle} \\ &\quad + \frac{(2\pi)^{\langle s \rangle} i^{s_2-s_1}}{\Gamma(s_1)\Gamma(s_2)} \sum_{l \neq 0} |l|^{\langle s \rangle-1} a_l(s; z) \end{aligned} \quad (2.7)$$

holds for $\operatorname{Re}\langle s \rangle > 1$. Here the l -sum on the right side converges absolutely for all $s \in \mathbb{C}^2$, and hence (2.7) provides the meromorphic continuation of $\tilde{\zeta}_{\mathbb{Z}}(s; z)$ to the whole s -space $\mathbb{C}^2$.

Theorem 1 yields the following transformation formula (Fourier-type expansion) for $\tilde{\zeta}_{\mathbb{Z}^2}(s; z)$.

Theorem 2. *Let $z = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$, define*

$$\begin{aligned} \mathcal{E}_0(s; z) &= \{1 + e^{\pi i(s_1-s_2)}\} \left\{ \zeta(\langle s \rangle) + 2\pi i^{2s_2-1} \frac{\Gamma(\langle s \rangle - 1)}{\Gamma(s_1)\Gamma(s_2)} \right. \\ &\quad \left. \times (z_1 - z_2)^{1-\langle s \rangle} \zeta(\langle s \rangle - 1) \right\}, \end{aligned} \quad (2.8)$$

and let $a_l(s; z)$ be as in Theorem 1. Then the formula

$$\tilde{\zeta}_{\mathbb{Z}^2}(s; z) = \mathcal{E}_0(s; z) + \frac{(2\pi)^{\langle s \rangle} i^{s_2-s_1}}{\Gamma(s_1)\Gamma(s_2)} \{1 + e^{\pi i(s_1-s_2)}\} \sum_{l \neq 0} \sigma_{\langle s \rangle-1}(l) a_l(s; z) \quad (2.9)$$

holds. Here the l -sum on the right side converges for all $s \in \mathbb{C}^2$, and hence (2.9) provides the meromorphic continuation of $\tilde{\zeta}_{\mathbb{Z}^2}(s; z)$ to the whole s -space $\mathbb{C}^2$.

For $s_j \in \mathbb{C}$, we write $s_j = \sigma_j + it_j$ ($j = 1, 2$). Further let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n be the shifted factorial of s , and

$$\Phi_{r,s}(e(w)) = \sum_{h,k=1}^{\infty} h^r k^s e(hkw) = \sum_{l=1}^{\infty} \sigma_{r-s}(l) l^s e(lw), \quad (2.10)$$

the function first introduced by Ramanujan [14]; this converges absolutely for all $(r, s) \in \mathbb{C}^2$ when $w \in \mathcal{H}^+$, and defines there an entire function.

Theorem 3. *Let $z = (z_1, z_2) \in \mathcal{H}^+ \times \mathcal{H}^-$ and $z_1 - z_2 = z \in \mathcal{H}^+$. Then for any integer $N_1 \geq 1$ and $N_2 \geq 1$, the asymptotic expansion*

$$\begin{aligned} \widetilde{\zeta_{Z^2}}(s; z) = & \mathcal{E}_0(s; z) + \frac{2(2\pi)^{\langle s \rangle}}{\Gamma(s_1)} \cos\{\pi(s_1 - s_2)/2\} \{S_{1,N_1}(s; z) + R_{1,N_1}(s; z)\} \\ & + \frac{2(2\pi)^{\langle s \rangle}}{\Gamma(s_2)} \cos\{\pi(s_1 - s_2)/2\} \{S_{2,N_2}(s; z) + R_{2,N_2}(s; z)\}, \end{aligned} \quad (2.11)$$

holds in the region of s with $-N_2 < \sigma_1 < 1 + N_1$ and $-N_1 < \sigma_2 < 1 + N_2$. Here $\mathcal{E}_0(s, z)$ is defined as in Theorem 2,

$$\begin{aligned} S_{1,N_1}(s; z) \\ = \sum_{n=0}^{N_1-1} \frac{(-1)^n (s_2)_n (1-s_1)_n}{n!} \Phi_{s_1-n-1, -s_2-n}(e(z_1)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-s_2-n}, \end{aligned} \quad (2.12)$$

$$\begin{aligned} S_{2,N_2}(s; z) \\ = \sum_{n=0}^{N_2-1} \frac{(-1)^n (s_1)_n (1-s_2)_n}{n!} \Phi_{s_2-n-1, -s_1-n}(e(-z_2)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-s_1-n}, \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} R_{1,N_1}(s; z) = & \frac{(-1)^{N_1} (s_2)_{N_1} (1-s_1)_{N_1}}{(N_1-1)!} \sum_{h,k=1}^{\infty} e(hkz_1) h^{\langle s \rangle-1} \\ & \times \int_0^1 \xi^{-s_2-N_1} (1-\xi)^{N_1-1} U(s_2 + N_1, \langle s \rangle; 2\pi hke^{-\pi i/2} z/\xi) d\xi, \end{aligned} \quad (2.14)$$

$$\begin{aligned} R_{2,N_2}(s; z) = & \frac{(-1)^{N_2} (s_1)_{N_2} (1-s_2)_{N_2}}{(N_2-1)!} \sum_{h,k=1}^{\infty} e(-hkz_2) h^{\langle s \rangle-1} \\ & \times \int_0^1 \xi^{-s_1-N_2} (1-\xi)^{N_2-1} U(s_1 + N_2, \langle s \rangle; 2\pi hke^{-\pi i/2} z/\xi) d\xi. \end{aligned} \quad (2.15)$$

Let $\theta = \arg(-iz)$, then the expansion above gives the asymptotic series in the descending order of z , and R_{j,N_j} ($j = 1, 2$) is the remainder terms satisfying the estimates

$$R_{1,N_1}(s; z) = O\left(e^{-2\pi \text{Im}(z_1)} |z|^{-\sigma_2-N_1}\right), \quad (2.16)$$

$$R_{2,N_2}(s; z) = O\left(e^{2\pi \text{Im}(z_2)} |z|^{-\sigma_1-N_2}\right), \quad (2.17)$$

as $z \rightarrow \infty$ through the sector $\delta \leq \arg z \leq \pi - \delta$ with any small $\delta > 0$. Here the O -constants depend on N_j, t_j ($j = 1, 2$) and θ .

From Theorems 2 and 3, we can determine the singularities of $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$, and to derive the functional equations under some natural conditions. In order to describe our results, we put for any $m \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{M}_m &= \{s = (s_1, s_2) \in \mathbb{C}^2; s_1 - s_2 = m\}, \\ \mathcal{N}_m &= \{s = (s_1, s_2) \in \mathbb{C}^2; \langle s \rangle = s_1 + s_2 = m\}, \\ \mathcal{P} &= \{s = (s_1, s_2) \in \mathbb{C}^2; \langle s \rangle \in \{2, 1, 0, -2, -4, \dots\}\}, \\ \mathcal{Q} &= \{s = (s_1, s_2) \in \mathbb{C}^2; s_1 \in \{0, -1, -2, \dots\} \text{ or } s_2 \in \{0, -1, -2, \dots\}\}.\end{aligned}$$

Generally, it is possible to explain that the singularities (and a part of zeros) and the functional equation come from $\mathcal{E}_0(s; z)$, which is called “the constant term” of the Eisenstein series. In our case, this fact is stated as follows. :

Corollary 1. (i) For any $l \in \mathbb{Z}$ and any $s^* = (s_1^*, s_2^*) \in \mathcal{M}_{2l+1}$ with $s^* \notin \mathcal{P}$,

$$\widetilde{\zeta}_{\mathbb{Z}^2}(s^*; z) = 0,$$

namely, $\mathcal{M}_{2l+1} \setminus \mathcal{P}$ is the set of the zeros of $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$.

(ii) $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$ has singularities on $s \in \mathcal{P}$. In particular, $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$ has indeterminacy singularities on $s \in \mathcal{P} \cap \mathcal{Q}$ or on $s \in \mathcal{P} \cap \mathcal{M}_{2l+1}$ for any $l \in \mathbb{Z}$.

The functional equation of $\widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$ come into existence under the condition $s \in \mathcal{M}_{2k}$ ($k \in \mathbb{Z}$), which seems to be a natural generalization of the functional equation of the non-holomorphic Eisenstein series $E_k(s, z)$ attached to $SL(2, \mathbb{Z})$ (see Corollary 5 below). It is also possible to obtain a functional equation under a different kind of the condition that $s \in \mathcal{N}_{2l+1}$ ($l \in \mathbb{Z}$) upon subtracting the constant term $\mathcal{E}_0(s; z)$ of the double Eisenstein series. In the following, we use the notation $\widehat{1 - s} = (1 - s_2, 1 - s_1)$ for $s = (s_1, s_2)$.

Corollary 2. (i) The functional equation

$$\widetilde{\zeta}_{\mathbb{Z}^2}(\widehat{1 - s}; z) = \left(\frac{2\pi i}{z}\right)^{1 - \langle s \rangle} \frac{\Gamma(s_1)}{\Gamma(1 - s_2)} \widetilde{\zeta}_{\mathbb{Z}^2}(s; z)$$

holds on the hyper-plane $\mathcal{M}_{2k}$ ($k \in \mathbb{Z}$) except on the poles.

(ii) The functional equation

$$\widetilde{\zeta}_{\mathbb{Z}^2}(\widehat{1 - s}; z) - \mathcal{E}_0(\widehat{1 - s}; z) = \left(\frac{2\pi i}{z}\right)^{1 - \langle s \rangle} \frac{\Gamma(s_1)}{\Gamma(1 - s_2)} \left\{ \widetilde{\zeta}_{\mathbb{Z}^2}(s; z) - \mathcal{E}_0(s; z) \right\}$$

holds on the hyper-planes $\mathcal{M}_{2k}$ ($k \in \mathbb{Z}$) or $\mathcal{N}_{2l+1}$ ($l \in \mathbb{Z}$) except on the poles.

From Theorem 3, we obtain the following closed form expressions for specific values of $\widetilde{\zeta}_{\mathbb{Z}^2}(s, z)$ at positive integer lattice arguments:

Corollary 3. For any $\mathbf{m} = (m_1, m_2) \in \mathbb{N}^2$ ($\mathbb{N} = \{1, 2, \dots\}$) with $\langle \mathbf{m} \rangle > 2$,

$$\widetilde{\zeta_{\mathbf{Z}^2}}(\mathbf{m}; z) = \mathcal{E}_0(\mathbf{m}; z) + 2(2\pi)^{\langle \mathbf{m} \rangle} \cos\{\pi(m_1 - m_2)/2\} \left\{ \frac{T_1(\mathbf{m}; z)}{(m_1 - 1)!} + \frac{T_2(\mathbf{m}; z)}{(m_2 - 1)!} \right\}. \quad (2.18)$$

Here

$$\begin{aligned} \mathcal{E}_0(\mathbf{m}; z) = & -\frac{(2\pi)^{\langle \mathbf{m} \rangle} i^{m_1 - m_2} \cos\{\pi(m_1 - m_2)/2\} B_{\langle \mathbf{m} \rangle}}{\cos(\pi \langle \mathbf{m} \rangle / 2) \langle \mathbf{m} \rangle!} \\ & + \frac{4\pi i^{\langle \mathbf{m} \rangle - 1} (\langle \mathbf{m} \rangle - 2)! \cos\{\pi(m_1 - m_2)/2\} \zeta(\langle \mathbf{m} \rangle - 1)}{(m_1 - 1)!(m_2 - 1)!(z_1 - z_2)^{\langle \mathbf{m} \rangle - 1}}, \end{aligned} \quad (2.19)$$

where B_m is the m -th Bernoulli number and

$$T_1(\mathbf{m}; z) = \sum_{n=0}^{m_1-1} \binom{m_1-1}{n} (m_2)_n \Phi_{m_1-n-1, -m_2-n}(e(z_1)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-m_2-n}, \quad (2.20)$$

$$T_2(\mathbf{m}; z) = \sum_{n=0}^{m_2-1} \binom{m_2-1}{n} (m_1)_n \Phi_{m_2-n-1, -m_1-n}(e(-z_2)) \left(2\pi e^{-\frac{1}{2}\pi i} z\right)^{-m_1-n}. \quad (2.21)$$

From Theorems 1 and 2, the following two corollaries are readily derived.

Corollary 4 (Maass [10]). Let $z = x + iy \in \mathcal{H}^+$, and define

$$\begin{aligned} a_n(y; s_1, s_2) = & i^{s_2 - s_1} (2\pi)^{\langle \mathbf{s} \rangle} \\ & \times \begin{cases} n^{\langle \mathbf{s} \rangle - 1} \Gamma(s_1)^{-1} U(s_2, \langle \mathbf{s} \rangle; 4\pi n y) & \text{if } n > 0, \\ |n|^{\langle \mathbf{s} \rangle - 1} \Gamma(s_2)^{-1} U(s_1, \langle \mathbf{s} \rangle; 4\pi |n| y) & \text{if } n < 0, \\ \Gamma(s_1)^{-1} \Gamma(s_2)^{-1} \Gamma(\langle \mathbf{s} \rangle - 1) (4\pi y)^{1 - \langle \mathbf{s} \rangle} & \text{if } n = 0. \end{cases} \end{aligned}$$

Then the formula

$$\sum_{m=-\infty}^{\infty} (z + m)^{-s_1} (\bar{z} + m)^{-s_2} = \sum_{n=-\infty}^{\infty} a_n(y; s_1, s_2) e(nx + i|n|y)$$

holds, where the n -sum (with $n \neq 0$) on the right side converges for all $(s_1, s_2) \in \mathbb{C}^2$, and this formula provides the meromorphic continuation of the left side to the whole (s_1, s_2) -space $\mathbb{C}^2$.

Corollary 5 (Maass [10]). Under the same notation as in Corollary 4, we have

$$\begin{aligned} E_k(s, z) = & 1 + \frac{\zeta(k + 2s - 1)}{\zeta(k + 2s)} a_0(y; k + s, s) \\ & + \frac{1}{\zeta(k + 2s)} \sum_{l \neq 0} \sigma_{1-k-2s}(l) a_l(y; k + s, s) e(lx + i|l|y), \end{aligned}$$

and the functional equation

$$\begin{aligned} \pi^{-s} \Gamma(s) \zeta(2s) y^s E_k(s, z) \\ = \pi^{-1+s+k} \Gamma(1-s-k) \zeta(2-2s-2k) y^{1-s-k} E_k(1-s-k, z). \end{aligned}$$

Remark. The double Lipschitz formula and the Fourier series expansion of non-holomorphic Eisenstein series were shown by Maass [10], in which he described his results in terms of Whittaker functions. Equivalent statements described in terms of confluent hypergeometric functions can be found, for e.g., in [15, p. 132].

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