

LOCAL SOLVABILITY OF SECOND ORDER FUCHSIAN TYPE
EQUATIONS (Collaboration with C.PARENTI)

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§1. DISCUSSION IN A EXAMPLE.

Let $(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$ and let us consider

$$P = t\partial_t^2 - \Delta_x + a(t, x)\partial_t + \sum_{i=1}^n b_i(t, x)\partial_i + c(t, x) \quad (1.1)$$

near the origin, where Δ_x is the Laplacian, $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$ ($1 \leq i \leq n$). Note that P is hyperbolic on $\{t > 0\}$ and elliptic on $\{t < 0\}$, that is, P is so-called a mixed type operator.

To illustrate our argument, let us treat here the above operator P and show the local solvability of $Pu = f$ in hyperfunctions & distributions.

1-1. Solvability in hyperfunctions. Let P be the operator in (1.1) and assume that $a(t, x)$, $b_i(t, x)$ and $c(t, x)$ are analytic functions near the origin. Let $\mathcal{B}$ be the stalk of sheaf of hyperfunctions at the origin. Then, we have:

Theorem 1. If $1 - a(0, x) \notin \{1, 2, \dots\}$, the map $P : \mathcal{B} \rightarrow \mathcal{B}$ is surjective.

Sketch of proof. To prove this, it is sufficient to show the following:

(i) $P : \mathcal{A} \rightarrow \mathcal{A}$ is surjective,

(ii) $P : \mathcal{B}/\mathcal{A} \rightarrow \mathcal{B}/\mathcal{A}$ is surjective.

(i) is clear from the Cauchy-Kowalewski type theorem for Fuchsian type equations. (ii) follows from the following two facts:

(ii-1) when $p(\pm) = (0, 0; \tau = \pm 1, \xi = 0)$, $P : \mathcal{C}_{p(\pm)} \rightarrow \mathcal{C}_{p(\pm)}$ is surjective,

(ii-2) when $q = (0, 0; \tau, \xi \neq 0)$, $P : \mathcal{C}_q \rightarrow \mathcal{C}_q$ is bijective, where $\mathcal{C}_p$ denotes the stalk of sheaf of microfunctions at p .

Q.E.D.

1-2. Solvability in distributions. Let P be the operator in (1.1) and assume that $a(t, x)$, $b_1(t, x)$ and $c(t, x)$ are C^∞ functions near the origin. Let $\mathcal{D}'$ be the stalk of sheaf of distributions at the origin. Then, we have:

Theorem 2. If $1 - a(0, x) \notin \mathbb{Z}$, the map $P : \mathcal{D}' \rightarrow \mathcal{D}'$ is surjective.

Sketch of proof. Put

$$\mathcal{D}'|_{t>0} = \{ u|_{t>0} ; u \in \mathcal{D}' \},$$

$$\mathcal{D}'|_{t<0} = \{ u|_{t<0} ; u \in \mathcal{D}' \}.$$

Then, Theorem 2 follows from the following three facts:

(i) $P : \mathcal{D}'|_{t>0} \rightarrow \mathcal{D}'|_{t>0}$ is surjective,

(ii) $P : \mathcal{D}'|_{t<0} \rightarrow \mathcal{D}'|_{t<0}$ is surjective,

(iii) for any $\varphi(x) \in \mathcal{D}'$, there exists a unique solution

$u(t, x) \in C^\infty([0, T], \mathcal{D}')$ such that

$$Pu = 0 \text{ on } t > 0, \quad u|_{t=0} = \varphi(x).$$

In fact, we can get Theorem 2 as follows. Let $f \in \mathcal{D}'$. Then, by (i) and (ii) we can choose $v \in \mathcal{D}'$ such that $Pv = f + \delta(t) \otimes \psi(x)$. By (iii) we solve

$$\begin{cases} Pw = 0 & \text{on } t > 0, \\ w|_{t=0} = (a(0, x) - 1)^{-1} \psi(x). \end{cases}$$

Then, we have $P(Y(t)w) = \delta(t) \otimes \psi(x)$. Thus, by putting $u = v - Y(t)w$ we get a solution $u \in \mathcal{D}'$ of $Pu = f$. Q.E.D.

Remark. By the same argument as in the proof of Theorem 1, we can easily get that $P : \mathcal{D}'/C^\infty \rightarrow \mathcal{D}'/C^\infty$ is surjective. But, unfortunately, we do not know whether $P : C^\infty \rightarrow C^\infty$ is surjective or not. This is the reason why we proved Theorem 2 in a different way from the proof of Theorem 1.

§2. FURTHER RESULTS.

Let us treat here somewhat more general Fuchsian type operators. Let P_1 , P_2 and P_3 be of the form

$$\begin{aligned} P_1 &= t \partial_t^2 - t^k A(t, x, \partial_x) + a(t, x) \partial_t \\ &\quad + t^h \sum_{i=1}^n b_i(t, x) \partial_i + c(t, x), \\ P_2 &= t^2 \partial_t^2 - t^p A(t, x, \partial_x) + a(t, x) t \partial_t \\ &\quad + t^q \sum_{i=1}^n b_i(t, x) \partial_i + c(t, x), \\ P_3 &= t^2 \partial_t^2 + t^p A(t, x, \partial_x) + a(t, x) t \partial_t \\ &\quad + t^q \sum_{i=1}^n b_i(t, x) \partial_i + c(t, x), \end{aligned}$$

where $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ ($1 \leq i \leq n$), $k, h, p, q \in \mathbb{Z}_+ (= \{0, 1, 2, \dots\})$,

$$A(t, x, \partial_x) = \sum_{i,j=1}^n a_{ij}(t, x) \partial_i \partial_j$$

is a real elliptic differential operator such that

$$A(t, x, \xi/|\xi|) > 0 \quad \text{for } \forall(t, x), \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

and $a_{ij}(t,x)$, $a(t,x)$, $b_i(t,x)$ and $c(t,x)$ are C^∞ functions near the origin. Let $\rho(x)=1-a(0,x)$ and let $\rho_1(x), \rho_2(x)$ be the roots of $\rho(\rho-1)+a(0,x)\rho+c(0,x)=0$. Note that $\rho(x)$ is the non-trivial characteristic exponent of P_1 and that $\rho_1(x), \rho_2(x)$ are the characteristic exponents of P_2 and P_3 . Then, we have:

Theorem 3. (1) In case P_1 , if $k \geq 0$, $h \geq (k-1)/2$ and $\rho(0) \notin \mathbb{Z}$ hold, the map $P_1 : \mathcal{D}' \rightarrow \mathcal{D}'$ is surjective.

(2) In case P_2 , if $p \geq 1$, $q \geq p/2$ and $\rho_1(0), \rho_2(0) \notin \{-1, -2, \dots\}$, the map $P_2 : \mathcal{D}' \rightarrow \mathcal{D}'$ is surjective.

(3) In case P_3 , if $p \geq 1$, $q \geq p/2$ and $\rho_1(0), \rho_2(0) \notin \{-1, -2, \dots\}$, the map $P_3 : \mathcal{D}' \rightarrow \mathcal{D}'$ is surjective.

The proof of this result is quite similar to that of Theorem 2.

REFERENCES

- [1] A.Bove, J.E.Lewis, C.Parenti and H.Tahara : Cauchy problem for Fuchsian hyperbolic operators, II. J. Fac. Sci. Univ. Tokyo, 34-1 (1987).
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要 約

本稿では、2階のフックス型線形偏微分方程式の hyper-function ($\mathcal{B}$) 或いは distribution ($\mathcal{D}'$) での局所可解性が論じられている。

① $(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$ (原点の近傍) とし、例えば

$$P = t\partial_t^2 - \Delta_x + a(t, x)\partial_t + \sum_{i=1}^n b_i(t, x)\partial_{x_i} + c(t, x)$$

という作用素を考えてみる。 P は $\{t > 0\}$ では双曲型であり $\{t < 0\}$ では楕円型となっていて、いわゆる混合型作用素と呼ばれているものになる。 次が成り立つ。

(i) P の係数が解析的 とする。 この時、原点の近傍で

$$1 - a(0, x) \notin \{1, 2, \dots\} \Rightarrow P: \mathcal{B} \rightarrow \mathcal{B} \text{ は全射。}$$

(ii) P の係数が C^∞ クラス とする。 この時、原点の近傍で

$$1 - a(0, x) \notin \{0, \pm 1, \pm 2, \dots\} \Rightarrow P: \mathcal{D}' \rightarrow \mathcal{D}' \text{ は全射。}$$

② (ii) の証明は L^2 -評価をベースとした議論による。

同様の議論によって 次の様な作用素の $\mathcal{D}'$ での局所可解性を扱かうことも出来る。

$$P_1 = t\partial_t^2 - t^k A(t, x, \partial_x) + a(t, x)\partial_t + t^h \sum_{i=1}^n b_i(t, x)\partial_{x_i} + c(t, x),$$

$$P_2 = t^2\partial_t^2 - t^p A(t, x, \partial_x) + a(t, x)t\partial_t + t^q \sum_{i=1}^n b_i(t, x)\partial_{x_i} + c(t, x),$$

$$P_3 = t^2\partial_t^2 + t^p A(t, x, \partial_x) + a(t, x)t\partial_t + t^q \sum_{i=1}^n b_i(t, x)\partial_{x_i} + c(t, x).$$

但し、 $A(t, x, \partial_x)$ は 2 階の実係数楕円型作用素、 $k, h, p, q \in \mathbb{Z}_+ (= \{0, 1, 2, \dots\})$ で $h \geq (k-1)/2$, $p \geq 1$, $q \geq p/2$ とする。