A General Framework for Dynamic Epistemic Logic

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to my family

who supported my academic life

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Statement of Originality

I declare that I have written the present thesis solely by myself. The results in Chapters 3 and 4 are based on my published paper [10], which they expand to encompass further technical results.

Abstract

We propose a general framework for dynamic epistemic logics (DELs). This consists of a generic language for DELs and a class of structures, called model transition systems (MTSs), that describe model transformations in a static way. An MTS can be viewed as a two-layered Kripke model and consequently inherits various standard concepts, such as bisimulation and bounded morphism, from ordinary Kripke models. In the second half of the thesis we add the global operator to the language, which enables us to define notions of a canonical MTS and canonicity of a DEL formula for a property of MTSs. Using these notions, we clarify correspondences between axioms of DELs and properties of MTSs.

Contents

Chapter 1

Introduction

1.1 What is Dynamic Epistemic Logic?

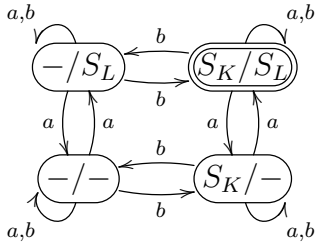
Dynamic Epistemic Logic (*DEL*) is a branch of modal logic for reasoning about knowledge changes or belief revisions caused by communication.

History of DEL The pioneering study on DEL is *Public Announcement Logic* (*PAL*) [11], which deals with truthful public announcements (see Section 2.4.1). This logic is simple and thus still used as a litmus test in many situations. Subsequently, *Epistemic Action* (*EA*) [3] gave a richer framework which expresses various kinds of actions of communication (Section 2.4.2). Ever since a greater variety of communication turned out to be expressible, the research field has become much more active. Up to the present day, many DELs for various kinds of actions have been proposed and studied [14, 17, 18, 19, 20, 23].

Epistemic logic A DEL consists of two components: a static base, called *epistemic logic*, and dynamic machinery. We offer here a scenario as an example, to illustrate epistemic logic:

> *Alice (a) and Bob (b) are in Kyoto (K) and London (L), respectively. Today, it is sunny in both cities.*

The situation of knowledge of Alice and Bob in the above scenario is represented as follows. Let S_K and S_L indicate that it is sunny in Kyoto and London, respectively. Then, there are four possibilities, which we depict as oblong circles in the following *pointed Kripke model* (where '$-$' indicates the negation of S_K or S_L):

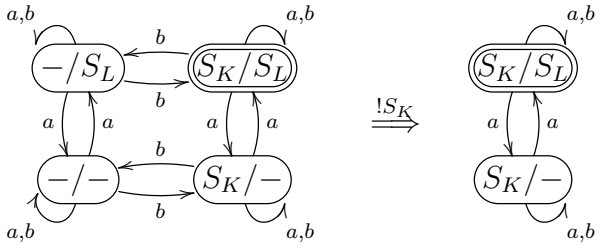


Each circle is called a *possible world*; and the world $(-/S_L)$, for example, is a possible world where it is sunny in London but not in Kyoto. The double circle means the *actual* world where it is sunny in both cities. Each arrow expresses that the target world is considered possible from the viewpoint of the source world. Let us consider, as an example, the two b-labelled arrows from the actual world to the actual world itself and $(-/S_L)$. These arrows indicate that Bob (in London), in the actual world, thinks that the two worlds (S_K/S_L) and $(-/S_L)$ are possible. In addition, in both of these two possible worlds, it is sunny in London; therefore, we may say that he knows that it is sunny in London, which we write as $[b]S_L$ in the formal language for epistemic logic. Similarly, in each of the target worlds of the a-labelled arrows from the actual world, it is sunny in Kyoto. Hence, we may also say that Alice knows that it is sunny in Kyoto, that is $[a]S_K$.

Addition of dynamism In DELs, every time an action of communication takes place, the model is transformed into another, according to the change of the epistemic situation. This is called a *model transformation*. Let us consider the following action, which happens in the setting of the above scenario:

Now, Alice calls Bob and tells him that it is sunny in Kyoto.

Note that we may consider this action as a public announcement to all the participants involved, which in the current situation just consists of Alice and Bob. Hence, we write this action as $!S_K$ by using the language of PAL. By $!S_K$, all worlds that are incompatible with the content S_K are eliminated; thus, the model above is transformed as follows:



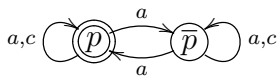
(1) before (2) after

Observe that the b-arrow from the actual world to $(-/S_L)$ is eliminated. Hence, $[b]S_K$ holds in the new actual world in (2), indicating that Bob knows that it is sunny in Kyoto. From the viewpoint of the original actual world in (1), what happens is that, after the phone call telling him that it is sunny in Kyoto, Bob knows this fact, which we write in PAL language as $[!S_K][b]S_K$.

Epistemic Action Then, how does Epistemic Action express various kinds of actions? To illustrate this, we consider another scenario involving a more complicated action:

> *An author (a) submitted a paper to a conference and he is waiting for the notification of acceptance. The committee (c) of the conference has decided to accept it (p). Now the committee is about to send an email of notification to the author.*

There are two possibilities before the sending of the email: either the paper is accepted (p), or it is not ($\overline{p}$). The committee knows which one of them is actually the case, whilst the author does not. This situation of knowledge is expressed by a pointed Kripke model below (this time, $\overline{p}$ indicates the negation of p):

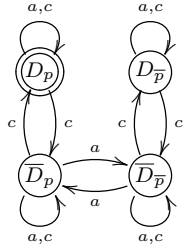


(i) before

We assume here that the electronic network via which the email is delivered is not necessarily robust. Hence, we may consider the following four possibilities:

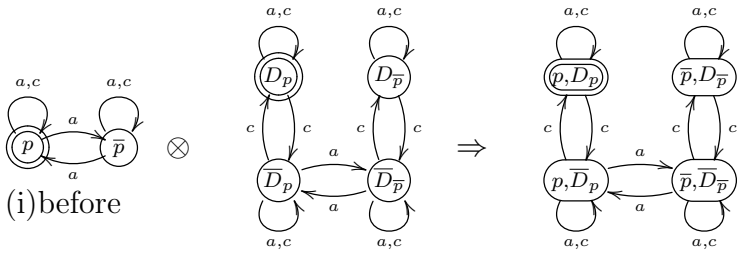
1. The email imparting p is sent and delivered (D_p);
2. The email imparting $\overline{p}$ is sent and delivered ($D_{\overline{p}}$);
3. The email imparting p is sent but not delivered ($\overline{D}_p$);
4. The email imparting $\overline{p}$ is sent but not delivered ($\overline{D}_{\overline{p}}$).

These actions are illustrated by the *pointed action model* below:



(ii) action

We also assume here that the action actually executed is D_p, which is indicated by the double circle. The arrows in the model (iii) express uncertainty. For example, the c-arrow from D_p to $\overline{D}_p$ expresses that, when D_p happens, the committee cannot exclude the possibility that $\overline{D}_p$ might happen. Meanwhile, the a-arrow from $\overline{D}_p$ to $\overline{D}_{\overline{p}}$ means that, if $\overline{D}_p$ happens, the author thinks that $\overline{D}_{\overline{p}}$ might happen. Applying this pointed action model to each possible world in the model (i) properly, we obtain the following model transformation:



The resulting model (iii) expresses the knowledge situation after the action D_p was executed. Observe that there is no a-arrow from the actual world in (iii) to any world where $\overline{p}$ holds. This means that the author knows p ($[a]p$). On the other hand, the committee does not know this fact ($\neg[c][a]p$). This is because the authors does not know p in the world at the lower left, which is the target world of a c-arrow from the actual world. Therefore, in the actual world of the model (i), we may say that, after the action (ii), the author knows p whilst the committee does not know this fact. EA expresses actions in this manner.

Other examples Below are some other examples of DELs:

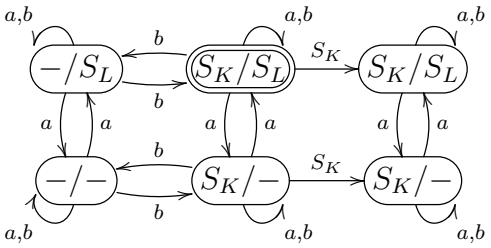
- Belief Revision [14]. This logic treats belief and its revision.
- *Temporal Public Announcement Logic* (*TPAL*) [16] and *Temporal Epistemic Action* (*TEA*) [7]. In reality, sometimes we cannot make an announcement, even if the content is true; for instance, we cannot say 'the professor is stupid' in a public situation. TPAL deals with such situations for a restricted set of public announcements. Its semantics is not dynamic but uses a static representation of model transformations, which is related to our contributions. Similarly, EA has been extended to TEA as well.[1]
- *Preference Upgrade* [17]. This treats preferences, which are upgraded by suggestions. A typical sentence the logic deals with is, 'after a suggestion of φ, she prefers ψ'.

[1]The original name is *Temporal Dynamic Epistemic Logic* (*TDEL*).

- Evidence-based Beliefs [18]. If one accepts evidence for φ, he or she should accept ψ, when the evidence of φ implies ψ. This logic treats such situations.
- Ontic Logic [19]. This is an extension of EA which treats not only actions of communication but also ones that change facts, such as flipping a coin.
- Logic of Lying [20]. A public announcement should be truthful in PAL; however, announcements are not always true. Even worse, it may be possible that we cannot determine whether they are true or not. This logic considers such situations, by introducing actions of false announcement, i.e. lying, and combining a true announcement and its false counterpart in a non-deterministic way.
- Command Logic [23]. This expresses and models sentences such as 'after her boss requesting φ, she has to see to it that φ is true'.

1.2 Our Contributions

1. Static representation of model transformations Notice that, in the above examples, changes of Kripke models are explained by *dynamic* transformations of models. Another way of explanation is to consider a class of pointed Kripke models, which are linked by *update relations* expressing model transformations. Such a structure is proposed with the name of *update universe* by van Benthem in [15]. Below is an intuitive image of a part of an update universe, which combines the two models (1) and (2) in the example of PAL. The S_K-labelled relation is (a segment of) an update relation.



In addition to van Benthem's update universe, several studies on static representation of model transformations have been proposed in the literature [7, 16, 21, 22]. In this thesis, we propose a new formalism, called *model transition systems* (*MTSs*) by modifying update universes; other static representations in [7, 16, 21, 22] are also expressible by MTSs. Our new models have the advantages explained below.

2. Unifying framework for DELs By virtue of the modification, an MTS obtains a higher degree of abstractness, since an MTS expresses model transformations by labelled relations, rather than procedures of changing pointed Kripke models. We take this advantage to introduce a general and abstract framework for DELs which consists of a generic language for DELs on the syntactic side and MTSs on the semantic side. This framework encompasses various DELs, including PAL, EA, TPAL, TEA and so forth (see Sections 3.2.2, 3.2.3 and 3.5).

3. Structure of a two-layered Kripke model In addition, an MTS has the structure of a *two-layered Kripke model*, as we shall see in Section 3.2.1. Therefore, MTSs readily inherit many notions from ordinary Kripke models, e.g. a *bisimulation* and a *bounded morphism* (see Section 3.3). For such notions we establish some basic properties, such as the *Hennessy-Milner property*, stating that model-theoretic equivalence and syntactic equivalence coincide under a specific condition.

4. Systematic study of modal correspondence in DEL Our framework, with an addition of the auxiliary operator called the *global operator*, naturally leads to the notions of a *canonical MTS* and *canonicity of a DEL formula for a property of MTSs*. Using these concepts, we clarify which formula expresses which property of actions. For instance, an axiom, called $\mathsf{D_t}$, expresses determinism; another example is axiom P, which denotes the property of changing only epistemic situations and preserving the truth value of the atomic facts. The results are summarised in Section 4.3.4.

5. Completeness theorems via MTSs Wang and Cao [22] and Wang and Aucher [21] gave alternative proofs to the completeness theorems for PAL and EA by defining intermediary semantics using static representations of model transformations. Their methods can be generalised in our framework; that is, we first express given logic and model transformation in our framework, and then prove completeness by using our technical tools, including the new bounded morphisms and the correspondence results above. This provides a more conspicuous, general and flexible method for proving completeness. By using our technique, we generalise two known results in [7, 16]. That is to say, we give dynamic semantics to TPAL and TEA for the *full* sets of public announcements and pointed action models, and prove their completeness (Section 3.5).

6. New dynamic epistemic logics MTSs enable us to define specific model transformations in an abstract way. This expands the range of model transformations. We take this advantage to introduce two logics of independent interest: a sublogic *LE* of PAL, called the logic of *eliminative update*, and a sublogic *LSP* of EA, called the logic of *subproduct updates*. Their completeness is also proved by using the method above (Sections 4.3.2 and 4.3.3).

1.3 Organisation

Chapter 2: Preliminaries We recall several notions and results in ordinary modal logic to which we refer in subsequent chapters. We also review the mathematical treatment of PAL and EA.

Chapter 3: General Framework for DELs We define a generic language and MTSs, and at the same time, we see that PAL and EA can be expressed in this framework. We then introduce the notions of a bisimulation and a bounded morphism for MTSs and prove basic properties associated with them. At the end of this chapter, we show that MTSs can express other static representations. In passing, we give fully-extended dynamic semantics to TPAL and TEA, for which completeness is proved by using MTSs.

Chapter 4: The Global Operator and Canonicity In order to define the notions of a canonical MTS and canonicity of a DEL formula, we add the global operator to the generic language. We then clarify and formulate correspondences between axioms of DELs and properties of MTSs. We also introduce LE and LSP and prove their completeness. Conservativity of adding the global operator is also discussed at the end.

Chapter 2

Preliminaries

We give preliminaries on several notions and facts in epistemic logic and dynamic epistemic logics, PAL and EA, in order that we may refer to them in subsequent chapters. We refer the reader to standard textbooks, such as [4], for detailed explanations and proofs for Sections 2.1 and 2.2, and to [5] for further information on the topics in Section 2.4.

2.1 Epistemic Logic

Epistemic Logic is a kind of modal logic which is used to reason about knowledge, belief, preference and so forth. Its mathematical treatment is the same as that of ordinary modal logic.

Definition 2.1.1. Let $\mathcal{P}$ be a set of *atomic propositions* and $\mathcal{E}$ a set of *epistemic expressions*. The language $\mathcal{L}(\mathcal{E})$ is defined by the following rule:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle e \rangle \varphi,$$

where p ranges over $\mathcal{P}$ and e over $\mathcal{E}$. ⊣

Here, $\langle e \rangle$ is generally called a *modality* or an *operator*, while in the context of epistemic logic we call it an *epistemic operator*. The dual operator $[e]$ of $\langle e \rangle$ is given by $\neg\langle e \rangle\neg$. The boolean connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual. The intended meaning of $[e]\varphi$ varies from logic to logic: for example, agent e knows, belives or prefers φ.

Remark 2.1.1. In Preference Upgrade [17], each agent n has two operators $\langle k_n \rangle \varphi$ and $\langle p_n \rangle \varphi$ for its knowledge and preference, respectively. In such a case, it is enough to consider the disjoint union $\mathcal{E}_K \uplus \mathcal{E}_P$ as the set $\mathcal{E}$ of epistemic expressions, where $\mathcal{E}_K$ is the set of epistemic expressions for knowledge and $\mathcal{E}_P$ is that for preference.

The language is interpreted by *a Kripke model*:

Definition 2.1.2. A *Kripke model* M for $\mathcal{L}(\mathcal{E})$ is a triple $(S, (R_e)_{e\in\mathcal{E}}, V)$ consisting of:

- a set S;
- an $\mathcal{E}$-indexed family of binary relations R_e on S;
- a function $V : \mathcal{P} \to \wp(\mathcal{S})$. ⊣

An element w of S is called a *world* or a *state*, and the function V is called a *valuation*. In addition, when wR_ev, we call the state v an *e-successor* of w. A *pointed Kripke model* is a pair (M, w) of a Kripke model M and a state w of it. A *Kripke frame* $(S, (R_e)_{e\in\mathcal{E}})$ is a Kripke model without a valuation; the *underlying Kripke frame* $U(M)$ of a Kripke model $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ is its frame $(S, (R_e)_{e\in\mathcal{E}})$.

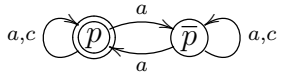
Definition 2.1.3. Suppose that $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ is a Kripke model and that w is a state of M. We inductively give the condition for a formula $\varphi \in \mathcal{L}(\mathcal{E})$ to be *satisfied* at w in M (notation: $M, w \Vdash \varphi$) as follows:

$M, w \Vdash \top$		always
$M, w \Vdash p$	iff	$w \in V(p)$
$M, w \Vdash \neg\varphi$	iff	$M, w \nVdash \varphi$
$M, w \Vdash \varphi \vee \psi$	iff	$M, w \Vdash \varphi$ or $M, w \Vdash \psi$
$M, w \Vdash \langle e\rangle\varphi$	iff	$M, v \Vdash \varphi$ for some $v \in M$ with wR_ev

For a set Σ of formulae in $\mathcal{L}(\mathcal{E})$, we write $M, w \Vdash \Sigma$ when all formulae in Σ is satisfied at w in M. ⊣

We say that a formula $\varphi \in \mathcal{L}(\mathcal{E})$ is *valid* in M (or that M *validates* φ) and write $M \Vdash \varphi$ if $M, w \Vdash \varphi$ holds for all states $w \in M$. For a Kripke frame $F = (S, (R_e)_{e\in\mathcal{E}})$, $F, w \Vdash \varphi$ means that $M, w \Vdash \varphi$ holds for any Kripke model M with $U(M) = F$; and a formula φ is *valid* in F if any Kripke model M with $U(M) = F$ validates φ. Further, we write $\Vdash \varphi$ when all Kripke models, thus all Kripke frames, validate the formula φ.

Example 2.1.1. Assume that $\mathcal{P} = \{p\}$ and $\mathcal{E} = \{a, c\}$. The figure (i) in the previous chapter



(i) before

is expressed as the Kripke model $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ given below:

- $S = \{w, v\}$ (w and v indicate the left and right circles, respectively);

- $R_a = \{(w,w),(w,v),(v,v),(v,w)\}$;
- $R_c = \{(w,w),(v,v)\}$;
- $V(p) = \{w\}$.

Then, we have $M, w \Vdash \neg[a]p$ and $M, w \Vdash [c]p$, which means that the author does not know p and that the committee knows p, respectively. ⊣

2.2 Completeness

The base logic for the above semantics is multimodal logic K in the language $\mathcal{L}(\mathcal{E})$,[1,2] which axiomatises the class of all Kripke models, as we shall see in this section.

Multimodal logic K

	all instances of propositional tautologies
K for $e \in \mathcal{E}$	$[e](\varphi \to \psi) \to [e]\varphi \to [e]\psi$
Dual for $e \in \mathcal{E}$	$\langle e\rangle\varphi \leftrightarrow \neg[e]\neg\varphi$

Modus ponens $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ Generalisation for $e \in \mathcal{E}$ $\dfrac{\varphi}{[e]\varphi}$

The addition of an axiom A to the logic K is expressed by $\mathsf{K} \oplus \mathsf{A}$. Modal logic has a well-known method to prove completeness for many logics. We review here the method and demonstrate a proof of the completeness of K by using it.

Basic modal logics A *basic modal logic* is a set of formulae containing all axioms of the logic K and closed under both modus ponens and the generalisation rule.[3] We often identify a proof system and the set of all formulae provable in the system. Thus, any logic of the form $\mathsf{K} \oplus \mathsf{A}$ is a basic modal logic.

We first recall several notions associated with a basic modal logic.

Definition 2.2.1. Let Λ be a basic modal logic.

- A formula $\varphi \in \mathcal{L}(\mathcal{E})$ is a *theorem* of Λ (notation: $\vdash_\Lambda \varphi$) if Λ contains φ.

[1] Some readers familiar with modal logic might notice the absence of the substitution rule. The rule is admissible, since we consider all instances of axioms. The reason of the absence is related to an additional axiom (see also footnote 6 in Section 2.4.1).

[2] The Dual axiom in the logic K is really needed. Our $[e]$ are defined operators: $[e] = \neg\langle e\rangle\neg$. Thus, if we did not have Dual, the outermost operators $\langle e\rangle$ of proved formulae would always be followed by the negation symbol $\neg$, since the K axiom and the generalisation rule only involve the defined operator $[e]$ (see also [4]).

[3] A basic modal logic is not a conventional notion. The conventional one is a *normal modal logic*, which requires that the set of formulae be closed under uniform substitution as well. We relax the condition for the same reason as that for the absence of the substitution rule (see footnote 1 above).

- A formula $\varphi \in \mathcal{L}(\mathcal{E})$ is *deducible* in Λ from a set of formulae $\Sigma \subseteq \mathcal{L}(\mathcal{E})$ (notation: $\Sigma \vdash_\Lambda \varphi$) if there exist finitely many formulae $\varphi_1, \ldots, \varphi_n$ in Σ such that

$$\vdash_\Lambda \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi.$$

⊣

Definition 2.2.2 (Soundness and Completeness)**.** Let Λ be a basic modal logic and $\mathscr{C}$ a class of Kripke models.

- We write $\Vdash_{\mathscr{C}} \varphi$ if all models in $\mathscr{C}$ validate φ.
- Λ is *sound* with respect to $\mathscr{C}$ if, for any formula $\varphi \in \mathcal{L}(\mathcal{E})$,

$$\vdash_\Lambda \varphi \text{ implies } \Vdash_{\mathscr{C}} \varphi.$$

- Λ is *weakly complete* with respect to $\mathscr{C}$ if, for any formula $\varphi \in \mathcal{L}(\mathcal{E})$,

$$\Vdash_{\mathscr{C}} \varphi \text{ implies } \vdash_\Lambda \varphi.$$

- Let Σ be a set of formulae in $\mathcal{L}(\mathcal{E})$. A formula $\varphi \in \mathcal{L}(\mathcal{E})$ is a *local semantic consequence of* Σ *over* $\mathscr{C}$ (notation: $\Sigma \Vdash_{\mathscr{C}} \varphi$) if, for any Kripke model M in $\mathscr{C}$ and a state w in M,

$$M, w \Vdash \Sigma \text{ implies } M, w \Vdash \varphi.$$

- Λ is *strongly complete* with respect to $\mathscr{C}$ if, for any formula $\varphi \in \mathcal{L}(\mathcal{E})$ and set Σ of formulae in $\mathcal{L}(\mathcal{E})$,

$$\Sigma \vdash_\Lambda \varphi \text{ implies } \Sigma \Vdash_{\mathscr{C}} \varphi.$$

⊣

Maximal consistent sets Maximal consistent sets play an important role when proving completeness.

Definition 2.2.3 (Maximal Consistent Sets)**.** Let Λ be a normal modal logic.

- A set of formulae Σ is Λ-*inconsistent* if $\Sigma \vdash_\Lambda \bot$. Otherwise, Σ is Λ-*consistent*.
- A *maximal* Λ-*consistent set* (Λ-*MCS*) is a Λ-consistent set which does not have any strictly larger Λ-consistent set. ⊣

Below are some important properties of MCSs:

Proposition 2.2.1 (Properties of Maximally Consistent Sets)**.** *Each* Λ-*MCS* Γ *has the following properties:*

- Γ *is closed under modus ponens;*

- Γ *includes* Λ*;*
- *For any formula* $\varphi \in \mathcal{L}(\mathcal{E})$*, either* φ *or* $\neg\varphi$ *belongs to* Γ*;*
- *If* Γ *contains* $\varphi \vee \psi$*,* φ *or* ψ *belongs to* Γ*.*

In addition, Λ-MCSs satisfy the following lemma:

Lemma 2.2.1 (Lindenbaum's Lemma)**.** *Any* Λ*-consistent set* Γ *has a* Λ*-MCS* Γ^+ *that includes* Γ*.*

Canonical Kripke models The first step towards completeness is to construct a Kripke model consisting of all the MCSs:

Definition 2.2.4. Let Λ be a basic modal logic. A *canonical Kripke model* $\mathcal{M}_\Lambda = (S, (R_e)_{e\in\mathcal{E}}, V)$ for Λ is given as follows:

- S is the set of all Λ-MCSs;
- R_e $(e \in \mathcal{E})$ is the relation such that wR_ev iff $\varphi \in v$ implies $\langle e\rangle\varphi \in w$ for any formula $\varphi \in \mathcal{L}(\mathcal{E})$;
- $V(p) := \{w \in S \mid p \in w\}$ for each atomic proposition $p \in \mathcal{P}$. $\dashv$

The *canonical Kripke frame* $\mathcal{F}_\Lambda$ for a basic modal logic Λ is the underlying Kripke frame $U(\mathcal{M}_\Lambda)$ of the canonical Kripke model $\mathcal{M}_\Lambda$. Canonical Kripke models have good properties; in particular, they satisfy the following two lemmata:

Lemma 2.2.2 (Existence Lemma)**.** *Let* Λ *be a basic modal logic and* $\mathcal{M}_\Lambda$ *its canonical Kripke model. Then, for any state* $w \in \mathcal{M}_\Lambda$*,*

$\langle e\rangle\varphi \in w$ *implies that there exists a state* $v \in \mathcal{M}_\Lambda$ *such that* wR_ev *and* $\varphi \in v$*.*

Lemma 2.2.3 (Truth Lemma)**.** *Let* Λ *be a basic modal logic. Then,*

$$\mathcal{M}_\Lambda, w \Vdash \varphi \text{ iff } \varphi \in w$$

for any state $w \in \mathcal{M}_\Lambda$ *and formula* $\varphi \in \mathcal{L}(\mathcal{E})$*.*

By using them we obtain:

Theorem 2.2.1 (Canonical Model Theorem)**.** *Any basic modal logic is sound and strongly complete with respect to its canonical Kripke model.*

Proof. The soundness is straightforward. We show (the contraposition of) the completeness. Let Λ be a basic modal logic and suppose $\Sigma \nvdash_\Lambda \varphi$. Then, we can readily see that $\Sigma \cup \{\neg\varphi\}$ is Λ-consistent. Therefore, there exists a Λ-MCS $w = (\Sigma \cup \{\neg\varphi\})^+$ that includes $\Sigma \cup \{\neg\varphi\}$. Hence, we see that $\Sigma \cup \{\neg\varphi\}$ is satisfied at $(\mathcal{M}_\Lambda, w)$, which implies $\Sigma \not\models_{\mathcal{M}_\Lambda} \neg\varphi$. $\square$

Corollary 2.2.1 (Completeness Theorem)**.** K *is sound and strongly complete with respect to the class of all Kripke models.*

2.2.1 Modal Correspondence

By using Canonical Model Theorem (Theorem 2.2.1), we can prove completeness for many basic modal logics.

Example 2.2.1. Some axioms have names; for example, (T) $\varphi \to \langle e \rangle \varphi$; ($\mathsf{4}$) $\langle e \rangle \langle e \rangle \varphi \to \langle e \rangle \varphi$; ($\mathsf{B}$) $\varphi \to [e] \langle e \rangle \varphi$. The soundness and strong completeness holds for the following pairs:

- $\mathsf{K} \oplus \mathsf{T}$ and the class of reflexive Kripke models;
- $\mathsf{K} \oplus \mathsf{4}$ and the class of transitive Kripke models;
- $\mathsf{K} \oplus \mathsf{B}$ and the class of symmetric Kripke models.

Here, a *reflexive* Kripke model is a model whose relations R_e are *reflexive*. Similar remarks apply to the latter two. ⊣

As above, there exists a correspondence between axioms and properties. It is formally expressed by the following concept:

Definition 2.2.5 (Canonicity for a Property)**.** Let A be an axiom schema and P a property of Kripke models. A is *canonical* for P if:

- the canonical Kripke model for A satisfies P; and
- any Kripke model with the property P validates A. ⊣

Soundness and strong completeness for an axiom A and a property P are immediate from the canonicity of A for P. In fact, the axioms T, $\mathsf{4}$ and B are canonical for transitivity, reflexivity and symmetry, respectively; and their soundness and completeness are obtained from this. More importantly, the canonicity of axioms can be combined. Thus, the combined axiom $\mathsf{S5{=}T4B}$ is canonical for the property that the relations R_e are equivalence relations.

We shall extend the notion of canonicity to DELs and prove various canonicity results similar to the above in Section 4.3.

2.3 Bisimulations and Bounded Morphisms

For Kripke models, we have the notions of a relation and a function that preserve the structure. They are called a bisimulation and a bounded morphism, respectively. We recall here their definitions and basic properties.

2.3.1 Bisimulations

Definition 2.3.1 (Bisimulations)**.** Let $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ and $N = (S', (Q_e)_{e\in\mathcal{E}}, V')$ be Kripke models. A *bisimulation*[4] is a binary relation $Z \subseteq S \times S'$ such that, for any epistemic expression $e \in \mathcal{E}$,

1. if wZv and wR_ew' hold, then there exists $v' \in S'$ such that vQ_ev' and $w'Zv'$;
2. if wZv and vQ_ev' hold, then there exists $w' \in S$ such that wR_ew' and $w'Zv'$;
3. every two states w and v with wZv satisfy exactly the same atomic propositions.

The first two conditions are depicted as follows:

1. (M, w) Z (N, v) — e — (M, w') Z $^{\exists}(N, v')$: the *zig condition* for $e \in \mathcal{E}$

2. (M, w) Z (N, v) — e — $^{\exists}(M, w')$ Z (N, v') : the *zag condition* for $e \in \mathcal{E}$

The third condition is often called *atomic harmony*. A *bisimulation between Kripke frames* is defined by the zig-zag condition above. ⊣

When (M, w) and (N, v) are related by a bisimulation, we write $(M, w) \leftrightarrows (N, v)$.

Bisimilarity implies modal equivalence.

Definition 2.3.2 (Modal Equivalence)**.** Let (M, w) and (N, v) be pointed Kripke models. If they satisfy exactly the same formulae in $\mathcal{L}(\mathcal{E})$, we say that they are *modally equivalent for the language* $\mathcal{L}(\mathcal{E})$ and write $(M, w) \leftrightsquigarrow (N, v)$ or simply $w \leftrightsquigarrow v$. ⊣

Proposition 2.3.1. *Let (M, w) and (N, v) be pointed Kripke models. Then,*

$$(M, w) \leftrightarrows (N, v) \text{ implies } (M, w) \leftrightsquigarrow (N, v).$$

Proof. This is proved by induction on the complexity of formulae. We show here an $\langle e\rangle$-step and see how the zig-zag condition for $e \in \mathcal{E}$ is used. Let $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ and $N = (S', (Q_e)_{e\in\mathcal{E}}, V')$, and Z be a bisimulation that relates $w \in M$ and $v \in N$. Suppose $M, w \Vdash \langle e\rangle\varphi$. Thus, there is a state $w' \in M$ such that wR_ew' and $M, w' \Vdash \varphi$. Then, by the zig-zag condition, there exists $v' \in N$ such that vQ_ev' and $w'Zv'$. Therefore, (N, v') satisfies φ by the induction hypothesis, and thus we obtain $N, v \Vdash \langle e\rangle\varphi$. □

[4] In this thesis, we allow the bisimulation based on the empty relation.

2.3.2 The Hennessy-Milner Property

The reverse direction of Proposition 2.3.1 does not necessarily hold:

Example 2.3.1. Assume, for simplicity, that $\mathcal{E}$ is a singleton (thus, we suppress subscripts for it) and that $\mathcal{P} = \{p\}$.

- Let $M = (S, R, V)$ be a Kripke model such that:
  1. $S = \{(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid i \geq j\} \cup \{0\}$;
  2. $R = \{((i, j), (i', j')) \in S \times S \mid (i', j') = (i, j + 1)\} \cup \{(0, (i, 1)) \mid i \in \mathbb{Z}_{>0}\}$;
  3. $V(p) = S$.
- Let $N = (S', Q, V')$ be a model such that:
  1. $S' = S \cup \{(\infty, j) \mid j \in \mathbb{Z}_{>0}\}$;
  2. $Q = R \cup \{((\infty, j), (\infty, j + 1)) \mid j \in \mathbb{Z}_{>0}\} \cup \{(0, (\infty, 1))\}$;
  3. $V'(p) = S'$.

Then, $(M, 0)$ and $(N, 0)$ are modally equivalent but there does not exist any bisimulation. ⊣

Therefore, the following property is worth defining here:

Definition 2.3.3. Let $\mathscr{C}$ be a class of Kripke models. $\mathscr{C}$ has the *Hennessy-Milner property* if, for any pair of Kripke models M and N in $\mathscr{C}$ and states $w \in M$ and $v \in N$,

$$(M, w) \leftrightsquigarrow (N, v) \text{ implies } (M, w) \leftrightarrows (N, v).$$

⊣

The following classes of Kripke models have the Hennessy-Milner property:

Definition 2.3.4. Let $M = (S, (R_e)_{e \in \mathcal{E}}, V)$ be a Kripke model.

- M is *image-finite* if any state $w \in M$ has finitely many, including nought, e-successors for each epistemic expression $e \in \mathcal{E}$;
- M is *finite* if its carrier set S is finite. ⊣

Note that a finite Kripke model is image-finite.

Theorem 2.3.1 (Hennessy-Milner Theorem)**.** *The class of image-finite Kripke models has the Hennessy-Milner property. Hence, so does that of finite Kripke models.*

Proof. We prove that, for two image-finite Kripke models $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ and $N = (S', (Q_e)_{e\in\mathcal{E}}, V')$, the relation $\leftrightsquigarrow$ restricted to $S\times S'$ gives rise to a bisimulation between M and N. Atomic harmony is trivial by definition. Accordingly, we show here the zig condition; the zag condition can be proved analogously. Suppose $w \leftrightsquigarrow v$ and wR_ew'. If there does *not* exist a state v' such that vQ_ev' and $w' \leftrightsquigarrow v'$, then any e-successor v' of v has a formula $\varphi_{v'}$ such that $N, v' \Vdash \varphi_{v'}$ but $M, w' \nVdash \varphi_{v'}$. By virtue of the image-finiteness, their disjunction $\bigvee\{\varphi_{v'} \mid vR_ev'\}$ is legitimate, and we have $N, v \Vdash [e]\bigvee\{\varphi_{v'} \mid vR_ev'\}$. Considering the modal equivalence between w and v, we see that $M, w \Vdash [e]\bigvee\{\varphi_{v'} \mid vR_ev'\}$. Thus, w' satisfies at least one $\varphi_{v'}$, which is a contradiction. □

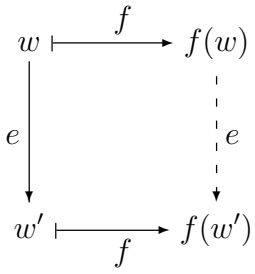
We shall extend the Hennessy-Milner property to the setting of DELs in Section 3.3.1.

2.3.3 Bounded Morphisms

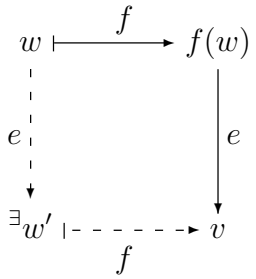
Definition 2.3.5 (Bounded Morphisms)**.** Suppose that $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ and $N = (S', (Q_e)_{e\in\mathcal{E}}, V')$ are Kripke models. A *bounded morphism* $f : M \to N$ is a function $f : S \to S'$ which satisfies:

1. wR_ew' implies $f(w)Q_ef(w')$;
2. if $f(w)Q_ev$, then there exists $w' \in M$ such that wR_ew' and $f(w') = v$; and
3. atomic harmony.

The former two conditions are illustrated as follows:

1. 

the *homomorphic condition* for $e \in \mathcal{E}$

2. 

the *back condition* for $e \in \mathcal{E}$

A *bounded morphism between Kripke frames* is defined by the homomorphic and back conditions.

⊣

A bounded morphism amounts to a bisimulation which is a function. Thus, a bounded morphism $f : M \to N$ implies the modal equivalence between a state $w \in M$ and its value $f(w) \in N$.

2.4 Dynamic Epistemic Logic

We next review two dynamic epistemic logics: PAL and EA. We assume that Ag is a set of *agents*.

2.4.1 Public Announcement Logic

Public Announcement Logic (PAL) is a logic proposed by Plaza in [11]. A public announcement is an action of information disclosure where all involved agents know that everyone receives the announcement. Examples of public announcements include not only ordinary announcements in front of an audience but also face-to-face talks and talking on the phone, *etc.*

The language and semantics are given as below:

Definition 2.4.1. The language $\mathcal{L}_{\mathsf{PAL}}$ of PAL is given by the following grammar:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle n\rangle\varphi \mid \langle\!\langle !\varphi\rangle\!\rangle\psi,$$

where p ranges over $\mathcal{P}$ and n over Ag. ⊣

$\langle\!\langle !\varphi\rangle\!\rangle$ is called an *announcement operator*. The dual operator $[[!\varphi]]$ of $\langle\!\langle !\varphi\rangle\!\rangle$ is given by $[[!\varphi]]\psi := \neg\langle\!\langle !\varphi\rangle\!\rangle\neg\psi$, whose intended meaning is that ψ holds after a truthful public announcement of φ.

Definition 2.4.2 (Plaza's Semantics)**.** The inherited connectives and operators are interpreted as before, whilst each announcement operator $\langle\!\langle !\varphi\rangle\!\rangle$ is interpreted as

$$M, w \Vdash \langle\!\langle !\varphi\rangle\!\rangle\psi \iff M|_{!\varphi}, w \Vdash \psi.$$

Here, $M|_{!\varphi} = (S', (R'_n)_{n\in Ag}, V')$ is the submodel of $M = (S, (R_n)_{n\in Ag}, V)$ whose carrier set S' is $\{v \in M \mid M, v \Vdash \varphi\}$.[5] ⊣

Note that the example of PAL in Section 1.1 is also technically materialised in the language and semantics above.

PAL is axiomatised by the logic PAL, which is defined to be the logic K (for operators of both kinds $\langle n\rangle$ and $\langle\!\langle !\varphi\rangle\!\rangle$) augmented with the following axioms:[6]

[5] Generally, a *submodel* $M' = (S', (R'_e)_{e\in\mathcal{E}}, V')$ of a Kripke model $M = (S, (R_e)_{e\in\mathcal{E}}, V)$ is a Kripke model such that (i) S' is a subset of S, (ii) $R'_e = R_e \cap (S' \times S')$ for each $e \in \mathcal{E}$ and (iii) $V'(p) = V(p) \cap S'$ for each $p \in \mathcal{P}$.

[6] Notice that the axiom $\mathsf{R_P}$ refers to an atomic proposition p. Hence, logics involving $\mathsf{R_P}$ are not closed under substitution.

$\mathsf{R_T}$: $\langle\!\langle !\varphi\rangle\!\rangle \top \leftrightarrow \varphi$; $\quad$ $\mathsf{R_P}$: $\langle\!\langle !\varphi\rangle\!\rangle p \leftrightarrow \langle\!\langle !\varphi\rangle\!\rangle \top \wedge p$;
$\mathsf{R_N}$: $\langle\!\langle !\varphi\rangle\!\rangle \neg\psi \leftrightarrow \langle\!\langle !\varphi\rangle\!\rangle \top \wedge \neg\langle\!\langle !\varphi\rangle\!\rangle \psi$; $\quad$ $\mathsf{R_E}$: $\langle\!\langle !\varphi\rangle\!\rangle \langle e\rangle \psi \leftrightarrow \langle\!\langle !\varphi\rangle\!\rangle \top \wedge \langle e\rangle \langle\!\langle !\varphi\rangle\!\rangle \psi$.

They are called *reduction axioms*, since they induce a rewriting system that translates any PAL formula φ into an equivalent one $t(\varphi)$ which does not contain any announcement operators. For example, $\langle\!\langle !p\rangle\!\rangle q$ is rewritten as follows:

$$\langle\!\langle !p\rangle\!\rangle q \overset{\mathsf{R_P}}{\Longrightarrow} \langle\!\langle !p\rangle\!\rangle \top \wedge q \overset{\mathsf{R_T}}{\Longrightarrow} p \wedge q.$$

The completeness is proved by using this rewriting system:

Theorem 2.4.1 (Completeness)**.** PAL *is sound and strongly complete with respect to Plaza's semantics.*

Proof. The soundness is straightforward. We prove the completeness. By the reduction axioms, we see $\vdash_{\mathsf{PAL}} \varphi \leftrightarrow t(\varphi)$ and hence $\Vdash \varphi \leftrightarrow t(\varphi)$ by the soundness. Therefore, the completeness of PAL is reduced to that of the logic K for $\mathcal{E}$ with respect to the class of Kripke models:

$$\Vdash \varphi \;\Rightarrow\; \Vdash t(\varphi) \;\Rightarrow\; \vdash_{\mathsf{K}} t(\varphi) \;\Rightarrow\; \vdash_{\mathsf{PAL}} \varphi.$$

□

We call a proof of this kind *proof by reduction*.

2.4.2 Epistemic Action

Epistemic Action (EA), which was a proposed by Baltag, Moss, Solecki in [3], is a framework to express various kinds of actions, including public announcements. We recall here its formal definition.

Definition 2.4.3. The language $\mathcal{L}_{\mathsf{EA}}$ of EA is given by the following grammar:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle n\rangle\varphi \mid \langle\!\langle (U,s)\rangle\!\rangle\varphi,$$

where p ranges over $\mathcal{P}$ and n over Ag. (U,s) is a pointed action model. An *action model* $U = (U, (\to_n)_{n\in Ag}, \mathrm{Pre}_U)$ is a Kripke frame $(U, (\to_n)_{n\in Ag})$ equipped with a *precondition function* $\mathrm{Pre}_U : U \to \mathcal{L}_{\mathsf{EA}}$; and a *pointed action model* (U,s) is a pair of an action model U and a state $s \in U$. ⊣

We denote the set of all action models by $|\mathcal{A}_{\mathsf{EA}}|$. The dual operator $[[(U,s)]]\varphi := \neg\langle\!\langle (U,s)\rangle\!\rangle\neg\varphi$ is read as 'φ holds after an epistemic action (U,s)'.

Definition 2.4.4 (Baltag *et al.*'s Semantics)**.** The connectives and operators other than $\langle\!\langle (U,s) \rangle\!\rangle$ are interpreted as usual. For a Kripke model $M = (S, (R_n)_{n\in Ag}, V)$, a state $w \in S$ and a pointed action model (U, s) with $U = (U, \{\rightarrow_n\}_{n\in Ag}, \mathrm{Pre}_U)$, the interpretation of $\langle\!\langle (U,s) \rangle\!\rangle \varphi$ is given as follows:

$$M, w \Vdash \langle\!\langle (U,s) \rangle\!\rangle \varphi \iff M \otimes U, (w,s) \Vdash \varphi.$$

Here, $M \otimes U = (S', (R'_n)_{n\in Ag}, V')$ is given by:

- $S' = \{(v,t) \in S \times U \mid M, v \Vdash \mathrm{Pre}_U(t)\}$;
- $(v,t)R'_n(v',t') \iff vR_nv'$ and $t{\rightarrow_n}t'$;
- $(v,t) \in V'(p) \iff v \in V(p)$. ⊣

This semantics is axiomatised by EA, which is K together with the following reduction axioms:

$$\begin{array}{lcl}
\mathsf{R_{T(EA)}} & : & \langle\!\langle (U,s) \rangle\!\rangle \top \leftrightarrow \mathrm{Pre}_U(s); \\
\mathsf{R_P} & : & \langle\!\langle (U,s) \rangle\!\rangle p \leftrightarrow \langle\!\langle (U,s) \rangle\!\rangle \top \wedge p; \\
\mathsf{R_N} & : & \langle\!\langle (U,s) \rangle\!\rangle \neg\varphi \leftrightarrow \langle\!\langle (U,s) \rangle\!\rangle \top \wedge \neg\langle\!\langle (U,s) \rangle\!\rangle \varphi; \\
\mathsf{R_{E(EA)}} & : & \langle\!\langle (U,s) \rangle\!\rangle \langle n \rangle \varphi \leftrightarrow \langle\!\langle (U,s) \rangle\!\rangle \top \wedge \bigvee \{\langle n \rangle \langle\!\langle (U,t) \rangle\!\rangle \varphi \mid s \rightarrow_n t \text{ in } U\}.
\end{array}$$

Note that the axiom $\mathsf{R_{E(EA)}}$ can be simplified by introducing a defined operator $\langle\!\langle U^n_s \rangle\!\rangle \varphi := \bigvee \{\langle\!\langle (U,t) \rangle\!\rangle \varphi \mid s \rightarrow_n t\}$:

$$\mathsf{R_{E(EA)}} : \langle\!\langle (U,s) \rangle\!\rangle \langle n \rangle \varphi \leftrightarrow \langle\!\langle (U,s) \rangle\!\rangle \top \wedge \langle n \rangle \langle\!\langle U^n_s \rangle\!\rangle \varphi.$$

Theorem 2.4.2 (Completeness)**.** EA *is sound and strongly complete with respect to Baltag et al.'s semantics.*

Proof. The completeness is obtained via proof by reduction. □

Chapter 3

General Framework for DELs

This chapter introduces a general framework for DELs. It consists of a generic language and large-scale Kripke models, called model transition systems, wherein model transformations of DELs are expressed. We also take PAL and EA (Sections 2.4.1 and 2.4.2, *supra*) as leading examples and illustrate how our framework expresses them in an appropriate way. Subsequently, we consider the notions of a bisimulation and a bounded morphism for MTSs. At the end of this chapter, we see that many static representations of model transformations in the literature can be expressed in our framework.

3.1 Generic Language and Proof Systems

We begin with the generic language for DELs.

Definition 3.1.1. Let $\mathcal{P}$ be a set of atomic propositions, $\mathcal{E}$ a set of epistemic expressions and $\mathcal{A}$ a set of *action expressions*. We define a *DEL language* $\mathcal{L}(\mathcal{E}, \mathcal{A})$ by the following rule:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle e \rangle \varphi \mid \langle\langle \alpha \rangle\rangle \varphi,$$

where p ranges over $\mathcal{P}$, e over $\mathcal{E}$ and α over $\mathcal{A}$. ⊣

The new operators $\langle\langle \alpha \rangle\rangle$ are called *action operators*, and their dual operators $[[\alpha]]$ are given by $\neg\langle\langle \alpha \rangle\rangle\neg$. Further, we write $\langle\langle \gamma \rangle\rangle$ for a string $\gamma = \alpha_1 \cdots \alpha_n \in \mathcal{A}^*$ to denote $\langle\langle \alpha_1 \rangle\rangle \cdots \langle\langle \alpha_n \rangle\rangle$.

Note that the languages of PAL and EA can be regarded as instances of $\mathcal{L}(\mathcal{E}, \mathcal{A})$. $\mathcal{L}_{\mathsf{PAL}}$ can be written as $\mathcal{L}(Ag, \mathcal{A}_{\mathsf{PAL}})$ with $\mathcal{A}_{\mathsf{PAL}}$ the set of all announcements $!\varphi$, whereas $\mathcal{L}_{\mathsf{EA}}$ is $\mathcal{L}(Ag, \mathcal{A}_{\mathsf{PAL}})$ with $\mathcal{A}_{\mathsf{EA}}$ the set of pointed action models (U, s). In addition,

the language $\mathcal{L}(\mathcal{E})$ (Definition 2.1.1, *supra*) is also an instance $\mathcal{L}(\mathcal{E}, \emptyset)$ of the generic language.

Our base logic is the multimodal logic K in the language $\mathcal{L}(\mathcal{E}, \mathcal{A})$.

Multimodal logic K for $\mathcal{L}(\mathcal{E}, \mathcal{A})$

	all instances of propositional tautologies
K for $e \in \mathcal{E}$	$[e](\varphi \to \psi) \to [e]\varphi \to [e]\psi$
K for $\alpha \in \mathcal{A}$	$[[\alpha]](\varphi \to \psi) \to [[\alpha]]\varphi \to [[\alpha]]\psi$
Dual for $e \in \mathcal{E}$	$\langle e \rangle \varphi \leftrightarrow \neg[e]\neg\varphi$
Dual for $\alpha \in \mathcal{A}$	$\langle\!\langle \alpha \rangle\!\rangle \varphi \leftrightarrow \neg[[\alpha]]\neg\varphi$

Modus ponens $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ Generalisation for $e \in \mathcal{E}$ $\dfrac{\varphi}{[e]\varphi}$ Generalisation for $\alpha \in \mathcal{A}$ $\dfrac{\varphi}{[[\alpha]]\varphi}$

Some reduction axioms (see Section 2.4) can be written in $\mathcal{L}(\mathcal{E}, \mathcal{A})$ in general forms:

$\mathsf{R_P} : \langle\!\langle \alpha \rangle\!\rangle p \leftrightarrow \langle\!\langle \alpha \rangle\!\rangle \top \wedge p$; $\quad \mathsf{R_N} : \langle\!\langle \alpha \rangle\!\rangle \neg\varphi \leftrightarrow \langle\!\langle \alpha \rangle\!\rangle \top \wedge \neg\langle\!\langle \alpha \rangle\!\rangle \varphi$;
$\mathsf{R_E} : \langle\!\langle \alpha \rangle\!\rangle \langle e \rangle \varphi \leftrightarrow \langle\!\langle \alpha \rangle\!\rangle \top \wedge \langle e \rangle \langle\!\langle \alpha \rangle\!\rangle \varphi$.

3.2 Model Transition Systems

In [15] van Benthem proposed a static representation of model transformations, called an *update universe*. It consists of a family of pointed Kripke models linked by *update relations*. In this section we introduce a variant of an update universe, called a model transition system (MTS), which better fits our generic approach. We then illustrate how MTSs express model transformations which take place in PAL and EA.

3.2.1 Model Transition Systems and Interpretation

Definition 3.2.1 (Model Transition Systems)**.** A *model transition system* (*MTS*) for $\mathcal{L}(\mathcal{E}, \mathcal{A})$ is a quadruple $\mathsf{M} = (I, M, \Phi, R)$ such that:

1. I is an index set;
2. M is an I-indexed family of Kripke models $M_i = (S_i, (R_i^e)_{e \in \mathcal{E}}, V_i)$ $(i \in I)$, where each M_i is allowed to be empty[1];
3. Φ is an $\mathcal{A}$-indexed family of binary relations Φ_α on I;

[1]By the *empty* Kripke model we mean the Kripke model with the empty carrier set.

4. R assigns to each action $\alpha \in \mathcal{A}$ and each pair $(i,j) \in \Phi_\alpha$ a binary relation $R^\alpha_{(i,j)} \subseteq M_i \times M_j$.

For an MTS $\mathsf{M} = (I, M, \Phi, R)$, when $wR^\alpha_{(i,j)}v$, we write $(M_i, w) \xrightarrow{\alpha} (M_j, v)$ and call (M_j, v) an *α-successor* of (M_i, w). ⊣

We can view an MTS $\mathsf{M} = (I, M, \Phi, R)$ as an *outer frame* $(I, \Phi) = (I, (\Phi_\alpha)_{\alpha\in\mathcal{A}})$, consisting of the index set I and the *outer relations* $\Phi_\alpha \subseteq I \times I$, that is enriched by the *inner Kripke models* M_i and the *inner relations* $R^\alpha_{(i,j)} \subseteq M_i \times M_j$. In this sense, an MTS can be regarded as a *two-layered Kripke model*.

A *frame transition system* (*FTS*) is defined analogously by replacing Kripke models $M = (M_i)_{i\in I}$ with Kripke frames $F = (F_i)_{i\in I}$. The *underlying FTS* of an MTS $\mathsf{M} = (I, M, \Phi, R)$ is obtained by replacing each M_i with its underlying Kripke frame $U(M_i)$. We denote the resulting FTS by $U(\mathsf{M})$.

An MTS (I, M, Φ, R) expresses model transformations as follows. If $(i,j) \in \Phi_\alpha$, it means that model M_i can be transformed into model M_j by action α. For such (i,j), $wR^\alpha_{(i,j)}v$ expresses that state w in M_i is sent to state v in M_j. In consequence, the MTS expresses the transformation of a pointed Kripke model (M_i, w) into (M_j, v) such that $(i,j) \in \Phi_\alpha$ and $wR^\alpha_{(i,j)}v$. $(M_i, w) \xrightarrow{\alpha} (M_j, v)$ denotes this model transformation.

The language $\mathcal{L}(\mathcal{E}, \mathcal{A})$ is interpreted *in* an MTS:

Definition 3.2.2. Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS, i an index in I and w a state in $M_i = (S_i, (R^e_i)_{e\in\mathcal{E}}, V_i)$. Below are the conditions which define when a formula φ is *satisfied* at state $w \in M_i$ in M (notation: $\mathsf{M}, M_i, w \models \varphi$):

$\mathsf{M}, M_i, w \models \top$		always
$\mathsf{M}, M_i, w \models p$	iff	$w \in V_i(p)$
$\mathsf{M}, M_i, w \models \neg\varphi$	iff	$\mathsf{M}, M_i, w \not\models \varphi$
$\mathsf{M}, M_i, w \models \varphi \vee \psi$	iff	$\mathsf{M}, M_i, w \models \varphi$ or $\mathsf{M}, M_i, w \models \psi$
$\mathsf{M}, M_i, w \models \langle e\rangle\varphi$	iff	$\mathsf{M}, M_i, v \models \varphi$ for some $v \in M_i$ with $wR^e_i v$
$\mathsf{M}, M_i, w \models \langle\!\langle\alpha\rangle\!\rangle\varphi$	iff	$\mathsf{M}, M_j, v \models \varphi$ for some α-successor (M_j, v) of (M_i, w)

⊣

We say that $\mathsf{M} = (I, M, \Phi, R)$ *validates* φ if $\mathsf{M}, M_i, w \models \varphi$ holds for any index $i \in I$ and state $w \in M_i$; and an FTS F *validates* φ if any MTS M with $U(\mathsf{M}) = \mathsf{F}$ validates φ.

An MTS (I, M, Φ, R) is called a *functional MTS* (*f-MTS*) if the outer relations Φ_α are (total) functions. For f-MTSs, we use the following notations: $\Phi_\alpha(i) := j$ for $(i,j) \in \Phi_\alpha$; $M^\alpha_i := M_{\Phi_\alpha(i)}$; $R^\alpha_i := R^\alpha_{(i,\Phi_\alpha(i))}$. Further, we naturally extend these notations to sequences of actions $\gamma = \alpha_1 \cdots \alpha_n \in \mathcal{A}^*$: $\Phi_\gamma(i)$, M^γ_i and R^γ_i.

An MTS whose outer relations are partial functions can be transformed into an 'equivalent' f-MTS by adding the empty Kripke model:

Definition 3.2.3. For an MTS $\mathsf{M} = (I, M, \Phi, R)$ whose outer relations are partial functions, we define an f-MTS $\mathsf{M}^+ = (I^+, M^+, \Phi^+, R^+)$ as follows:

- $I^+ = I \cup \{*\}$;
- $M_i^+ = M_i$ for $i \in I$ and M_*^+ is the empty Kripke model;
- $\Phi_\alpha^+(i) = j$ if $(i,j) \in \Phi_\alpha$ and $\Phi_\alpha^+(i) = \Phi_\alpha^+(*) = *$ otherwise;
- $(R^+)_i^\alpha = R_{(i,j)}^\alpha$if $(i,j) \in \Phi_\alpha$ and $(R^+)_i^\alpha = (R^+)_*^\alpha = \emptyset$ otherwise. ⊣

The original MTS M and the resulting f-MTS M^+ are modally equivalent in the sense that

$$\mathsf{M}, M_i, w \models \varphi \text{ iff } \mathsf{M}^+, M_i^+, w \models \varphi$$

for any index $i \in I$, state $w \in M_i$ and formula $\varphi \in \mathcal{L}(\mathcal{E}, \mathcal{A})$.

3.2.2 PAL Transition Systems

MTSs are used to express model transformations of DELs in a static way. We here take PAL as an example and see how its model transformation is expressed by MTSs.

Firstly, by abuse of notation (see also Definition 2.4.1, *supra*), we write $M_i|_{!\varphi}$ to denote the submodel of M_i whose carrier set is $\{w \in M_i \mid \mathsf{M}, M_i, w \models \varphi\}$, when M_i is an inner model of an MTS M.

A *PAL transition system* $\mathsf{M} = (I, M, \Phi, R)$ is an f-MTS such that: for any index $i \in I$ and announcement $!\varphi \in \mathcal{A}_{\mathsf{PAL}}$,

- $M_i^{!\varphi} = M_i|_{!\varphi}$;
- $R_i^{!\varphi} = \{(w,w) \subseteq M_i \times M_i^{!\varphi} \mid w \in M_i^{!\varphi} \subseteq M_i\}$.

It is easy to verify that a PAL transition system $\mathsf{M} = (I, M, \Phi, R)$ indeed simulates Plaza's model transformation. It suffices to prove, by induction on φ, that (i) $\mathsf{M}, M_i, w \models \varphi \Leftrightarrow M_i, w \Vdash \varphi$ and that (ii) the two Kripke models expressed by $M_i|_{!\varphi}$ are equal, simultaneously.

Furthermore, given a Kripke model N, we can readily construct a PAL transition system (I, M, Φ, R) as follows:

- I is the Kleene closure $\mathcal{A}_{\mathsf{PAL}}^*$ of $\mathcal{A}_{\mathsf{PAL}}$;
- M_γ for $\gamma = !\varphi_1 \cdots !\varphi_m \in I$ is $N|_\gamma$, the model obtained by successively applying Plaza's model transformations $!\varphi_1, \ldots, !\varphi_m$ to N;
- $\Phi_{!\varphi}(\gamma) = \gamma!\varphi$ for $!\varphi \in \mathcal{A}_{\mathsf{PAL}}$ and $\gamma \in I$;
- $R_\gamma^{!\varphi} = \{(w,w) \in M_\gamma \times M_{\gamma!\varphi} \mid w \in M_{\gamma!\varphi} \subseteq M_\gamma\}$ for $\gamma \in I$ and $!\varphi \in \mathcal{A}_{\mathsf{PAL}}$.

We call this system the *PAL transition system generated from* N.

3.2.3 EA Transition Systems

We next consider a class of MTSs which simulate model transformations in EA. The transformations are expressed by MTSs as follows.

Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS for $\mathcal{L}_{\mathsf{EA}}$. For each inner Kripke model $M_i = (S, (R_n)_{n\in Ag}, V)$ and an action model $U = (U, (\to_n)_{n\in Ag}, \mathrm{Pre}_U)$, we first define the transformed Kripke model $M_i \otimes U = (S', (R'_n)_{n\in Ag}, V')$ by:

- $S' = \{(w, s) \in S \times U \mid \mathsf{M}, M_i, w \models \mathrm{Pre}_U(s)\}$;
- $(w, s)R'_n(v, t) \iff wR_nv$ and $s\to_n t$;
- $(w, s) \in V'(p) \iff w \in V(p)$.

Based on this model transformation, we define an *EA transition system* to be an f-MTS $\mathsf{M} = (I, M, \Phi, R)$ for $\mathcal{L}_{\mathsf{EA}}$ such that: for any index $i \in I$ and action model $U \in |\mathcal{A}_{\mathsf{EA}}|$,

- $\Phi_{(U,s)}(i) = \Phi_{(U,t)}(i)$ for any action states s and t in U (the *state-independence condition*[2]);
- $M_i^U = M_i \otimes U$;
- $R_i^{(U,s)} = \{(w, (w, s)) \in M_i \times M_i^U \mid (w, s) \in M_i^U\}$ for any action state $s \in U$.

We can readily see that EA transition systems indeed simulate Baltag *et al.*'s model transformation, as in the case of PAL.

Given a Kripke model N, we can also construct an EA transition system (I, M, Φ, R):

- $I = |\mathcal{A}_{\mathsf{EA}}|^*$;
- $M_{\vec{U}} = N \otimes \vec{U}$;
- $\Phi_U(\vec{U}) = \vec{U}U$;
- $R_{\vec{U}}^{(U,s)} = \{(w, (w, s)) \in M_{\vec{U}} \times M_{\vec{U}U} \mid (w, s) \in M_{\vec{U}U}\}$.

Here, $N \otimes \vec{U}$ for $\vec{U} = U_1 \cdots U_m$ denotes the result of successively applying $U_1, \ldots, U_m$ to N. This system is called the *EA transition system generated from* N.

3.3 Two-layered Bisimulations and Bounded Morphisms

As we have seen, an MTS has the structure of a two-layered Kripke model. Accordingly, we can define the notions of a two-layered bisimulation and a two-layered bounded morphism.

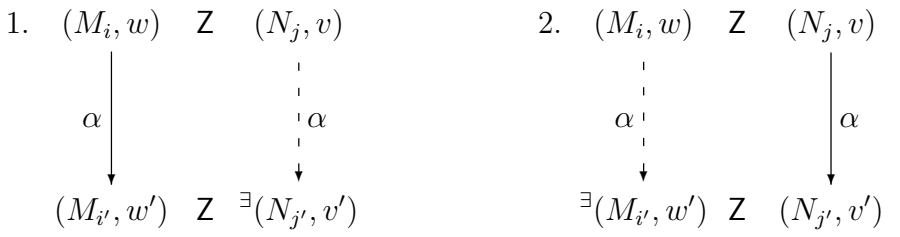
[2]Generally, if an MTS $\mathsf{M} = (I, M, \Phi, R)$ for $\mathcal{L}_{\mathsf{EA}}$ meets the *state-independence condition* ($\Phi_{(U,s)} = \Phi_{(U,t)}$), we call it *state-independent* and use the notation Φ_U for $\Phi_{(U,s)}$. In addition, we write M_i^U for $M_{\Phi_U(i)}$ if M is a state-independent f-MTS. These notations are naturally extended to $\Phi_{\vec{U}}$ and $M_i^{\vec{U}}$, where $\vec{U}$ is a sequence of action models.

3.3.1 Two-layered Bisimulations

Definition 3.3.1 (Two-layered Bisimulations)**.** Suppose that $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ are MTSs. A *two-layered bisimulation* (*2-bisimulation*) $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ between M and N is a pair of

- a (frame) bisimulation $Z \subseteq I \times J$ between the outer frames (I, Φ) and (J, Ψ), and
- a Z-indexed family of (model) bisimulations $Z_{(i,j)}$ between M_i and N_j[3]

which satisfies the zig-zag condition for $\alpha \in \mathcal{A}$: (we write $(M_i, w)\mathsf{Z}(N_j, v)$ if $wZ_{(i,j)}v$ holds)



the *zig condition for* $\alpha \in \mathcal{A}$ the *zag condition for* $\alpha \in \mathcal{A}$

We call a 2-bisimulation $(Z, (Z_{(i,j)})_{(i,j)\in Z})$ *empty* if Z is empty.[4] ⊣

We write $(M_i, w) \leftrightarrows (N_j, v)$ for MTSs M and N when there is a 2-bisimulation between M and N which relates (M_i, w) and (N_j, v).

Example 3.3.1 (Sabotage Bisimulations)**.** The notion of an *s-bisimulation* in sabotage modal logic [2] can be rephrased in terms of 2-bisimulations.[5] For simplicity, we here assume that $\mathcal{E}$ and $\mathcal{A}$ are singletons (thus we omit subscripts and superscripts for them). An *s-transition system* $\mathsf{M} = (I, M, \Phi, R)$ *generated from* a Kripke model $N = (S, Q, V)$ is an MTS such that:

- $I = \{P \subseteq S \times S \mid P \subseteq Q\}$;
- $M_P = (S, P, V)$ for $P \in I$;
- $(P, P') \in \Phi \Leftrightarrow P' = P - \{(w, v)\}$ for some pair $(w, v) \in P$;
- $R_{(P,P')} = \{(w, w) \mid w \in S\}$.

An *s-bisimulation* amounts to a *non-empty* 2-bisimulation between two generated s-transition systems.[6] ⊣

[3]Note that Z is a bisimulation with respect to the action expressions $\alpha \in \mathcal{A}$, whilst $Z_{(i,j)}$ is with respect to the epistemic expressions $e \in \mathcal{E}$.

[4]Recall that we allow bisimulations to be empty (see footnote 4 in Section 2.3.1).

[5]Slightly modified in order that the proof of Proposition 3 of [2] goes through.

[6]Here, a *non-empty* 2-bisimulation relates at least one pair of states.

Just as do ordinary bisimulations (Proposition 2.3.1), 2-bisimilarity also implies modal equivalence.

Definition 3.3.2. Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be MTSs, and (M_i, w) and (N_j, v) be states in M and N, respectively. We call (M_i, w) and (N_j, v) *modally equivalent for the language* $\mathcal{L}(\mathcal{E}, \mathcal{A})$ (notation: $(M_i, w) \lhd\!\rhd (N_j, v)$) if, for any formula φ in $\mathcal{L}(\mathcal{E}, \mathcal{A})$,

$$\mathsf{M}, M_i, w \models \varphi \text{ iff } \mathsf{N}, N_j, v \models \varphi.$$

$\dashv$

Proposition 3.3.1. *Let* M *and* N *be MTSs, and* (M_i, w) *and* (N_j, v) *be states in* M *and* N*. Then,*

$$(M_i, w) \leftrightarrows (N_j, v) \text{ implies } (M_i, w) \lhd\!\rhd (N_j, v).$$

Proof. This is proved as Proposition 2.3.1. □

Operations on 2-bisimulations On 2-bisimulations, we may consider various operations. For example, let $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ be a 2-bisimulation between M and N, and $\mathsf{Z}' = (Z', (Z'_{(j,k)})_{(j,k)\in Z'})$ be one between N and L. The *composition*[7] $\mathsf{Z} \circ \mathsf{Z}' = (Z'', (Z''_{(i,k)})_{(i,k)\in Z''})$ is then given by:

- $Z'' = Z \circ Z'$;
- $Z''_{(i,k)} = \bigcup\{Z_{(i,j)} \circ Z'_{(j,k)} \mid (i,j) \in Z \text{ and } (j,k) \in Z'\}$.

Likewise, we can define the *identity 2-bisimulation* Id_{M} for each MTS M. Therefore, MTSs and 2-bisimulations constitute a category $\mathbb{BIS}$. In addition, for each 2-bisimulation $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$, the *inverse* $\mathsf{Z}^{-1} = (Z^{-1}, (Z^{-1}_{(i,j)})_{(j,i)\in Z^{-1}})$ can be defined.[8] Thus, $\leftrightarrows$ is reflexive, transitive and symmetric.

Let us next fix two MTSs M and N and consider the set $\mathbb{BIS}(\mathsf{M}, \mathsf{N})$ of all 2-bisimulations between them. Actually, $\mathbb{BIS}(\mathsf{M}, \mathsf{N})$ can be regarded as a complete lattice. Firstly, this set can be partially ordered as follows: given two 2-bisimulations $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ and $\mathsf{Z}' = (Z', (Z'_{(i,j)})_{(i,j)\in Z'})$, we define $\mathsf{Z} \leq \mathsf{Z}'$ if

- $Z \subseteq Z'$ and
- $Z_{(i,j)} \subseteq Z'_{(i,j)}$ for all $(i,j) \in Z$.

Secondly, $\mathbb{BIS}(\mathsf{M}, \mathsf{N})$ is closed under arbitrary unions. Given a 2-bisimulation $\mathsf{Z}^\lambda = (Z^\lambda, (Z^\lambda_{(i,j)})_{\lambda\in Z^\lambda}) \in \mathbb{BIS}(\mathsf{M}, \mathsf{N})$ for each $\lambda \in \Lambda$, the *union* $\bigcup_{\lambda\in\Lambda} \mathsf{Z}^\lambda = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ is defined by:

[7] It is crucial here that our bisimulations are allowed to be empty, since Z'' or $Z''_{(i,k)}$ may be empty.

[8] However, Z^{-1} is not necessarily the inverse of Z as a morphism in the category $\mathbb{BIS}$.

- $Z = \bigcup\{Z^\lambda \mid \lambda \in \Lambda\}$;
- $Z_{(i,j)} = \bigcup\{Z^\lambda_{(i,j)} \mid (i,j) \in Z^\lambda\}$.

Furthermore, the least bisimulation in $\mathbb{BIS}(\mathsf{M}, \mathsf{N})$ is the empty 2-bisimulation, whilst the greatest one $\mathsf{Z}^{\max}$ is the union $\bigcup \mathbb{BIS}(\mathsf{M}, \mathsf{N})$ of all 2-bisimulations between M and N. The latter admits another description in terms of $\leftrightarrows$ and $\leftrightarrows$:

- $Z^{\max} = \{(i,j) \in I \times J \mid ((I,\Phi), i) \leftrightarrows ((J,\Psi), j)\}$;
- $Z^{\max}_{(i,j)} = \{(w,v) \in M_i \times N_j \mid (M_i, w) \leftrightarrows (N_j, v)\}$.

The Hennessy-Milner property Having introduced basic concepts and shown elementary results, let us now move on to the Hennessy-Milner property (cf. 2.3.2, *supra*):

Definition 3.3.3. A class $\mathscr{C}$ of MTSs has the *Hennessy-Milner property* if, for every two MTSs M and N in $\mathscr{C}$ and any states (M_i, w) in M and (N_j, v) in N,

$$(M_i, w) \leftrightsquigarrow (N_j, v) \text{ implies } (M_i, w) \leftrightarrows (N_j, v). \qquad \dashv$$

The class of all MTSs does not have the Hennessy-Milner property. A counterexample is obtained from an ordinary one for Kripke models (e.g. Example 2.3.1), since an ordinary Kripke model can be viewed as an MTS for the language $\mathcal{L}(\mathcal{E}, \emptyset)$ which consists of only one inner Kripke model.

We prove here the Hennessy-Milner property for the following two kinds of f-MTSs:

Definition 3.3.4. Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS.

- M is *image-finite* if each inner Kripke model is image-finite and if every state (M_i, w) has finitely many, including nought, α-successors for any action expression $\alpha \in \mathcal{A}$.
- M is *finitary* if its inner Kripke models are all finite. $\dashv$

Note that, for an f-MTS $\mathsf{M} = (I, M, \Phi, R)$, image-finiteness amounts to that of all inner relations R^α_i. Therefore, the class of finitary f-MTSs is a subclass of that of image-finite f-MTSs, and thus it suffices to prove the Hennessy-Milner property for the latter class.

Lemma 3.3.1. *Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be MTSs. For any indices $i \in I$ and $j \in J$, if M_i and N_j are image-finite in the ordinary sense, then the relation $\leftrightsquigarrow$ restricted to $M_i \times N_j$ is a bisimulation in the ordinary sense.*

Proof. This is proved in a similar way to the standard proof of Hennessy-Milner Theorem (Theorem 2.3.1). $\square$

Theorem 3.3.1. *The class of image-finite f-MTSs has the Hennessy-Milner property. As a consequence, so does the class of finitary f-MTSs.*

Proof. Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be f-MTSs in the class. We give a 2-bisimulation $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ as follows:

- $Z = \{(\Phi_\gamma(i), \Psi_\gamma(j)) \in I \times J \mid \gamma \in \mathcal{A}^*$ and $(M_i, w) \leftrightsquigarrow (N_j, v)$ for some $w \in M_i$ and $v \in N_j\}$;
- $Z_{(i,j)} = \{(w, v) \in M_i \times N_j \mid (M_i, w) \leftrightsquigarrow (N_j, v)\}$.

Z is obviously a bisimulation between the outer frames (I, Φ) and (J, Ψ), since all Φ_α and Ψ_α are (total) functions and $(i, j) \in Z$ implies $(\Phi_\alpha(i), \Psi_\alpha(j)) \in Z$. $Z_{(i,j)}$ $((i, j) \in Z)$ are also bisimulations by Lemma 3.3.1. Therefore, it remains to verify the zig-zag condition for each $\alpha \in \mathcal{A}$.

This is proved by the standard technique for the proof of Hennessy-Milner Theorem. Suppose $(M_i, w) \leftrightsquigarrow (N_j, v)$ and $(M_i, w) \xrightarrow{\alpha} (M_i^\alpha, w')$. In order to prove the condition by contradiction, let us also suppose that there does not exist $v' \in N_j^\alpha$ that satisfies $(N_j, v) \xrightarrow{\alpha} (N_j^\alpha, v')$ and $(M_i^\alpha, w') \leftrightsquigarrow (N_j^\alpha, v')$. Thus, for any state $v' \in N_j^\alpha$ with $(N_j, v) \xrightarrow{\alpha} (N_j^\alpha, v')$, there is a formula $\varphi_{v'}$ such that $\mathsf{N}, N_j^\alpha, v' \models \varphi_{v'}$ but $\mathsf{M}, M_i^\alpha, w' \not\models \varphi_{v'}$. With N_j image-finite, we can take the disjunction $\psi = \bigvee\{\varphi_{v'} \mid (N_j, v) \xrightarrow{\alpha} (N_j^\alpha, v')\}$, and we have

$$\mathsf{N}, N_j, v \models [[\alpha]]\psi \Rightarrow \mathsf{M}, M_i, w \models [[\alpha]]\psi \Rightarrow \mathsf{M}, M_i^\alpha, w' \models \psi.$$

Therefore, we obtain $\mathsf{M}, M_i^\alpha, w' \models \varphi_{v'}$ for some v', which is a contradiction. □

We shall address, in Section 4.2, a general case for MTSs which are not necessarily functional.

2-Bisimulations between PAL/EA transition systems Let us now focus on 2-bisimulations between PAL/EA transition systems. We can construct a 2-bisimulation between PAL/EA transition systems from a given bisimulation between a pair of inner Kripke models:

Proposition 3.3.2 (2-Bisimulations between PAL/EA Transition Systems)**.** *Suppose that* $\mathsf{M} = (I, M, \Phi, R)$ *and* $\mathsf{N} = (J, N, \Psi, Q)$ *are PAL/EA transition systems. Then, for any states* (M_i, w) *in* M *and* (N_j, v) *in* N*,*

$$(M_i, w) \leftrightarrows (N_j, v) \text{ iff } (M_i, w) \leftrightarrows (N_j, v).$$

Proof. The backward direction is trivial by definition. We sketch the proof of the forward one in the case of PAL transition systems.[9]

Let X be a bisimulation between M_{i_0} and N_{j_0} which relates $w_0 \in M_{i_0}$ and $v_0 \in N_{j_0}$. We demonstrate how to construct a 2-bisimulation $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ which relates (M_{i_0}, w_0) and (N_{j_0}, v_0).

Z is defined to be $\{(\Phi_\gamma(i_0), \Psi_\gamma(j_0)) \in I \times J \mid \gamma \in \mathcal{A}^*_{\mathsf{PAL}}\}$. The bisimulations $Z_{(i,j)}$ are defined in two stages. The first stage defines $Z_\gamma \subseteq M^\gamma_{i_0} \times N^\gamma_{j_0}$ for $\gamma \in \mathcal{A}^*_{\mathsf{PAL}}$ by $Z_\gamma = (R^\gamma_{i_0})^{-1} \circ X \circ Q^\gamma_{j_0}$. We then define $Z_{(i,j)} \subseteq M_i \times N_j$ by:

$$Z_{(i,j)} = \bigcup \{Z_\gamma \subseteq M_i \times N_j \mid \gamma \in \mathcal{A}^*_{\mathsf{PAL}} \text{ and } (\Phi_\gamma(i_0), \Psi_\gamma(j_0)) = (i,j)\}.$$

Z is indeed a bisimulation between M and N. It is trivial by definition that Z is a bisimulation for the outer frames. To see that each $Z_{(i,j)}$ is a bisimulation between M_i and N_j, it is enough to verify that each Z_γ is one. This is done by proving the following statement, by induction on φ, for all $\gamma \in \mathcal{A}^*_{\mathsf{PAL}}$ simultaneously:
If Z_γ is a bisimulation,

1. $\mathsf{M}, M^\gamma_{i_0}, w \models \varphi$ iff $\mathsf{N}, N^\gamma_{j_0}, v \models \varphi$ for any w and v with $wZ_\gamma v$; and
2. $Z_{\gamma!\varphi}$ is a bisimulation.

The zig-zag condition for each $!\varphi \in \mathcal{A}_{\mathsf{PAL}}$ is proved by reducing $wZ_{(i,j)}v$ to $wZ_\gamma v$ for some γ. □

Using the proposition above, we see that $\leftrightarrows$, $\Leftrightarrows$, $\leftrightsquigarrow$ and $\Lsh\!\!\Rsh$ are equivalent for PAL/EA transition systems under a reasonable condition:

Corollary 3.3.1. *Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be PAL/EA transition systems whose inner Kripke models are image-finite. Then, for any states (M_i, w) in M and (N_j, v) in N, the following are all equivalent:*

$$\begin{array}{ll} (M_i, w) \leftrightarrows (N_j, v); & (M_i, w) \Leftrightarrows (N_j, v); \\ (M_i, w) \leftrightsquigarrow (N_j, v); & (M_i, w) \Lsh\!\!\Rsh (N_j, v). \end{array}$$

Proof. By Propositions 3.3.1 and 3.3.2 and Theorem 2.3.1. □

[9] The background idea of the proof can be found in Proposition 6.21 (Preservation of bisimilarity) of [5].

3.3.2 Two-layered Bounded Morphisms

Let us next have a look at the definition of a two-layered bounded morphism.

Definition 3.3.5 (Two-layered Bounded Morphisms)**.** Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be MTSs. A *two-layered bounded morphism* (*2-bounded morphism*) $\mathsf{f} = (f, (f_i)_{i \in I}) : \mathsf{M} \to \mathsf{N}$ is a pair of

- a (frame) bounded morphism $f : (I, \Phi) \to (J, \Psi)$ and
- an I-indexed family of (model) bounded morphisms $f_i : M_i \to N_{f(i)}$

which satisfies the following conditions for any states and action α (here we use the notation $\mathsf{f}(M_i, w)$ to denote $(N_{f(i)}, f_i(w))$):

1. $(M_i, w) \xrightarrow{\alpha} (M_{i'}, w')$ implies $\mathsf{f}(M_i, w) \xrightarrow{\alpha} \mathsf{f}(M_{i'}, w')$.
2. If $\mathsf{f}(M_i, w) \xrightarrow{\alpha} (N_j, v)$, then there exists $(M_{i'}, w')$ such that $(M_i, w) \xrightarrow{\alpha} (M_{i'}, w')$ and $\mathsf{f}(M_{i'}, w') = (N_j, v)$.

1. $$\begin{array}{ccc} (M_i, w) & \overset{\mathsf{f}}{\longmapsto} & \mathsf{f}(M_i, w) \\ \alpha \downarrow & & \vdots\, \alpha \\ (M_{i'}, w') & \underset{\mathsf{f}}{\longmapsto} & \mathsf{f}(M_{i'}, w') \end{array}$$

   the *homomorphic condition* for $\alpha \in \mathcal{A}$

2. $$\begin{array}{ccc} (M_i, w) & \overset{\mathsf{f}}{\longmapsto} & \mathsf{f}(M_i, w) \\ \alpha \,\vdots & & \downarrow \alpha \\ {}^{\exists}(M_{i'}, w') & \underset{\mathsf{f}}{\dashmapsto} & (N_j, v) \end{array}$$

   the *back condition* for $\alpha \in \mathcal{A}$

⊣

A 2-bounded morphism amounts to a 2-bisimulation $(Z, (Z_{(i,j)})_{(i,j) \in Z})$ such that Z and all $Z_{(i,j)}$ are functions. Hence, a 2-bounded morphism $\mathsf{f} : \mathsf{M} \to \mathsf{N}$ implies the modal equivalence for the language $\mathcal{L}(\mathcal{E}, \mathcal{A})$ between a state (M_i, w) and its value $\mathsf{f}(M_i, w)$.

Further, the class of 2-bounded morphisms can be considered as a subclass of that of 2-bisimulations, and this subclass contains all identities Id_{M} and is closed under composition. In particular, the composition of two 2-bounded morphisms is written as follows:

Definition 3.3.6 (Compositions of 2-bounded Morphisms)**.** Given 2-bounded morphisms $\mathsf{f} = (f, (f_i)_{i \in I}) : \mathsf{M} \to \mathsf{N}$ and $\mathsf{g} = (g, (g_j)_{j \in J}) : \mathsf{N} \to \mathsf{L}$, the *composition* $\mathsf{g} \circ \mathsf{f} : \mathsf{M} \to \mathsf{L}$ is defined to be $(g \circ f, (g_{f(i)} \circ f_i)_{i \in I})$. ⊣

Unravellings of MTSs Let us now provide an example of a 2-bounded morphism. It is associated with an *unravelling* of an MTS.

We first recall the definition of unravelling an ordinary Kripke frame:

Definition 3.3.7. (See [4].) The *unravelling* $(\vec{I}, (\vec{\Phi}_\alpha)_{\alpha\in\mathcal{A}})$ of a Kripke frame $(I, (\Phi_\alpha)_{\alpha\in\mathcal{A}})$ around $i_0 \in I$ is the Kripke frame such that:

- $\vec{I}$ is the set of finite sequences $(i_0, \alpha_1, i_1, \ldots, \alpha_n, i_n)$ such that $(i_{k-1}, i_k) \in \Phi_{\alpha_k}$ for $k = 1, \ldots, n$;
- for s and t in $\vec{I}$, $(s,t) \in \vec{\Phi}_\alpha$ iff $t = s \cdot (\alpha, j)$ for some $j \in I$, where $\cdot$ denotes the concatenation operation. ⊣

An unravelling of an MTS is obtained by naturally enriching an unravelling of its outer frame:

Definition 3.3.8 (Unravellings of MTSs)**.** Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS. The *unravelling* of M around $i \in I$ is the MTS $\vec{\mathsf{M}} = (\vec{I}, \vec{M}, \vec{\Phi}, \vec{R})$ such that:

- the outer frame $(\vec{I}, \vec{\Phi})$ is the unravelling of (I, Φ) around i;
- $\vec{M}_s = M_{L(s)}$ for $s \in \vec{I}$, where $L(s)$ is the last index i_n of $s = (i, \alpha_1, i_1, \ldots, \alpha_n, i_n)$;
- $\vec{R}^\alpha_{(s,t)} = R^\alpha_{(L(s),L(t))}$ for $(s,t) \in \vec{\Phi}_\alpha$. ⊣

Proposition 3.3.3. *Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS and $\vec{\mathsf{M}} = (\vec{I}, \vec{M}, \vec{\Phi}, \vec{R})$ its unravelling around $i \in I$. Then, there is a bounded morphism $(L, (\mathrm{id}_s)_{s\in\vec{I}}) : \vec{\mathsf{M}} \to \mathsf{M}$ which consists of:*

- *the function $L : \vec{I} \to I$ which maps a sequence in $\vec{I}$ to its last index; and*
- *the $\vec{I}$-indexed family of the identity mappings $\mathrm{id}_s : \vec{M}_s \to M_{L(s)}$.*

Remark 3.3.1. When an MTS $\mathsf{M} = (I, M, \Phi, R)$ is functional, its unravelling around an index $i \in I$ can be identified with the f-MTS $\vec{\mathsf{M}} = (\mathcal{A}^*, \vec{M}, \Phi^{\mathcal{A}}, \vec{R})$ given as below:

$$\begin{array}{ll} \mathcal{A}^* \text{ is the Kleene closure of } \mathcal{A}, & \vec{M}_\gamma = M_i^\gamma, \\ \Phi^{\mathcal{A}}_\alpha(\gamma) = \gamma\alpha, & \vec{R}^\alpha_\gamma = R_{\Phi^\alpha_{\gamma(i)}}, \end{array}$$

where $\gamma \in \mathcal{A}^*$ and $\alpha \in \mathcal{A}$. In what follows, we use this expression for an unravelling of an f-MTS. ⊣

3.4 Completeness

We mention here three basic completeness results. Since we shall discuss the completeness of DELs with the global operator in great detail in Section 4.3, we omit the detailed proofs here.

Theorem 3.4.1. *Soundness and strong completeness hold for each of the following pairs:*

- K *and the class of all MTSs;*
- PAL *and the class of PAL transition systems;*
- EA *and the class of EA transition systems.*

The proof of the completeness of K is easy; the canonical Kripke model $\mathcal{M}_{\mathsf{K}} = (S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, V)$ can be naturally considered as an MTS consisting of one inner Kripke model. Note that this MTS can also be viewed as an f-MTS; thus, K is complete with respect to the class of f-MTSs as well. In Section 4.3, we shall see that the class of f-MTSs has a 'distinctive' axiomatisation in the language augmented with the global operator. The completeness of PAL and EA is proved by transforming the canonical Kripke models into PAL/EA transition systems.

Furthermore, by recalling the fact that validity in PAL/EA transition systems coincides with that in dynamic semantics (see Sections 3.2.2 and 3.2.3, *supra*), we obtain independent proofs of their completeness:

Corollary 3.4.1 ([3, 11])**.** *Soundness and strong completeness hold for the following pairs:*

- PAL *and Plaza's semantics;*
- EA *and Baltag et al.'s semantics.*

Remark 3.4.1. As we have seen in Theorems 2.4.1 and 2.4.2, we usually employ the technique called proof by reduction, when proving completeness for a DEL. Meanwhile, our proofs of completeness of PAL and EA are based on *semantic* understanding of PAL/EA axioms rather than *syntactic* rewriting based on reduction axioms. Thus, our proofs may be called *reduction-free*. Wang *et al.* gave more direct reduction-free proofs for PAL and EA in [22] and [21], respectively, which we shall look at in the next section. ⊣

3.5 Other Static Expressions of Model Transformations

In the literature, there are several structures that express model transformations in static ways [7, 16, 21, 22]. In this subsection we illustrate how they are expressed by MTSs, thus demonstrating the flexibility and applicability of our framework.

Extended Kripke models A reduction-free proof of the completeness of PAL was given by Wang and Cao in [22]. They proposed the notion of a *normal extended Kripke model* as an intermediary technical step towards the completeness. It amounts to an f-MTS $\mathsf{M} = (\{*\}, M, \Phi, R)$ for $\mathcal{L}_{\mathsf{PAL}}$ which satisfies the following two conditions:

- every inner relation $R_i^{!\varphi}$ is a partial bounded morphism;
- for any state, having a $!\varphi$-successor and satisfying φ are equivalent.

This model expresses Plaza's model transformation, since

$$(M_*|_{!\varphi}, w) \leftrightarrows (M_*, v)$$

holds for any states w and v in M_* with $(M_*, w) \xrightarrow{!\varphi} (M_*, v)$.

Normal extended Kripke models are related to our PAL transition systems, as the former can be 'unravelled' into the latter (see the proofs of Proposition 4.3.8 and Theorem 4.3.3, *infra*, for the 'unravelling' technique).

Similarly, normal extended Kripke models for EA have been defined as well by Wang and Aucher in [21].

PAL-generated ETL models In PAL, a (truthful) public announcement is possible if only the content is true. However, it is not necessarily the case in reality. In some public situations, one cannot announce 'the professor is stupid' or 'he is a bad boy' even if these are true. *PAL-generated ETL models* [16] deal with this kind of situation. The paper [16] only considers announcements of the pure epistemic formulae in $\mathcal{L}(\mathcal{E}, \emptyset)$; however, it is possible to extend the models to all announcements $\mathcal{A}_{\mathsf{PAL}} = \{!\varphi \mid \varphi \in \mathcal{L}_{\mathsf{PAL}}\}$.

We first extend the notion of a PAL protocol, which imposes a restriction on possible (sequences of) announcements:

Definition 3.5.1. A *PAL protocol* is a subset of $\mathcal{A}^*_{\mathsf{PAL}}$ closed under prefixes. We denote the set of all protocols by $\mathrm{Ptcl}(\mathcal{A}_{\mathsf{PAL}})$. A *state-dependent PAL protocol* for a Kripke model $N = (S, (Q_n)_{n\in Ag}, V)$ is a function $P : S \to \mathrm{Ptcl}(\mathcal{A}_{\mathsf{PAL}})$. ⊣

Although the original definition of a PAL-generated ETL model is directly given, we here define intermediary dynamic semantics, which is required in order to extend the definition. A Kripke model $N = (S, (Q_n)_{n\in Ag}, V)$ with a state-dependent PAL-protocol $P : S \to \mathrm{Ptcl}(\mathcal{A}_{\mathsf{PAL}})$ interprets PAL formulae φ as follows (notation: $(N, P), v \Vdash \varphi$). The boolean connectives and the epistemic operators are interpreted as usual in N. An announcement operator is interpreted by

$$(N, P), v \Vdash \langle\langle !\varphi\rangle\rangle\psi \iff (N, P)_{!\varphi}, v \Vdash \psi,$$

where $(N, P)_{!\varphi} = (N', P')$ consists of:

- the submodel N' of N whose carrier set is $\{w \in S \mid (N, P), w \Vdash \varphi \text{ and } !\varphi \in P(w)\}$; and
- the state-dependent PAL protocol P' for N' such that, for each $w \in N' \subseteq N$, $P'(w) = \{\gamma \in \mathcal{A}^*_{\mathsf{PAL}} \mid !\varphi\gamma \in P(w)\}$.

A PAL-generated ETL model is a structure which expresses the model transformation above in a static way:

Definition 3.5.2. Let $N = (S, (Q_n)_{n\in Ag}, V)$ be a Kripke model and $P : S \to \mathrm{Ptcl}(\mathcal{A}_{\mathsf{PAL}})$ a state-dependent PAL protocol. A *PAL-generated ETL model* amounts to the f-MTS $\mathsf{Forest}_{\mathsf{TPAL}}(N, P) = (\mathcal{A}^*_{\mathsf{PAL}}, M, \Phi^{\mathcal{A}_{\mathsf{PAL}}}, R)$ such that:[10]

- M_γ is the model part of $(N, P)_\gamma$;
- $R^{!\varphi}_\gamma = \{(w, w) \in M_\gamma \times M_{\gamma!\varphi} \mid w \in M_{\gamma!\varphi} \subseteq M_\gamma\}$.

Here, $(N, P)_\gamma$ for $\gamma = !\varphi_1 \cdots !\varphi_m \in \mathcal{A}^*_{\mathsf{PAL}}$ is the results of successively applying $!\varphi_1, \ldots, !\varphi_m$ to (N, P). ⊣

It is readily seen that the dynamic and the static interpretations of a PAL formula coincide; that is

$$\mathsf{Forest}_{\mathsf{TPAL}}(N, P), M_\gamma, w \models \varphi \text{ iff } (N, P)_\gamma, w \Vdash \varphi$$

for any index $\gamma \in \mathcal{A}^*_{\mathsf{PAL}}$, state $w \in M_\gamma$ and formula $\varphi \in \mathcal{L}_{\mathsf{PAL}}$.

The PAL-generated ETL models are axiomatised by the logic $\mathsf{TPAL} := \mathsf{K} \oplus \mathsf{R_P R_N R_E R^R_T}$, where $\mathsf{R^R_T}$ is the right direction $\langle\langle !\varphi\rangle\rangle\top \to \varphi$ of $\mathsf{R_T}$. In addition, we can also obtain the completeness of TPAL with respect to the dynamic semantics.

Theorem 3.5.1. TPAL *is sound and strongly complete with respect to the class of PAL-generated ETL models.*

[10] See Remark 3.3.1 regarding the notation $\Phi^{\mathcal{A}_{\mathsf{PAL}}}$.

Corollary 3.5.1. TPAL *is sound and strongly complete with respect to the class of pairs of a Kripke model and a state-dependent PAL protocol.*

We shall explain the proofs of Theorem 3.5.1 and Corollary 3.5.1 in the next chapter (Theorem 4.3.2 and Corollary 4.3.3) in the language augmented with the global operator. The proofs are inevitably reduction-free, because TPAL does not have a complete set of reduction axioms for rewriting formulae.

EA-generated ETL models PAL protocols and PAL-generated ETL models have been extended to EA protocols and EA-generated ETL models in [7].[11] Their dynamic semantics was proposed by Wang and Aucher in [21].

The dynamic semantics and the definition of an EA-generated ETL model can also be extended as in the case of PAL-generated ETL models. For later purposes, we give here their extended definitions.

An *EA-protocol* and a *state-dependent EA protocol* are defined in the trivial ways. The dynamic semantics is based on the following clause: given a pair (N, P) of a Kripke model $N = (S, (Q_e)_{e\in\mathcal{E}}, V)$ and a state-dependent EA-protocol $P : S \to \mathrm{Ptcl}(\mathcal{A}_{\mathsf{EA}})$,

$$(N, P), v \Vdash \langle\!\langle (U, s) \rangle\!\rangle \varphi \iff (N, P) \otimes U, v \Vdash \varphi,$$

Here, $(N, P) \otimes U = (N', P')$ consists of:

- the submodel N' of $N \otimes U$ in Baltag *et al.*'s semantics whose carrier set is $S' = \{(w, t) \in S \times U \mid (N, P), w \Vdash \mathrm{Pre}_U(t) \text{ and } (U, t) \in P(w)\}$, and
- the protocol P' given by $P'((w, t)) = \{\gamma \in \mathcal{A}^*_{\mathsf{EA}} \mid (U, t)\gamma \in P(w)\}$.

Using the semantics above, an EA-generated ETL models is given as follows:

Definition 3.5.3. Let $N = (S, (Q_n)_{n\in Ag}, V)$ be a Kripke model, $P : S \to \mathrm{Ptcl}(\mathcal{A}_{\mathsf{EA}})$ a state-dependent protocol. Then, an EA-generated ETL model corresponds to the f-MTS $\mathsf{Forest}_{\mathsf{TEA}}(N, P) = (|\mathcal{A}_{\mathsf{EA}}|^*, M, \Phi^{|\mathcal{A}_{\mathsf{EA}}|}, R)$[12] such that:

- $M_{\vec{U}}$ is given by the model part of $(N, P) \otimes \vec{U}$;
- $R^{(U,s)}_{\vec{U}} = \{(w, (w, s)) \in M_{\vec{U}} \times M_{\vec{U}U} \mid (w, s) \in M_{\vec{U}U}\}$.

Here, $(N, P) \otimes \vec{U}$ for $\vec{U} = U_1 \cdots U_n \in |\mathcal{A}_{\mathsf{EA}}|^*$ is obtained by applying $U_1, \cdots, U_n$ to (N, P).

The class of EA-generated ETL models are axiomatised by the logic $\mathsf{TEA} := \mathsf{K_g} \oplus \mathsf{R_P R_N R_{E(EA)} R^R_{T(EA)}}$, where $\mathsf{R^R_{T(EA)}}$ is $\langle\!\langle (U, s) \rangle\!\rangle \top \to \mathrm{Pre}_U(s)$:

[11] The original names are *DEL protocol* and *DEL-generated ETL model*.

[12] By $\Phi^{|\mathcal{A}_{\mathsf{EAg}}|}$, we mean the $|\mathcal{A}_{\mathsf{EAg}}|$-indexed family of the functions $\Phi_U^{|\mathcal{A}_{\mathsf{EAg}}|}$ given by $\Phi_U^{|\mathcal{A}_{\mathsf{EAg}}|}(\vec{U}) = \vec{U}U$.

Theorem 3.5.2. TEA *is sound and strongly complete with respect to the class of EA-generated ETL models.*

Corollary 3.5.2. TEA *is sound and strongly complete with respect to the class of pairs of a Kripke model and a state-dependent EA protocol.*

Their proofs will be discussed in the augmented language (see Theorems 4.3.4 and Corollary 4.3.5).

Chapter 4

The Global Operator and Canonicity

In this chapter we add an auxiliary operator, called the *global operator*, to the language $\mathcal{L}(\mathcal{E}, \mathcal{A})$. The addition leads to very natural definitions of a *canonical MTS* and *canonicity of a DEL formula for a property of MTSs*. Using these notions, we make an analysis of axiom schemata which define PAL and EA. We also discuss conservativity of addition of the global operator at the end.

4.1 Introduction of the Global Operator

We begin by specifying the language augmented with the global operator.

Definition 4.1.1.

- The *DEL language* $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$ *with the global operator* E is given by the following grammar:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle e \rangle \varphi \mid \mathrm{E}\varphi \mid \langle\!\langle \alpha \rangle\!\rangle \varphi.$$

  The dual operator A is defined to be $\neg\mathrm{E}\neg$.
- The global operator is interpreted as follows:

$$\mathsf{M}, M_i, w \models \mathrm{E}\varphi \;\text{ iff }\; \mathsf{M}, M_i, v \models \varphi \text{ for some state } v \in M_i. \qquad \dashv$$

Accordingly, we replace the base logic with the multimodal logic K_{g} with the global operator. K_{g} is defined to be the logic K for $e \in \mathcal{E}$, $\alpha \in \mathcal{A}$ and E together with the following axiom schemata (see also [4]):

1. $\varphi \to \mathrm{E}\varphi$ (reflexivity);
2. $\mathrm{EE}\varphi \to \mathrm{E}\varphi$ (transitivity);
3. $\varphi \to \mathrm{AE}\varphi$ (symmetry);
4. $\langle e \rangle \varphi \to \mathrm{E}\varphi$ (inclusion).

In this context we may consider a new reduction axiom which deals with the global operator:

$$\mathsf{R_G} : \langle\!\langle \alpha \rangle\!\rangle \mathrm{E}\varphi \leftrightarrow \langle\!\langle \alpha \rangle\!\rangle \top \wedge \mathrm{E}\langle\!\langle \alpha \rangle\!\rangle \varphi.$$

PALg and EAg PAL and EA augmented with the global operator are called *PALg* and *EAg*.

The language $\mathcal{L}_{\mathsf{PALg}}$ of PALg is defined to be $\mathcal{L}_{\mathsf{g}}(Ag, \mathcal{A}_{\mathsf{PALg}})$, where the content φ of an announcement $!\varphi$ in $\mathcal{A}_{\mathsf{PALg}}$ is written in the augmented language $\mathcal{L}_{\mathsf{PALg}}$. The language $\mathcal{L}_{\mathsf{EAg}}$ of EAg is $\mathcal{L}_{\mathsf{g}}(Ag, \mathcal{A}_{\mathsf{EAg}})$, where the precondition $\mathrm{Pre}_U(s)$ of each action (U, s) in $\mathcal{A}_{\mathsf{EAg}}$ is written in the language $\mathcal{L}_{\mathsf{EAg}}$. Again, we denote by $|\mathcal{A}_{\mathsf{EAg}}|$ the set of action models for $\mathcal{L}_{\mathsf{EAg}}$.

The proof systems PAL and EA are extended as follows:

- $\mathsf{PALg} := \mathsf{K_g} \oplus \mathsf{R_T}\mathsf{R_P}\mathsf{R_N}\mathsf{R_E}\mathsf{R_G}$;
- $\mathsf{EAg} := \mathsf{K_g} \oplus \mathsf{R_{T(EA)}}\mathsf{R_P}\mathsf{R_N}\mathsf{R_{E(EA)}}\mathsf{R_{G(EA)}}$,
  where $\mathsf{R_{G(EA)}}$ is $\langle\!\langle (U, s) \rangle\!\rangle \mathrm{E}\varphi \leftrightarrow \langle\!\langle (U, s) \rangle\!\rangle \top \wedge \bigvee\{\mathrm{E}\langle\!\langle (U, t) \rangle\!\rangle \varphi \mid t \in U\}$.

By using the defined operator $\langle\!\langle U \rangle\!\rangle \varphi := \bigvee\{\langle\!\langle (U, s) \rangle\!\rangle \varphi \mid s \in U\}$, we can simplify $\mathsf{R_{G(EA)}}$ as follows:

$$\mathsf{R_{G(EA)}} : \langle\!\langle (U, s) \rangle\!\rangle \mathrm{E}\varphi \leftrightarrow \langle\!\langle (U, s) \rangle\!\rangle \top \wedge \mathrm{E}\langle\!\langle U \rangle\!\rangle \varphi.$$

The notions of a PAL transition system and an EA transition system are naturally extended to their new languages. The resulting systems, called a *PALg transition system* and an *EAg transition system*, also simulate Plaza's model transformation and Baltag *et al.*'s one in their extended languages.

PALg and EAg are sound with respect to the classes of PALg transition systems and EAg transition systems, respectively. We shall see in Section 4.3 that completeness also holds for them.

Moreover, TPAL [respectively, TEA] in Section 3.5 is naturally extended with the global operator as well: namely PALg protocols; PALg-generated ETL models $\mathsf{Forest_{TPALg}}(N, P)$; $\mathsf{TPALg} := \mathsf{K_g} \oplus \mathsf{R_P}\mathsf{R_N}\mathsf{R_E}\mathsf{R_G}\mathsf{R_T^R}$ [respectively, EAg protocols; EAg-generated ETL models $\mathsf{Forest_{TEAg}}(N, P)$; $\mathsf{TEAg} := \mathsf{K_g} \oplus \mathsf{R_P}\mathsf{R_N}\mathsf{R_{E(EA)}}\mathsf{R_{G(EA)}}\mathsf{R^R_{T(EA)}}$].

Basic MTSs The augmented language $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$ is suited for describing the structure of an MTS's outer frame, especially in the case of a *basic* MTS.

An MTS $\mathsf{M} = (I, M, \Phi, R)$ is *basic* if:

- M_i is non-empty for each index $i \in I$; and
- $R^{\alpha}_{(i,j)}$ is also non-empty for any action expression $\alpha \in \mathcal{A}$ and any pair $(i, j) \in \Phi_{\alpha}$.

If an MTS $\mathsf{M} = (I, M, \Phi, R)$ is basic, then the following are equivalent:

- $i \in I$ has an α-successor;
- $(I, \Phi), i \Vdash \langle\langle \alpha \rangle\rangle \top$;
- $\mathsf{M}, M_i, w \models \mathrm{E}\langle\langle \alpha \rangle\rangle \top$ for some $w \in M_i$.

Basic MTSs are not oddities. Indeed, any MTS can be transformed into an 'equivalent' basic MTS:

Definition 4.1.2. Given an MTS $\mathsf{M} = (I, M, \Phi, R)$, we can define a basic MTS $\mathsf{M}^- = (I^-, M^-, \Phi^-, R^-)$ as follows:

$$\begin{array}{rclcrcll} I^- & = & \{i \in I \mid M_i \neq \emptyset\}; & & M_i^- & = & M_i & (i \in I^-); \\ \Phi_\alpha^- & = & \{(i,j) \in \Phi_\alpha \mid R^\alpha_{(i,j)} \neq \emptyset\}; & & R^{-\alpha}_{(i,j)} & = & R^\alpha_{(i,j)} & ((i,j) \in \Phi_\alpha^-). \end{array}$$

⊣

M and M^- are equivalent in the following sense:

$$\mathsf{M}, M_i, w \models \varphi \text{ iff } \mathsf{M}^-, M_i^-, w \models \varphi$$

for any state (M_i, w) in M and formula $\varphi \in \mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$.

4.2 g-bisimulations and g-bounded Morphisms

When the language contains the global operator, it is reasonable to consider *total* bisimulations and surjective bounded morphisms. Correspondingly, we here introduce such 2-bisimulations and 2-bounded morphisms and see that many results in Section 3.3 carry over to the new settings.

g-bisimulations We begin by recalling the definition of a total bisimulation (see also [5]).

Definition 4.2.1. A bisimulation Z between Kripke models M and N is *total* if

- for any $w \in M$, there is $v \in N$ such that wZv; and
- for any $v \in N$, there is $w \in M$ such that wZv. ⊣

We write $(M, w) \leftrightarrows_{\mathsf{g}} (N, v)$ if there is a total bisimulation between Kripke models M and N which relates w and v. Existence of a total bisimulation implies modal equivalence for the pure epistemic language $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \emptyset)$ with the global operator.

Accordingly, we consider here 2-bisimulations whose component bisimulations are total:

Definition 4.2.2. Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be MTSs. A *g-bisimulation* $(Z, (Z_{(i,j)})_{(i,j)\in Z})$ between M and N is a 2-bisimulation whose component bisimulations $Z_{(i,j)}$ are total. ⊣

We write $\leftrightarrows_{\mathsf{g}}$ to denote existence of a g-bisimulation. Note that we do *not* impose the condition that Z is total. Thus, the empty 2-bisimulation is a g-bisimulation.

A 2-bisimulation implies modal equivalence $\leftrightsquigarrow_{\mathsf{g}}$ for $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$.

Proposition 4.2.1. *Let* $\mathsf{M} = (I, M, \Phi, R)$ *and* $\mathsf{N} = (J, N, \Psi, Q)$ *be MTSs. Then, for any states* (M_i, w) *in* M *and* (N_j, v) *in* N*,*

$$(M_i, w) \leftrightarrows_{\mathsf{g}} (N_j, v) \text{ implies } (M_i, w) \leftrightsquigarrow_{\mathsf{g}} (N_j, v).$$

Proof. This is readily proved by induction on φ. Let us here follow the E-steps. Suppose that $\mathsf{Z} = (Z, (Z_{(i,j)})_{(i,j)\in Z})$ is a g-bisimulation relating (M_i, w) and (N_j, v) and that $\mathsf{M}, M_i, w \Vdash \mathrm{E}\varphi$. Then, there exists a state w' in M_i such that $\mathsf{M}, M_i, w' \Vdash \varphi$. By the totality of the bisimulation $Z_{(i,j)}$, there is a state v' in N_j that satisfies $w' Z_{(i,j)} v'$, and this v' satisfies φ. Therefore, we obtain $\mathsf{N}, N_j, v \Vdash \mathrm{E}\varphi$. □

Operations on g-bisimulations As before, the class of g-bisimulations is closed under the four operations, namely identity, composition, inverse and union. In addition, the g-bisimulations between two MTSs M and N still comprise a complete lattice $\mathbb{BIS}_{\mathsf{g}}(\mathsf{M}, \mathsf{N})$. This lattice is a sub-join-semilattice of $\mathbb{BIS}(\mathsf{M}, \mathsf{N})$. However, it is not a sublattice, since it does not preserve meets, as the example below illustrates:

Example 4.2.1. We assume, for simplicity, that $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{A}$ are empty. Consider an MTS $\mathsf{M} = (\{*\}, M, \emptyset, \emptyset)$ with M_* consisting of three points $\{1, 2, 3\}$. Then, $Z_{(*,*)} = \{(1,1), (2,2), (3,3)\}$ and $Z'_{(*,*)} = \{(1,2), (2,1), (3,3)\}$ are total bisimulations between M_* and itself. They can also be considered as g-bisimulations together with $Z = (*, *)$. Their meet is the empty bisimulation in $\mathbb{BIS}_{\mathsf{g}}(\mathsf{M}, \mathsf{M})$, whereas it is $((*, *), \{(3, 3)\})$ in $\mathbb{BIS}(\mathsf{M}, \mathsf{M})$. ⊣

The Hennessy-Milner property Let us next consider the Hennessy-Milner property. In contrast to the previous case (Theorem 3.3.1), it holds for a class of MTSs which are not necessarily functional. On the other hand, we have to assume that inner Kripke models are not just image-finite, but finite.

Lemma 4.2.1. *Let* $\mathsf{M} = (I, M, \Phi, R)$ *and* $\mathsf{N} = (J, N, \Psi, Q)$ *be MTSs, and let* i *and* j *be indices in* I *and* J*, respectively. If* M_i *and* N_j *are finite, then the relation* $\leftrightsquigarrow_{\mathsf{g}}$ *restricted to* M_i *and* N_j *is a total bisimulation unless it is empty.*

Proof. The proof is an analogue of the standard one (see also the proof of Theorem 2.3.1). We only prove the totality. By assumption, there are (M_i, w) and (N_j, v) such that $(M_i, w) \leftrightsquigarrow_{\mathsf{g}} (N_j, v)$. Now let $w' \in M_i$ and suppose, for contradiction, that there is no $v' \in N_j$ that satisfies $(M_i, w') \leftrightsquigarrow_{\mathsf{g}} (N_j, v')$. Then, for any $v' \in N_j$, there is a formula $\varphi_{v'} \in \mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$ such that $\mathsf{N}, N_j, v' \models \varphi_{v'}$ but $\mathsf{M}, M_i, w' \not\models \varphi_{v'}$. Therefore, we have $\mathsf{N}, N_j, v \models \mathrm{A} \bigvee \{\varphi_{v'} \mid v' \in N_j\}$. Owing to the finiteness of N_j, this formula is legitimate. By using the modal equivalence, we obtain that $\mathsf{M}, M_i, w \models \mathrm{A} \bigvee \{\varphi_{v'} \mid v' \in N_j\}$. Hence, (M_i, w') satisfies at least one $\varphi_{v'}$, which is a contradiction. □

A basic MTS, introduced in the previous subsection, does not have a 'hollow' transition. Using this advantageous attribute and the lemma above, we obtain a Hennessy-Milner theorem:

Theorem 4.2.1. *The class of finitary, image-finite and basic MTSs has the Hennessy-Milner property.*

Proof. Let $\mathsf{M} = (I, M, \Phi, R)$ and $\mathsf{N} = (J, N, \Psi, Q)$ be MTSs in the class. We prove that the following data give rise to a g-bisimulation $(Z, (Z_{(i,j)\in Z})_{(i,j)\in Z})$ between M and N:

- $Z = \{(i, j) \in I \times J \mid (M_i, w) \leftrightsquigarrow_{\mathsf{g}} (N_j, v) \text{ for some } w \in M_i \text{ and } v \in N_j\}$,
- $Z_{(i,j)} = \{(w, v) \in M_i \times N_j \mid (M_i, w) \leftrightsquigarrow_{\mathsf{g}} (N_j, v)\}$.

Each $Z_{(i,j)}$ is a total bisimulation by Lemma 4.2.1. The zig-zag condition for $\alpha \in \mathcal{A}$ is proved as in the proof of Theorem 3.3.1. Hence, it is enough to prove that Z is a bisimulation.

Let us suppose iZj and $(i, i') \in \Phi_\alpha$. Thus, we have $(M_i, w) \xrightarrow{\alpha} (M_{i'}, w')$ for some states (M_i, w) and $(M_{i'}, w')$ (see the diagram below), since M is basic. Then, owing to iZj and the totality of $Z_{(i,j)}$, there is a state (N_j, v) such that $(M_i, w) \leftrightsquigarrow_{\mathsf{g}} (N_j, v)$. Hence, we can apply the zig-zag condition to obtain a state $(N_{j'}, v')$ as in the diagram. Thus, we see $(j, j') \in \Psi_\alpha$ and $i'Zj'$.

$$\begin{array}{ccc} (M_i, w) & \leftrightsquigarrow_{\mathsf{g}} & {}^{\exists}(N_j, v) \\ \big\downarrow \alpha & & \big\downarrow \alpha \\ (M_{i'}, w') & \leftrightsquigarrow_{\mathsf{g}} & {}^{\exists}(N_{j'}, v') \end{array}$$

□

g-bounded morphisms In what follows, we use g-bounded morphisms rather than g-bisimulations. A *g-bounded morphism* is a 2-bounded morphism $(f, (f_i)_{i\in I})$ whose component bounded morphisms f_i are surjective. A typical example is the 2-bounded morphism from an unravelling of an MTS to the original MTS (Proposition 3.3.3). A g-bounded morphism can also be regarded as a special case of a g-bisimulation, since a surjective bounded morphism can be considered as a total bisimulation. Thus, A g-bounded morphism also implies modal equivalence for the language $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$.

4.3 Canonicity and Modal Correspondence

We have arrived at the crux of this chapter. We introduce here the notions of a canonical MTS and canonicity of a DEL formula for a property of MTSs and clarify correspondences between axioms of DELs and properties of MTSs.

We first verify the completeness of $\mathsf{K_g}$ with respect to the class of MTSs. We then show canonicity results for three important properties: partial functionality of outer relations, determinism and atomic harmony. In particular, the canonicity for the first property leads to a 'distinctive' axiomatisation of the class of f-MTSs, as we mentioned in Section 3.4. Subsequently, we move on to canonicity results for the axioms of PALg and EAg. This is important not only for establishing completeness of PALg [respectively, EAg] with regard to the PALg transition systems [respectively, the EAg transition systems] but also for identifying the exact semantic effect of each axiom, leading to a *complete* understanding of the axiomatic systems PALg and EAg. In passing, we also introduce two logics of independent interest: the logic LEg of *eliminative* updates and the logic LSPg of *subproduct* updates.

Canonical MTSs In what follows, a *basic DEL logic* is a set of formulae in the language $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A})$ which contains all axioms of $\mathsf{K_g}$ and is closed under modus ponens and the generalisation rules for $[e]$, $[[\alpha]]$ and A.

A basic DEL logic Λ can be considered as an ordinary basic modal logic (Section 2.2); hence, we can construct the canonical Kripke model $\mathcal{M}_\Lambda = (S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, R_{\mathrm{E}}, V)$. The carrier set S consists of the maximal consistent sets of formulae, and R_e, R_α and R_{E} are the canonically defined binary relations on S. Observe that, by virtue of the additional axioms for the global operator, the relation R_{E} is an equivalence relation and 'canonically' partitions the epistemic part $(S, (R_e)_{e\in\mathcal{E}}, V)$ of the model in such a way that it does not 'cut' any link $w \xrightarrow{e} v$ of any epistemic relation R_e.

Based on the observation above we define the *canonical MTS* M_Λ *for* Λ to be the basic MTS (I, M, Φ, R) given as follows:

- I is the set S/R_{E} of all equivalence classes for R_{E};
- M_i is the submodel of $(S, (R_e)_{e\in\mathcal{E}}, V)$ whose carrier set is $i \in S/R_{\mathrm{E}}$;
- Φ_α is $\{(i,j) \in I \times I \mid R_\alpha \cap (i \times j) \neq \emptyset\}$;
- $R^\alpha_{(i,j)}$ is $R_\alpha \cap (i \times j)$.

The *canonical FTS* F_Λ *for* Λ is defined to be the underlying FTS $U(\mathsf{M}_\Lambda)$ of the canonical MTS M_Λ.

Since the canonical MTS M_Λ for a basic DEL logic Λ is based on the canonical Kripke model for the logic, the truth lemma holds trivially: $\mathsf{M}_\Lambda, M_i, w \models \varphi$ iff $\varphi \in w$. Thus, completeness is immediate for the base case $\Lambda = \mathsf{K}_\mathsf{g}$:

Theorem 4.3.1. K_g *is sound and strongly complete with respect to the class of all MTSs.*

Canonicity for a property of MTSs The notion of *canonicity for a property of MTSs* is an extension of the standard notion of canonicity for a property (Definition 2.2.5).

Definition 4.3.1 (Canonicity for a Property of MTSs)**.** Let P be a property of MTSs. An axiom schema A is *canonical* for P if:

- The canonical MTS M_Λ validates P for any basic DEL logic Λ containing A; and
- A is valid in any MTS with the property P. ⊣

Thus, if A is canonical for P, one can straightforwardly prove the soundness and strong completeness of $\mathsf{K}_\mathsf{g} \oplus \mathsf{A}$ with respect to the class of MTSs satisfying P.[1]

4.3.1 Canonicity Results for Three Important Properties

We now discuss canonicity for several general properties. In what follows, we only prove the first condition for canonicity (Definition 4.3.1), since the second is straightforward.

[1]The soundness holds because the $[[\alpha]]$-generalisation rule preserves validity with respect to any class $\mathscr{C}$ of MTSs: that is $\models_{\mathscr{C}} \varphi \Rightarrow \models_{\mathscr{C}} [[\alpha]]\varphi$. This does not necessarily obtain in the case of ordinary dynamic semantics; for example, considering PAL and letting $M = (\{0,1\}, (R_n)_{n\in Ag}, V)$ be a Kripke model such that each R_n is $\{0,1\} \times \{0,1\}$ and $V(p) = \{1\}$, we have $M \Vdash \langle n\rangle p$ and $M \nVdash [[!\neg p]]\langle n\rangle p$, since state 0 does not satisfy $[[!\neg p]]\langle n\rangle p$.

Partial functionality of outer relations In many DELs, including PAL and EA, the transformed model is uniquely determined by the original model and the action. A model transformation of this kind is simulated by an MTS $\mathsf{M} = (I, M, \Phi, R)$ whose outer relations $\Phi_\alpha \subseteq I \times I$ are partial functions.

The axiom schema G below syntactically captures such MTSs:

$$\mathsf{G} : \mathrm{E}\langle\langle\alpha\rangle\rangle\varphi \to [[\alpha]]\mathrm{E}\varphi.$$

Proposition 4.3.1. G *is canonical for partial functionality of outer relations.*

Proof. Suppose that Λ is a basic DEL logic which contains G and that $\mathcal{M}_\Lambda = (S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, R_{\mathrm{E}}, V)$ is its canonical Kripke model . We first prove that the following diagram commutes for any states w, w', v and v' in S:

$$\begin{array}{ccc} w & \xrightarrow{\alpha} & v \\ {\scriptstyle \mathrm{E}}\downarrow & & \vdots\,{\scriptstyle \mathrm{E}} \\ w' & \xrightarrow[\alpha]{} & v' \end{array}$$

Take an arbitrary formula φ in v'. Then, we have

$$\varphi \in v' \Longrightarrow \mathrm{E}\langle\langle\alpha\rangle\rangle\varphi \in w \overset{\mathsf{G}}{\Longrightarrow} [[\alpha]]\mathrm{E}\varphi \in w \Longrightarrow \mathrm{E}\varphi \in v.$$

Hence, we obtain $vR_{\mathrm{E}}v'$.

Let us next consider the canonical MTS $\mathsf{M}_\Lambda = (I, M, \Phi, R)$. We prove the partial functionality of the outer relations Φ_α. Suppose $(i, j) \in \Phi_\alpha$ and $(i, j') \in \Phi_\alpha$. Then, by definition, there exist states $w \in i$, $v \in j$ with $wR_\alpha v$ and $w' \in i$, $v' \in j'$ with $w'R_\alpha v'$. Based on the diagram above, we see that v and v' are in the same equivalence class, which means $j = j'$. □

Using the proposition and the M^+-construction (Definition 3.2.3), we obtain the completeness for the class of f-MTSs:

Corollary 4.3.1. $\mathsf{K}_\mathsf{g} \oplus \mathsf{G}$ *is sound and strongly complete with respect to the class of f-MTSs.*

Proof. The soundness holds, since the outer relations of any f-MTS are functions. To prove the completeness, use $\mathsf{M}^+_{\mathsf{K}_\mathsf{g}\oplus\mathsf{G}}$. □

Determinism It happens in many DELs that, in addition to the uniqueness of model transformation, the transformed *pointed* Kripke model is also uniquely determined from the original pointed Kripke model and the action. This property of model transformations is called determinism. Deterministic model transformations are expressed in deterministic MTSs, which are defined below:

Definition 4.3.2. Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS. An action $\alpha \in \mathcal{A}$ is *deterministic* in M if, for any state (M_i, w) in M, $(M_i, w) \xrightarrow{\alpha} (M_j, v)$ and $(M_i, w) \xrightarrow{\alpha} (M_{j'}, v')$ imply $(M_j, v) = (M_{j'}, v')$. M is *deterministic* if any action $\alpha \in \mathcal{A}$ is deterministic therein. $\dashv$

Determinism corresponds to:

$$\mathsf{D_t} : \langle\!\langle \alpha \rangle\!\rangle \varphi \to [[\alpha]]\varphi.$$

Proposition 4.3.2. $\mathsf{D_t}$ *is canonical for determinism.*

Proof. Let Λ be a basic DEL logic containing $\mathsf{D_t}$, and $\mathcal{M}_\Lambda = (S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, R_{\mathrm{E}}, V)$ its canonical Kripke model. It is enough to prove that each R_α is a partial function. Suppose $wR_\alpha v$ and $wR_\alpha v'$. Take an arbitrary formula φ in v. Then, we see

$$\varphi \in v \Longrightarrow \langle\!\langle \alpha \rangle\!\rangle \varphi \in w \overset{\mathsf{D_t}}{\Longrightarrow} [[\alpha]]\varphi \in w \Longrightarrow \varphi \in v'.$$

Therefore, we see $v \subseteq v'$, and by symmetry we obtain $v = v'$. $\square$

In several DELs, including PALg and EAg, schema $\mathsf{D_t}$ appears as a $\mathsf{K_g}$-equivalent schema $\mathsf{R_N}$: $\langle\!\langle \alpha \rangle\!\rangle \neg\varphi \leftrightarrow \langle\!\langle \alpha \rangle\!\rangle \top \wedge \neg\langle\!\langle \alpha \rangle\!\rangle \varphi$. As we shall see in Sections 4.3.2 and 4.3.3, schema $\mathsf{R_N}$ plays an important role, when we consider other properties corresponding to axioms of PALg and EAg.

Atomic harmony An action of communication itself does not change the facts; for example, communication does not change the weather. This property can be paraphrased as follows: a pointed Kripke model and any of its transformed pointed Kripke models satisfy the same atomic propositions. Such a property is called atomic harmony. More precisely:

Definition 4.3.3. An action $\alpha \in \mathcal{A}$ satisfies *atomic harmony* in an MTS $\mathsf{M} = (I, M, \Phi, R)$ if any two states (M_i, w) and (M_j, v) with $(M_i, w) \xrightarrow{\alpha} (M_j, v)$ satisfy exactly the same atomic propositions. We also say that an MTS satisfies *atomic harmony* or is *epistemic* if any action $\alpha \in \mathcal{A}$ satisfies atomic harmony therein. $\dashv$

This property is axiomatised by:

$$\mathsf{P} : (p \to [[\alpha]]p) \wedge (\langle\langle\alpha\rangle\rangle p \to p).$$

Proposition 4.3.3. P *is canonical for atomic harmony.*

Proof. Let Λ be a basic DEL logic which contains P. It suffices to prove that every two states w and v with $wR_\alpha v$ in its canonical Kripke model $(S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, R_{\mathrm{E}}, V)$ satisfy exactly the same atomic propositions. Suppose that state w contains p. Then, w also contains $[[\alpha]]p$ by the left conjunct of P, and thus state v satisfies p. Conversely, if state v contains p, then w satisfies $\langle\langle\alpha\rangle\rangle p$ and hence p by the right conjunct of P. □

In several DELs, including PALg and EAg, schema P appears as reduction axiom $\mathsf{R_P} : \langle\langle\alpha\rangle\rangle p \leftrightarrow \langle\langle\alpha\rangle\rangle\top \wedge p$. In fact, we can easily see that these two are equivalent in $\mathsf{K_g} \oplus \mathsf{R_N}$, although they are *not* in $\mathsf{K_g}$ (see the following example).

Example 4.3.1. Let $\mathsf{M} = (I, M, \Phi, R)$ be an f-MTS which is given as follows (for simplicity, we assume that $\mathcal{P} = \{p\}$, $\mathcal{E} = \emptyset$ and $\mathcal{A} = \{\alpha\}$):

- $I = \{1, 2\}$;
- $M_1 = (\{w\}, V_1)$ with $V_1(p) = \{w\}$;
- $M_2 = (\{v, v'\}, V_2)$ with $V_2(p) = \{v\}$;
- $\Phi_\alpha(1) = 2$ and $\Phi_\alpha(2) = 2$;
- $R_1^\alpha = \{(w, v), (w, v')\}$, and R_2^α is empty.

Then, M validates $\mathsf{R_P}$ but not P. ⊣

4.3.2 Canonicity Results of the Axioms of PALg

We next focus on the axioms of $\mathsf{PALg} = \mathsf{K_g} \oplus \mathsf{R_T R_P R_N R_E R_G}$. We have already seen the following correspondence:

- $\mathsf{R_N}$ and determinism;
- $\mathsf{R_P}$ and atomic harmony (under $\mathsf{R_N}$).

It remains to consider:

- $\mathsf{R_G}$: $\langle\langle\alpha\rangle\rangle \mathrm{E}\varphi \leftrightarrow \langle\langle\alpha\rangle\rangle\top \wedge \mathrm{E}\langle\langle\alpha\rangle\rangle\varphi$;
- $\mathsf{R_E}$: $\langle\langle\alpha\rangle\rangle\langle e\rangle\varphi \leftrightarrow \langle\langle\alpha\rangle\rangle\top \wedge \langle e\rangle\langle\langle\alpha\rangle\rangle\varphi$;
- $\mathsf{R_T}$: $\langle\langle!\varphi\rangle\rangle\top \leftrightarrow \varphi$.

We look at these axioms one by one in this order, dividing each into the left direction ($\mathsf{R^L_G}$, $\mathsf{R^L_E}$ and $\mathsf{R^L_T}$) and the right ones ($\mathsf{R^R_G}$, $\mathsf{R^R_E}$ and $\mathsf{R^R_T}$). Note that axioms $\mathsf{R_G}$ and $\mathsf{R_E}$ can be written in the generic language whereas $\mathsf{R_T}$ is specific to $\mathcal{L}_{\mathsf{PALg}}$. We therefore discuss $\mathsf{R_G}$ and $\mathsf{R_E}$ in the general setting.

Remark 4.3.1. The semantic properties of axioms of PAL have been already discussed in [15] and [22]. Here we make a general and finer analysis, by using the generic language (as long as possible) and dividing the axioms into two directions. We then formulate the results in the form of canonicity. ⊣

$\mathsf{R}^{\mathsf{L}}_{\mathsf{G}}$ and G are equivalent under R_{N} It is often the case that an axiom attains a clear meaning when combined with R_{N}. A typical example is $\mathsf{R}^{\mathsf{L}}_{\mathsf{G}} : \langle\langle\alpha\rangle\rangle\top \wedge \mathrm{E}\langle\langle\alpha\rangle\rangle\varphi \to \langle\langle\alpha\rangle\rangle\mathrm{E}\varphi$, which turns out to be equivalent to G under the condition of R_{N}:

Proposition 4.3.4. $\mathsf{R}^{\mathsf{L}}_{\mathsf{G}} \leftrightarrow \mathsf{G}$ *is provable in* $\mathsf{K}_{\mathsf{g}} \oplus \mathsf{R}_{\mathsf{N}}$.

Proof. The canonicity of D_{t} $(= \mathsf{R}_{\mathsf{N}})$ for determinism implies the completeness of $\mathsf{K}_{\mathsf{g}} \oplus \mathsf{R}_{\mathsf{N}}$ for the class of deterministic MTSs. $\mathsf{R}^{\mathsf{L}}_{\mathsf{G}} \leftrightarrow \mathsf{G}$ is valid in this class. □

Corollary 4.3.2. $\mathsf{R}_{\mathsf{N}}\mathsf{R}^{\mathsf{L}}_{\mathsf{G}}$ *is canonical for determinism and partial functionality of outer relations.*

$\mathsf{R}^{\mathsf{R}}_{\mathsf{G}}$ and surjectivity of inner relations We next address the reverse direction $\mathsf{R}^{\mathsf{R}}_{\mathsf{G}} : \langle\langle\alpha\rangle\rangle\mathrm{E}\varphi \to \langle\langle\alpha\rangle\rangle\top \wedge \mathrm{E}\langle\langle\alpha\rangle\rangle\varphi$.

Proposition 4.3.5. $\mathsf{R}^{\mathsf{R}}_{\mathsf{G}}$ *is canonical for surjectivity of inner relations.*

Proof. (See also Proposition 47 of [22].) Let Λ be a basic DEL logic containing $\mathsf{R}^{\mathsf{R}}_{\mathsf{G}}$. We first prove that the canonical Kripke model $\mathcal{M}_\Lambda = (S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, R_{\mathrm{E}}, V)$ satisfies the property that if $wR_\alpha v$ and $vR_{\mathrm{E}}v'$, then there is a state w' such that $wR_{\mathrm{E}}w'$ and $w'R_\alpha v'$:

$$\begin{array}{ccc} w & \xrightarrow{\alpha} & v \\ {\scriptstyle \mathrm{E}}\,\vdots & & \downarrow{\scriptstyle \mathrm{E}} \\ {}^{\exists}w' & \xrightarrow{\alpha} & v' \end{array}$$

Let w^- be the union of the two sets of formulae: $\Phi = \{\varphi \in \mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A}) \mid \mathrm{A}\varphi \in w\}$ and $\Psi = \{\langle\langle\alpha\rangle\rangle\psi \in \mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{A}) \mid \psi \in v'\}$. To prove existence of such w', it suffices to show that w^- is Λ-consistent, owing to Lindenbaum's Lemma (Lemma 2.2.1). Let us suppose, towards a contradiction, that w^- is Λ-inconsistent, that is there are formulae $\varphi_1, \ldots, \varphi_n \in \Phi$ and $\langle\langle\alpha\rangle\rangle\psi_1, \ldots, \langle\langle\alpha\rangle\rangle\psi_m \in \Psi$ such that

$$\vdash_\Lambda \bigwedge_{i=1}^{n} \varphi_i \wedge \bigwedge_{j=1}^{m} \langle\langle\alpha\rangle\rangle\psi_j \to \bot.$$

We then obtain

$$\vdash_{\mathsf{\Lambda}} \bigwedge_{i=1}^{n} \varphi_i \wedge \langle\!\langle \alpha \rangle\!\rangle \bigwedge_{j=1}^{m} \psi_j \to \bot.$$

Here, $\bigwedge_{i=1}^{n} \varphi_i$ and $\langle\!\langle \alpha \rangle\!\rangle \bigwedge_{j=1}^{m} \psi_j$ belong to Φ and Ψ, respectively. Therefore, we may assume w.o.l.g. that $\vdash_{\mathsf{\Lambda}} \varphi \wedge \langle\!\langle \alpha \rangle\!\rangle \psi \to \bot$ holds for some $\varphi \in \Phi$ and $\langle\!\langle \alpha \rangle\!\rangle \psi \in \Psi$. Since ψ is in v', state w contains $\langle\!\langle \alpha \rangle\!\rangle \mathrm{E}\psi$, so that $\mathrm{E}\langle\!\langle \alpha \rangle\!\rangle \psi$ too by axiom $\mathsf{R}^{\mathsf{R}}_{\mathsf{G}}$. Moreover, w contains $\mathrm{A}\varphi$, since φ is an element of Φ. Therefore, there exists a state w'' which contains $\langle\!\langle \alpha \rangle\!\rangle \psi$ and φ. Hence, $\varphi \wedge \langle\!\langle \alpha \rangle\!\rangle \psi$ is $\mathsf{\Lambda}$-consistent, which contradicts the assumption.

We next consider the canonical MTS $\mathsf{M}_{\mathsf{\Lambda}} = (I, M, \Phi, R)$ and prove that each inner relation $R^{\alpha}_{(i,j)}$ is surjective. Let (i, j) be an arbitrary pair in Φ_α. Then, there are states $w \in i$ and $v \in j$ such that $wR_\alpha v$ by definition. Now take an arbitrary state v' in M_j, namely $v' \in j$. Thus, we have $vR_{\mathrm{E}}v'$, and hence we can apply the diagram above to obtain a state $w' \in i$ such that $w'R_\alpha v'$, which means $w'R^{\alpha}_{(i,j)}v'$. □

$\mathsf{R}^{\mathsf{L}}_{\mathsf{E}}$ and $\mathcal{E}$-preservation Schema $\mathsf{R}^{\mathsf{L}}_{\mathsf{E}} : \langle\!\langle \alpha \rangle\!\rangle \top \wedge \langle e \rangle \langle\!\langle \alpha \rangle\!\rangle \varphi \to \langle\!\langle \alpha \rangle\!\rangle \langle e \rangle \varphi$ in conjunction with R_{N} corresponds to the following condition:

Definition 4.3.4. Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS. We say that M satisfies *$\mathcal{E}$-preservation* if the following diagram is satisfied for any indices $i, j, j' \in I$, states $w, w' \in M_i$, $v \in M_j$, $v' \in M_{j'}$, epistemic expression $e \in \mathcal{E}$ and action expression $\alpha \in \mathcal{A}$:

$$\begin{array}{ccc} (M_i, w) & \xrightarrow{\alpha} & (M_j, v) \\ {\scriptstyle e}\downarrow & & \vdots\,{\scriptstyle e} \\ (M_i, w') & \xrightarrow[\alpha]{} & (M_{j'}, v') \end{array}$$

(and thus j and j' turn out to be equal). ⊣

This condition means that α-transitions never separate e-related states.

Proposition 4.3.6. $\mathsf{R}_{\mathsf{N}}\mathsf{R}^{\mathsf{L}}_{\mathsf{E}}$ *is canonical for determinism and $\mathcal{E}$-preservation.*

Proof. Let $\mathsf{\Lambda}$ be a basic DEL logic which contains R_{N} and $\mathsf{R}^{\mathsf{L}}_{\mathsf{E}}$. It is enough to show that its canonical Kripke model $\mathcal{M}_{\mathsf{\Lambda}} = (S, (R_e)_{e\in\mathcal{E}}, (R_\alpha)_{\alpha\in\mathcal{A}}, R_{\mathrm{E}}, V)$ satisfies the following diagram for any states w, w', v and v' in S:

$$\begin{array}{ccc} w & \xrightarrow{\alpha} & v \\ {\scriptstyle e}\downarrow & & \vdots\,{\scriptstyle e} \\ w' & \xrightarrow[\alpha]{} & v' \end{array}$$

Noting that v is the unique α-successor of w by $\mathsf{R_N}$, we can prove this in a similar way to the proof of Proposition 4.3.1. □

Remark 4.3.2. For f-MTSs, $\mathcal{E}$-preservation is equivalent to the *homomorphic condition*, which requires that each inner relation be homomorphic. ⊣

$\mathsf{R_E^R}$ and the back condition The reverse direction $\mathsf{R_E^R} : \langle\!\langle\alpha\rangle\!\rangle\langle e\rangle\varphi \to \langle\!\langle\alpha\rangle\!\rangle\top \wedge \langle e\rangle\langle\!\langle\alpha\rangle\!\rangle\varphi$ corresponds to the *back condition*:

Definition 4.3.5. Let $\mathsf{M} = (I, M, \Phi, R)$ be an MTS. M satisfies the *back condition* if it satisfies the following diagram for any states (M_i, w), (M_j, v), (M_j, v') in M and any epistemic expression $e \in \mathcal{E}$ and action expression $\alpha \in \mathcal{A}$:

$$\begin{array}{ccc} (M_i, w) & \xrightarrow{\ \alpha\ } & (M_j, v) \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e} \\ {}^{\exists}(M_i, w') & \overset{\cdot}{-}\!\!\succ_{\alpha} & (M_j, v') \end{array}$$

⊣

Proposition 4.3.7. $\mathsf{R_E^R}$ *is canonical for the back condition.*

Proof. This is proved as Proposition 4.3.5. □

The logic of eliminative updates Up to this point, we have shown canonicity results on $\mathsf{R_N}$, $\mathsf{R_P}$, $\mathsf{R_G}$ and $\mathsf{R_E}$. By combining them, we can prove that the logic $\mathsf{LEg} := \mathsf{K_g} \oplus \mathsf{R_N R_P R_G R_E}$ axiomatises the class of f-MTSs that express 'eliminative' updates. Thus, LEg may be regarded as the logic of eliminative updates.

By an eliminative update we mean a model transformation which eliminates some states from the original model. It is a typical way of updating knowledge, as exemplified by public announcement. To take another example, think of the computer game Minesweeper. Each time the player clicks on a cell on the board, the player's knowledge about the mines' locations is updated by eliminating possibilities where mines might be buried. In this sense, the action of clicking in Minesweeper can also be considered to cause an eliminative update.

We define functional MTSs which express such updates as follows (functionality is included for technical reasons (see Remark 4.3.3, *infra*)):

Definition 4.3.6 (Eliminative f-MTSs)**.** An f-MTS $\mathsf{M} = (I, M, \Phi, R)$ is called *eliminative* if: for any index $i \in I$ and action $\alpha \in \mathcal{A}$,

- the transformed model M_i^α is a submodel of M_i; and
- R_i^α is $\{(w, w) \in M_i \times M_i^\alpha \mid w \in M_i^\alpha \subseteq M_i\}$. $\dashv$

For example, PAL/PALg-generated ETL models as well as PAL/PALg transition systems are eliminative f-MTSs.

Proposition 4.3.8. LEg *is sound and strongly complete with respect to the class of eliminative f-MTSs.*

Proof. We prove the completeness only. Consider the canonical MTS $\mathsf{M}_{\mathsf{LEg}} = (I, M, \Phi, f)$. It follows from the previous results that each inner relation f_i^α is a surjective partial bounded morphism. Indeed, R_{N} confirms determinism, i.e. partial functionality, and other properties are obtained (with the help of R_{N}) as follows:

$\mathsf{R}_{\mathsf{E}}^{\mathsf{L}}$	the homomorphic condition	$\mathsf{R}_{\mathsf{E}}^{\mathsf{R}}$	the back condition
R_{P}	atomic harmony	$\mathsf{R}_{\mathsf{G}}^{\mathsf{R}}$	surjectivity

In addition, schema $\mathsf{R}_{\mathsf{G}}^{\mathsf{L}}$ assures the partial functionality of the outer relations $\mathsf{M}_{\mathsf{LEg}}$; therefore, we can regard $\mathsf{M}_{\mathsf{LEg}}$ as an f-MTS, based on the M^+-construction. Note that the additional inner relations are empty and thus considered as surjective partial bounded morphisms.

Let us next consider an arbitrary LEg-consistent set Γ, which is satisfied at some state (M_i, w) in $\mathsf{M}_{\mathsf{LEg}}$. Hence, considering the unravelling $\vec{\mathsf{M}}_{\mathsf{LEg}} = (\mathcal{A}^*, \vec{M}, \Phi^{\mathcal{A}}, \vec{f})$ around $i \in I$, it is satisfied at $(\vec{M}_\epsilon, w)$ (see Proposition 3.3.3, *infra*, and Section 4.2, *supra*).

We then transform $\vec{\mathsf{M}}_{\mathsf{LEg}}$ into an eliminative f-MTS $\mathsf{N}_{\mathsf{LEg}} = (\mathcal{A}^*, N, \Phi^{\mathcal{A}}, Q)$ by replacing their inner models and relations with those given as below:

- N_γ for $\gamma \in \mathcal{A}^*$ is the submodel of $\vec{M}_\epsilon$ whose carrier set is the domain of the partial function $\vec{f}_\epsilon^{\,\gamma} : \vec{M}_\epsilon \rightharpoonup \vec{M}_\gamma$;[2]
- Q_γ^α is $\{(v, v) \in N_\gamma \times N_{\gamma\alpha} \mid v \in N_{\gamma\alpha} \subseteq N_\gamma\}$ for $\gamma \in \mathcal{A}^*$ and $\alpha \in \mathcal{A}$.

Then, we have the g-bounded morphism $\mathsf{g}_{\mathsf{LEg}} = (\mathrm{id}_{\mathcal{A}^*}, (g_\gamma)_{\gamma \in \mathcal{A}^*}) : \mathsf{N}_{\mathsf{LEg}} \to \vec{\mathsf{M}}_{\mathsf{LEg}}$, where $g_\gamma : N_\gamma \to \vec{M}_\gamma$ is the restriction of $\vec{f}_\epsilon^{\,\gamma}$ to its domain N_γ. The function $\mathrm{id}_{\mathcal{A}^*}$ is trivially a frame bounded morphism. We can also readily see that $\mathsf{g}_{\mathsf{LEg}}$ satisfies the homomorphic and back conditions:

$$\begin{array}{ccc} (N_\gamma, v) & \xmapsto{\mathsf{g}_{\mathsf{LEg}}} & (\vec{M}_\gamma, \vec{f}_\epsilon^{\,\gamma}(v)) \\ \alpha \downarrow & & \vdots\ \alpha \downarrow \\ (N_{\gamma\alpha}, v) & \xmapsto[\mathsf{g}_{\mathsf{LEg}}]{} & (\vec{M}_{\gamma\alpha}, \vec{f}_\epsilon^{\,\gamma\alpha}(v)) \end{array} \qquad \begin{array}{ccc} (N_\gamma, v) & \xmapsto{\mathsf{g}_{\mathsf{LEg}}} & (\vec{M}_\gamma, \vec{f}_\epsilon^{\,\gamma}(v)) \\ \alpha \vdots & & \downarrow \alpha \\ {}^{\exists}(N_{\gamma\alpha}, v) & \vdash\!\!\!\!\!\!\rightarrow_{\mathsf{g}_{\mathsf{LEg}}} & (\vec{M}_{\gamma\alpha}, \vec{f}_\epsilon^{\,\gamma\alpha}(v)) \end{array}$$

[2] $\vec{f}_\epsilon^{\,\gamma}$ for a sequence $\gamma = \alpha_1 \cdots \alpha_n \in \mathcal{A}^*$ is the natural extension of $\vec{f}_\epsilon^{\,\alpha}$ for $\alpha \in \mathcal{A}$ (see Section 3.2.1).

As for the component bounded morphisms g_γ, all inner relations $\vec{f}^{\alpha}_{\gamma}$ of $\vec{\mathsf{M}}_{\mathsf{LEg}}$ are surjective partial bounded morphisms and, as a consequence, so are $\vec{f}^{\gamma}_{\epsilon}$. Hence, each g_γ is a surjective (total) bounded morphism.

We therefore obtain:

$$\mathsf{g}_{\mathsf{LEg}} : \mathsf{N}_{\mathsf{LEg}} \to \vec{\mathsf{M}}_{\mathsf{LEg}}.$$

Hence, we can conclude that Γ is satisfiable in $\mathsf{N}_{\mathsf{LEg}}$ and thus in the class of eliminative f-MTSs. □

Remark 4.3.3. It is possible to extend the definition of eliminativity to non-functional MTSs. However, R_{E} is no longer suitable for axiomatising eliminativity, as the following example illustrates, and it is not clear whether there is a complete axiomatisation of the class of eliminative MTSs. On the other hand, if we consider the class of *eliminative f-FTSs*, the logic $\mathsf{K}_{\mathsf{g}} \oplus \mathsf{R}_{\mathsf{N}}\mathsf{R}_{\mathsf{G}}\mathsf{R}_{\mathsf{E}}$ axiomatises the class. ⊣

Example 4.3.2. Assume that $\mathcal{P} = \{p\}$ and $\mathcal{A} = \{\alpha\}$, and that $\mathcal{E}$ is a singleton (thus we suppress superscripts for it). Consider the following MTS $\mathsf{M} = (I, M, \Phi, R)$:

- $I = \{1, 2, 3\}$;
- $M_1 = (\{w, v\}, R_1, V_1)$ with $R_1 = \{(w, v)\}$ and $V_1(p) = \{v\}$,
- $M_2 = (\{w\}, R_2, V_2)$ with $R_2 = \emptyset$ and $V_2(p) = \emptyset$,
- $M_3 = (\{v\}, R_3, V_3)$ with $R_3 = \emptyset$ and $V_3(p) = \{v\}$;
- $\Phi_\alpha = \{(1, 2), (1, 3)\}$;
- $wR^{\alpha}_{(1,2)}w$ and $vR^{\alpha}_{(1,3)}v$.

Then, this MTS is eliminative but state w does not satisfy $\mathsf{R}^{\mathsf{L}}_{\mathsf{E}} : \langle\!\langle\alpha\rangle\!\rangle\top \wedge \langle e\rangle\langle\!\langle\alpha\rangle\!\rangle p \to \langle\!\langle\alpha\rangle\!\rangle\langle e\rangle p$ for the atomic proposition p. ⊣

Properties corresponding to $\mathsf{R}^{\mathsf{L}}_{\mathsf{T}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{T}}$ Let us return to our exploration of canonicity. Two schemata remain to be considered: $\mathsf{R}^{\mathsf{L}}_{\mathsf{T}} : \varphi \to \langle\!\langle !\varphi\rangle\!\rangle\top$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{T}} : \langle\!\langle !\varphi\rangle\!\rangle\top \to \varphi$. Notice that these schemata are specific to the language $\mathcal{L}_{\mathsf{PALg}}$. Therefore, so are their corresponding properties, which are given below:

Proposition 4.3.9.

1. $\mathsf{R}^{\mathsf{L}}_{\mathsf{T}}$ *is canonical for the property that, for any state, satisfying φ implies having a $!\varphi$-successor.*
2. $\mathsf{R}^{\mathsf{R}}_{\mathsf{T}}$ *is canonical for the property that, for any state, having a $!\varphi$-successor implies satisfying φ.*

The logic TPALg contains $\mathsf{R}^{\mathsf{R}}_{\mathsf{T}}$ (see Section 4.1). Its completeness is proved by using the proposition above:

Theorem 4.3.2. TPALg *is sound and strongly complete with respect to the class of PALg-generated ETL models.*

Proof. We prove the completeness only. By the construction used in the proof of Proposition 4.3.8, we have an eliminative f-MTS $\mathsf{N}_{\mathsf{TPAL}} = (\mathcal{A}^*_{\mathsf{PALg}}, N, \Phi^{\mathcal{A}_{\mathsf{PALg}}}, Q)$ and a g-bounded morphism $\mathsf{f}_{\mathsf{TPALg}}$ for TPALg:

$$\mathsf{f}_{\mathsf{TPALg}} : \mathsf{N}_{\mathsf{TPALg}} \to \vec{\mathsf{M}}_{\mathsf{TPALg}}.$$

We consider here the state-dependent PAL protocol $P : N_\epsilon \to \mathrm{Ptcl}(\mathcal{A}_{\mathsf{PALg}})$ defined by $P(v) = \{\gamma \in \mathcal{A}^*_{\mathsf{PALg}} \mid \langle\langle\gamma\rangle\rangle\top \in v\}$, and prove $\mathsf{N}_{\mathsf{TPALg}} = \mathsf{Forest}_{\mathsf{TPALg}}(N_\epsilon, P)$. It is enough to verify the following (1) and (2), by induction on φ, for all γ simultaneously:

1. $(N_\gamma, P_\gamma)_{!\varphi} = (N_{\gamma!\varphi}, P_{\gamma!\varphi})$;
2. $\mathsf{N}_\mathsf{\Lambda}, N_\gamma, v \models \varphi \iff (N_\gamma, P_\gamma), v \Vdash \varphi$ for any $v \in N_\gamma$.

Here, P_γ is the state-dependent PAL protocol given by $P_\gamma(v) = \{\gamma' \in \mathcal{A}^*_{\mathsf{PALg}} \mid \gamma\gamma' \in P(v)\}$. The canonicity results for $\mathsf{R}^{\mathsf{R}}_{\mathsf{T}}$ is used to prove that the item (2) implies (1) for the $\langle\langle!\varphi\rangle\rangle$-steps. □

Since the dynamic and static interpretations of a formula coincide, we also obtain:

Corollary 4.3.3. TPALg *is sound and strongly complete with respect to the class of pairs of a Kripke model and a state-dependent PALg protocol.*

Completeness of PALg Let us now prove the completeness of PALg.

Theorem 4.3.3. PALg *is sound and strongly complete with respect to the class of PALg transition systems.*

Proof. Based on the construction in the proof of Proposition 4.3.8, we obtain an eliminative f-MTS and a g-bounded morphism for PALg:

$$\mathsf{f}_{\mathsf{PALg}} : \mathsf{N}_{\mathsf{PALg}} \to \vec{\mathsf{M}}_{\mathsf{PALg}}.$$

Since $\vec{\mathsf{M}}_{\mathsf{PALg}}$ validates R_T, so does $\mathsf{N}_{\mathsf{PALg}}$. Using this fact, we can easily see that $\mathsf{N}_{\mathsf{PALg}}$ is a PALg transition system by Proposition 4.3.9. □

This leads to a reduction-free proof of Plaza's completeness theorem, since PALg transition systems simulate Plaza's model transformation:

Corollary 4.3.4 ([11, 22])**.** PALg *is sound and strongly complete with respect to Plaza's semantics.*

4.3.3 Canonicity Results of the Axioms of EAg

Let us now leave PALg and turn to the axioms of $\mathsf{EAg} = \mathsf{K_g} \oplus \mathsf{R_{T(EA)}R_PR_NR_{E(EA)}R_{G(EA)}}$. Note that, unlike before, the following discussion is *specific* to the language $\mathcal{L}_{\mathsf{EAg}}$, since the axiom schemata are written in the language. We therefore focus on which axiom corresponds to which property of EAg transition systems.

We consider here state-independent MTSs, in accordance with EAg transition systems,[3] and correspondingly we modify the notion of a canonical MTS. For a basic DEL logic Λ in $\mathcal{L}_{\mathsf{EAg}}$, the *EA-canonical MTS* $\mathsf{M}_\Lambda = (I, M, \Phi, R)$ is defined to be the state-independent MTS such that:

- I, M and R are the same as those of the ordinary canonical MTS;
- Φ is defined by

$$(i,j) \in \Phi_U \iff (i \times j) \cap \bigcup \{R_{(U,s)} \mid s \in U\} \neq \emptyset,$$

  where $R_{(U,s)}$ is the binary relation in the canonical Kripke model $\mathcal{M}_\Lambda$ associated to action (U, s).

The definition of canonicity is also modified:

Definition 4.3.7. Let A be a schema and P a property. A is *EA-canonical* for P if:

- the EA-canonical MTS $\mathsf{M}_\Lambda = (I, M, \Phi, R)$ satisfies P for any basic DEL logic Λ containing A; and
- A is valid in any state-independent MTS that satisfies P. ⊣

Again, from the EA-canonicity of a schema A for a property P, we can immediately obtain the soundness and strong completeness of the logic $\mathsf{K_g} \oplus \mathsf{A}$ with respect to the class of state-independent MTSs with the property P.

The general canonicity results in Section 4.3.1, *supra*, induce the following EA canonicity results:

- $\mathsf{D_t}$ ($\mathsf{R_N}$) is EA-canonical for determinism;
- P is EA-canonical for atomic harmony;
- $\mathsf{R_NR_P}$ is EA-canonical for determinism and atomic harmony.

Therefore, we consider here:[4]

- $\mathsf{R_{G(EA)}} : \langle\!\langle (U,s) \rangle\!\rangle \mathrm{E}\varphi \leftrightarrow \langle\!\langle (U,s) \rangle\!\rangle \top \wedge \mathrm{E}\langle\!\langle U \rangle\!\rangle \varphi$;
- $\mathsf{R_{E(EA)}} : \langle\!\langle (U,s) \rangle\!\rangle \langle n \rangle \varphi \leftrightarrow \langle\!\langle (U,s) \rangle\!\rangle \top \wedge \langle n \rangle \langle\!\langle U^n_s \rangle\!\rangle \varphi$;
- $\mathsf{R_{T(EA)}} : \langle\!\langle (U,s) \rangle\!\rangle \top \leftrightarrow \mathrm{Pre}_U(s)$.

As before, we divide them into the left directions ($\mathsf{R^L_{G(EA)}}$, $\mathsf{R^L_{E(EA)}}$ and $\mathsf{R^L_{T(EA)}}$) and the right ones ($\mathsf{R^R_{G(EA)}}$, $\mathsf{R^R_{E(EA)}}$ and $\mathsf{R^R_{T(EA)}}$), and give a canonicity result for each.

[3] See footnote 2 in Section 3.2.3, regarding notations for state-independent MTSs.

[4] See Sections 2.4.2 and 4.1 for the definitions of $\langle\!\langle U^s_n \rangle\!\rangle$ and $\langle\!\langle U \rangle\!\rangle$, respectively.

Properties corresponding to $\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$ We consider $\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$ at a time. The canonicity results for them are proved in parallel.

We first consider the defined operator $\langle\!\langle U\rangle\!\rangle\varphi = \bigvee\{\langle\!\langle (U,s)\rangle\!\rangle\varphi \mid s \in U\}$. It corresponds to each union $R^U_{(i,j)} := \bigcup\{R^{(U,s)}_{(i,j)} \mid s \in U\}$ of inner relations in a state-independent MTS and the union $R_U := \bigcup\{R_{(U,s)} \mid s \in U\}$ in a canonical Kripke model.[5] In addition, the relation R_U is canonical for $\langle\!\langle U\rangle\!\rangle$ in the following sense (cf. Definition 2.2.4):

Lemma 4.3.1. *Suppose that Λ is a basic DEL logic in the language $\mathcal{L}_{\mathsf{EAg}}$ and that $\mathcal{M}_\Lambda = (S, (R_n)_{n\in Ag}, (R_{(U,s)})_{(U,s)\in\mathcal{A}_{\mathsf{EAg}}}, R_{\mathrm{E}}, V)$ is its canonical Kripke model. Then, for any states w and v in S,*

$$wR_Uv \text{ iff } \varphi \in v \text{ implies } \langle\!\langle U\rangle\!\rangle\varphi \in w \text{ for any formula } \varphi \in \mathcal{L}_{\mathsf{EAg}}.$$

Proof. The right direction is straightforward. We prove the left one. Suppose, towards a contradiction, that there is no $s \in U$ such that $wR_{(U,s)}v$. Thus, for each $s \in U$, there exists a formula $\varphi_s \in v$ such that $\langle\!\langle (U,s)\rangle\!\rangle\varphi_s \notin w$. Then, the conjunction $\bigwedge\varphi_s$ of all φ_s ($s \in U$) belongs to v and therefore $\langle\!\langle U\rangle\!\rangle\bigwedge\varphi_s \in w$. Hence, $\langle\!\langle (U,t)\rangle\!\rangle\bigwedge\varphi_s$ is in w for some $t \in U$, so that we have $\langle\!\langle (U,t)\rangle\!\rangle\varphi_t \in w$, which is a contradiction. □

Based on the observation above, we can see that schema $\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$ in conjunction with R_{N} corresponds to the functionality of outer relations and $\mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$ to the surjectivity of the defined inner relations $R^U_{(i,j)}$. We call the latter property *EA-surjectivity*.

Lemma 4.3.2.

1. $\mathrm{E}\langle\!\langle U\rangle\!\rangle\varphi \to [[U]]\mathrm{E}\varphi$ *is EA-canonical for partial functionality of outer relations.*
2. $\langle\!\langle U\rangle\!\rangle\mathrm{E}\varphi \to \mathrm{E}\langle\!\langle U\rangle\!\rangle\varphi$ *is EA-canonical for EA-surjectivity.*

Proof. $\mathrm{E}\langle\!\langle U\rangle\!\rangle\varphi \to [[U]]\mathrm{E}\varphi$ and $\langle\!\langle U\rangle\!\rangle\mathrm{E}\varphi \to \mathrm{E}\langle\!\langle U\rangle\!\rangle\varphi$ play the roles of G and $\mathsf{R}^{\mathsf{R}}_{\mathsf{G}}$, respectively (see Propositions 4.3.1 and 4.3.5). □

Proposition 4.3.10.

1. $\mathsf{R}_{\mathsf{N}}\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$ *is EA-canonical for determinism and partial functionality of outer relations.*
2. $\mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$ *is EA-canonical for EA-surjectivity.*

Proof. The schemata $\mathsf{R}_{\mathsf{N}}\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$ imply $\mathrm{E}\langle\!\langle U\rangle\!\rangle\varphi \to [[U]]\mathrm{E}\varphi$ and $\langle\!\langle U\rangle\!\rangle\mathrm{E}\varphi \to \mathrm{E}\langle\!\langle U\rangle\!\rangle\varphi$, respectively. □

[5] Therefore, an EA-canonical MTS $\mathsf{M} = (I, M, \Phi, R)$ can be viewed as 'basic' for the defined operators $\langle\!\langle U\rangle\!\rangle$. Actually, we have the following fact: $\mathsf{M}, M_i, w \models \mathrm{E}\langle\!\langle U\rangle\!\rangle\top$ for some $w \in M_i \iff (I,\Phi), i \models \langle\!\langle U\rangle\!\rangle\top$ (see the subheading 'Basic MTSs' in Section 4.1).

Properties corresponding to $\mathsf{R}^{\mathsf{L}}_{\mathsf{E(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{E(EA)}}$ Again, we consider two schemata $\mathsf{R}^{\mathsf{L}}_{\mathsf{E(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{E(EA)}}$ simultaneously.

This time, the defined operator $\langle\langle U^n_s\rangle\rangle\varphi = \bigvee\{\langle\langle (U,t)\rangle\rangle\varphi \mid s \to_n t\}$ corresponds to $R^{U^n_s}_{(i,j)} := \bigcup\{R^{(U,t)}_{(i,j)} \mid s \to_n t\}$ in a state-independent MTS and $R_{U^n_s} := \bigcup\{R_{(U,t)} \mid s \to_n t\}$ in a canonical Kripke model. For the latter relation, the 'canonicity' holds again:

Lemma 4.3.3. *Suppose that Λ is a basic DEL logic in the language $\mathcal{L}_{\mathsf{EAg}}$ and $\mathcal{M}_\Lambda = (S, (R_n)_{n\in Ag}, (R_{(U,s)})_{(U,s)\in\mathcal{A}_{\mathsf{EAg}}}, R_{\mathrm{E}}, V)$ is its canonical Kripke model. Then, for any states w and v in S,*

$$wR_{U^n_s}v \text{ iff } \varphi \in v \text{ implies } \langle\langle U^n_s\rangle\rangle\varphi \in w \text{ for any formula } \varphi \in \mathcal{L}_{\mathsf{EAg}}.$$

Proof. This is proved as Lemma 4.3.1. □

The schemata $\mathsf{R}^{\mathsf{L}}_{\mathsf{E(EA)}}$, with R_{N}, and $\mathsf{R}^{\mathsf{R}}_{\mathsf{E(EA)}}$ correspond to the conditions below, respectively, and this is proved as in the cases of $\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$.

Definition 4.3.8. Let $\mathsf{M} = (I, M, \Phi, R)$ be a state-independent MTS.

- M satisfies *EA $\mathcal{E}$-preservation* if it satisfies the following diagram:

$$\begin{array}{ccc} (M_i, w) & \xrightarrow{(U,s)} & (M_j, v) \\ {\scriptstyle n}\big\downarrow & & \big\downarrow{\scriptstyle n} \\ (M_i, w') & \xrightarrow[U^n_s]{} & (M_{j'}, v') \end{array}$$

  for any indices $i, j, j' \in I$, states $w, w' \in M_i$, $v \in M_j$, $v' \in M_{j'}$, agent $n \in Ag$ and action $(U, s) \in \mathcal{A}_{\mathsf{EAg}}$.
- M satisfies the *EA back condition* if it satisfies the diagram below:

$$\begin{array}{ccc} (M_i, w) & \xrightarrow{(U,s)} & (M_j, v) \\ {\scriptstyle n}\big\downarrow & & \big\downarrow{\scriptstyle n} \\ {}^{\exists}(M_i, w') & \xrightarrow[U^n_s]{} & (M_j, v') \end{array}$$

  for any states (M_i, w), (M_j, v), (M_j, v') in M, agent $n \in Ag$ and action $(U, s) \in \mathcal{A}_{\mathsf{EAg}}$. ⊣

Lemma 4.3.4.

1. *$\langle n\rangle\langle\langle U^n_s\rangle\rangle\varphi \to [[(U, s)]]\langle n\rangle\varphi$ is EA-canonical for EA $\mathcal{E}$-preservation.*
2. *$\langle\langle (U, s)\rangle\rangle\langle n\rangle\varphi \to \langle n\rangle\langle\langle U^n_s\rangle\rangle\varphi$ is EA-canonical for the EA back condition.*

Proof. Items 1 and 2 are proved in similar ways to proofs of Propositions 4.3.1 and 4.3.5, respectively. □

Proposition 4.3.11.

1. $\mathsf{R_N R^L_{E(EA)}}$ *is EA-canonical for determinism and EA* $\mathcal{E}$*-preservation.*
2. $\mathsf{R^R_{E(EA)}}$ *is EA-canonical for the EA back condition.*

Proof. $\mathsf{R_N R^L_{E(EA)}}$ and $\mathsf{R^R_{E(EA)}}$ imply $\langle n\rangle\langle\!\langle U^n_s\rangle\!\rangle\varphi \to [[(U,s)]]\langle n\rangle\varphi$ and $\langle\!\langle (U,s)\rangle\!\rangle\langle n\rangle\varphi \to \langle n\rangle\langle\!\langle U^n_s\rangle\!\rangle\varphi$, respectively. □

The logic of subproduct updates Let $\mathsf{LSPg} := \mathsf{K_g} \oplus \mathsf{R_N R_P R_{G(EA)} R_{E(EA)}}$. It almost corresponds to the logic LEg, discussed in Proposition 4.3.8, but this time completeness holds for the class of subproduct f-MTSs, which we define below; thus, LSPg may be regarded as the logic of subproduct updates.

Definition 4.3.9 (Subproduct f-MTSs)**.** We first define a model transformation. Given a Kripke model $N = (S, (Q_n)_{n\in Ag}, V)$ and an action model $U = (U, (\to_n)_{n\in Ag}, \mathrm{Pre}_U)$, the transformed model $N \times U = (S', (Q'_n)_{n\in Ag}, V')$ is given by:

- $S' = S \times U$;
- $(v,s)Q'_n(w,t) \iff vQ_nw$ and $s \to_n t$;
- $(v,s) \in V'(p) \iff v \in V(p)$.

A *subproduct f-MTS* (I, M, Φ, R) is defined to be a state-independent f-MTS such that: for any index $i \in I$ and action model $U \in |\mathcal{A}_{\mathsf{EAg}}|$,

- M_i^U is a submodel of $M_i \times U$; and
- $R_i^{(U,s)} = \{(w,(w,s)) \in M_i \times M_i^U \mid (w,s) \in M_i^U\}$ for any state $s \in U$. ⊣

Examples of subproduct f-MTSs include EAg transition systems and EAg-generated ETL models.

Proposition 4.3.12. LSPg *is sound and strongly complete with respect to the class of subproduct f-MTS.*

Proof. The proof is analogous to that of Proposition 4.3.8. We give here a proof sketch.

Let $\mathsf{M} = (I, M, \Phi, R)$ be the EA-canonical MTS for LSPg. By its EA-canonicity result, $\mathsf{R_N}$ ensures determinism of M. Further, the following properties are also obtained (with the help of $\mathsf{R_N}$) by their EA-canonicity results:

$\mathsf{R^L_{E(EA)}}$	EA $\mathcal{E}$-preservation	$\mathsf{R^R_{E(EA)}}$	the EA back condition
$\mathsf{R_P}$	atomic harmony	$\mathsf{R^R_{G(EA)}}$	EA-surjectivity

Moreover, owing to $\mathsf{R}^{\mathsf{L}}_{\mathsf{G(EA)}}$, the outer relations of M are partial functions and thus we may regard it as an f-MTS.

We next take an arbitrary LSPg-consistent set Γ. This set Γ is satisfied at some state (M_i, w) in M. Hence, it is also satisfied at $(\vec{M}_\epsilon, w)$ in the 'unravelling' $\vec{\mathsf{M}} = (|\mathcal{A}_{\mathsf{EAg}}|^*, \vec{M}, \Phi^{|\mathcal{A}_{\mathsf{EAg}}|}, \vec{R})$ of M around $i \in I$.

We then transform $\vec{\mathsf{M}}$ into a subproduct f-MTS $\mathsf{N} = (|\mathcal{A}_{\mathsf{EAg}}|^*, N, \Phi^{|\mathcal{A}_{\mathsf{EAg}}|}, Q)$ by replacing the inner Kripke models and inner relations with the following models and relations: for each $\vec{U} \in |\mathcal{A}_{\mathsf{EAg}}|^*$,

- the submodel $N_{\vec{U}}$ of $\vec{M}_\epsilon \times \vec{U}$ with the carrier set

$$\{(v, \vec{s}) \in \vec{M}_\epsilon \times \vec{U} \mid (\vec{M}_\epsilon, v) \text{ has a } (\vec{U}, \vec{s})\text{-successor in } \vec{\mathsf{M}}\};^{6}$$

- $Q^{(U,s)}_{\vec{U}} = \{((v, \vec{s}), (v, \vec{s}, s)) \in N_{\vec{U}} \times N_{\vec{U}U} \mid (v, \vec{s}, s) \in N_{\vec{U}U}\}$ for $(U, s) \in \mathcal{A}_{\mathsf{EAg}}$.

Here, $\vec{M}_\epsilon \times \vec{U}$ is the result of successively applying the sequence $\vec{U}$ of action models to $\vec{M}_\epsilon$ (see Definition 4.3.9).

Then, we have the g-bounded morphism $\mathsf{f} = (\mathrm{id}_{|\mathcal{A}_{\mathsf{EAg}}|^*}, (f_{\vec{U}})_{\vec{U} \in |\mathcal{A}_{\mathsf{EAg}}|^*}) : \mathsf{N} \to \vec{\mathsf{M}}$, where $f_{\vec{U}} : N_{\vec{U}} \to \vec{M}_{\vec{U}}$ assigns to $(v, \vec{s})$ the unique $(\vec{U}, \vec{s})$-successor of $(\vec{M}_\epsilon, v)$ in $\vec{\mathsf{M}}$.

From the discussion above, we can conclude Γ is satisfiable in N and hence in the class of subproduct f-MTSs. □

Remark 4.3.4. We can also axiomatise the class of *subproduct f-FTSs* by $\mathsf{K_g} \oplus \mathsf{R_N R_{G(EA)} R_{E(EA)}}$. On the other hand, it is not clear whether the class of *subproduct MTSs* can be axiomatised (cf. Remark 4.3.3). ⊣

Properties corresponding to $\mathsf{R}^{\mathsf{L}}_{\mathsf{T(EA)}}$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{T(EA)}}$ The remaining schemata are $\mathsf{R}^{\mathsf{L}}_{\mathsf{T(EA)}} : \mathrm{Pre}_U(s) \to \langle\!\langle (U, s) \rangle\!\rangle \top$ and $\mathsf{R}^{\mathsf{R}}_{\mathsf{T(EA)}} : \langle\!\langle (U, s) \rangle\!\rangle \top \to \mathrm{Pre}_U(s)$. They correspond to the following properties:

Proposition 4.3.13.

- $\mathsf{R}^{\mathsf{L}}_{\mathsf{T(EA)}}$ *is EA-canonical for the property that, for any state, satisfying* $\mathrm{Pre}_U(s)$ *implies having a* (U, s)*-successor.*
- $\mathsf{R}^{\mathsf{R}}_{\mathsf{T(EA)}}$ *is EA-canonical for the property that, for any state, having a* (U, s)*-successor implies satisfying* $\mathrm{Pre}_U(s)$*.*

Logic TEAg contains $\mathsf{R}^{\mathsf{R}}_{\mathsf{T(EA)}}$ (see Section 4.1), and its completeness is obtained by using the proposition above:

[6] To denote a sequence of pointed action models, we write $(\vec{U}, \vec{s})$, instead of $(U_1, s_1) \cdots (U_m, s_m)$.

Theorem 4.3.4. TEAg *is sound and strongly complete with respect to the class of EAg-generated ETL models.*

Proof. This is proved as Theorem 4.3.2, by using Proposition 4.3.13 and the construction used in the proof of Proposition 4.3.12. □

Corollary 4.3.5. TEAg *is sound and strongly complete with respect to the class of pairs of a Kripke model and a state-dependent EAg protocol.*

Completeness of EAg As Theorem 4.3.3, *supra*, by combining Proposition 4.3.13 and the construction in the proof of Proposition 4.3.12, we obtain:

Theorem 4.3.5. EAg *is sound and strongly complete with respect to the class of EAg transition systems.*

Consequently, we also obtain a reduction-free proof of the completeness for Baltag *et al.*'s semantics:

Corollary 4.3.6 ([3])**.** EAg *is sound and strongly complete with respect to Baltag et al.'s semantics.*

4.3.4 Summary of the Canonicity and Completeness Results

We summarise the canonicity and completeness results we have obtained so far.

Canonicity Canonicity results in each group below can be freely combined like ordinary canonicity results (cf. Subsection 2.2.1).

General Canonicity

G	partial functionality of outer relations
$\mathsf{D_t}$ ($\mathsf{R_N}$)	determinism
P	atomic harmony
$\mathsf{R_N R_P}$	determinism and atomic harmony
$\mathsf{R_N R^L_G}$	determinism and partial functionality of outer relations
$\mathsf{R^R_G}$	surjectivity of inner relations
$\mathsf{R_N R^L_E}$	determinism and $\mathcal{E}$-preservation
$\mathsf{R^R_E}$	the back condition

Canonicity in $\mathcal{L}_{\mathsf{PALg}}$

$\mathsf{R^L_T}$	satisfying φ implies having a $!\varphi$-successor
$\mathsf{R^R_T}$	having a $!\varphi$-successor implies satisfying φ

EA-canonicity	
$\mathsf{D_t}$ ($\mathsf{R_N}$)	determinism
P	atomic harmony
$\mathsf{R_N R_P}$	determinism and atomic harmony
$\mathsf{R_N R^L_{G(EA)}}$	determinism and partial functionality of outer relations
$\mathsf{R^R_{G(EA)}}$	EA-surjectivity
$\mathsf{R_N R^L_{E(EA)}}$	determinism and EA $\mathcal{E}$-preservation
$\mathsf{R^R_{E(EA)}}$	the EA back condition
$\mathsf{R^L_{T(EA)}}$	satisfying $\mathrm{Pre}_U(s)$ implies having a (U,s)-successor
$\mathsf{R^R_{T(EA)}}$	having a (U,s)-successor implies satisfying $\mathrm{Pre}_U(s)$

Completeness The above canonicity results are accompanied by corresponding soundness and strong completeness results. Taking less obvious facts (Propositions 4.3.8 and 4.3.12, *etc.*) into account, we also obtain the following correspondence:

$\mathsf{K_g}$	MTSs
$\mathsf{K_g \oplus G}$	f-MTSs
$\mathsf{K_g \oplus GP}$	epistemic f-MTSs
$\mathsf{K_g \oplus GR_N}$	deterministic f-MTSs
$\mathsf{K_g \oplus GR_N R_P}$	deterministic and epistemic f-MTSs
$\mathsf{K_g \oplus R_N R_G R_E}$	eliminative f-FTSs
$\mathsf{K_g \oplus R_N R_G R_E R_P}$ (LEg)	eliminative f-MTSs
TPALg	PALg-generated ETL models
PALg	PALg transition systems
$\mathsf{K_g \oplus R_N R_{G(EA)} R_{E(EA)}}$	subproduct f-FTSs
$\mathsf{K_g \oplus R_N R_{G(EA)} R_{E(EA)} R_P}$ (LSPg)	subproduct f-MTSs
TEAg	EAg-generated ETL models
EAg	EAg transition systems

4.4 Conservativity

The global operator was introduced as an auxiliary operator. Therefore, it is expected that the operator conservatively extends the original logics. In this subsection we see that many logics, including those which appeared in the previous section, are indeed conservative.[7]

Throughout this section, we assume that $\mathcal{A}$ and $\mathcal{B}$ are sets of action expressions such that $\mathcal{A} \subseteq \mathcal{B}$ (e.g. $\mathcal{A} = \mathcal{A}_{\mathsf{PAL}}$ and $\mathcal{B} = \mathcal{A}_{\mathsf{PALg}}$). Thus, we have $\mathcal{L}(\mathcal{E}, \mathcal{A}) \subseteq \mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$.

[7]Conservativity gives a justification for the use of the global operator in our exploration of canonicity, since the conservativity of $\mathsf{K_g \oplus A}$ over $\mathsf{K \oplus A}$ ensures that the canonicity of A for P implies the soundness and strong completeness of $\mathsf{K \oplus A}$ with respect to the class of MTSs with P.

General method for conservativity We first give a general method to prove conservativity under a specific condition.

In our proof, we use the following fact: a Kripke model for $\mathcal{L}(\mathcal{E}, \mathcal{A})$ can be naturally regarded as one for the extended language $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$ by associating the empty relation to each action in $\mathcal{B} - \mathcal{A}$. This leads to an obvious conservativity result:

Proposition 4.4.1. K_{g} *in* $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$ *is conservative over* K *in* $\mathcal{L}(\mathcal{E}, \mathcal{A})$.

Proof. Let us consider the canonical Kripke model $\mathcal{M}_{\mathsf{K}}$ for $\mathcal{L}(\mathcal{E}, \mathcal{A})$. This model can be viewed as a model for $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$, and K_{g} is sound with respect to this model. Therefore, if a formula φ in $\mathcal{L}(\mathcal{E}, \mathcal{A})$ is provable in K_{g}, then $\mathcal{M}_{\mathsf{K}}$ validates φ and thus φ is also provable in K. □

Let us next think about adding axioms. An additional axiom to K or K_{g} is usually given in the form of a schema. We prove here the conservativity of logics defined by *canonical* schemata (other cases are reserved for future work):

Definition 4.4.1. Let A be a schema written in $\mathcal{L}(\mathcal{E}, \mathcal{A})$ and closed under substitution.[8] We call A *canonical* if, for any basic DEL logic $\Lambda \supseteq \mathsf{K} \oplus \mathsf{A}$, its canonical Kripke frame $\mathcal{F}_{\Lambda}$ validates A. Here, a *basic DEL logic in* $\mathcal{L}(\mathcal{E}, \mathcal{A})$ is a set of formulae in $\mathcal{L}(\mathcal{E}, \mathcal{A})$ containing all axioms in K and closed under modus ponens and the generalisation rules for $[e]$ and $[[\alpha]]$. ⊣

Using the notion of canonicity, we first prove the following lemma:

Lemma 4.4.1. *Let* S *be a set of canonical schemata written in* $\mathcal{L}(\mathcal{E}, \mathcal{A})$. *Then, the canonical Kripke model* $\mathcal{M}_{\mathsf{K} \oplus \mathsf{S}}$ *validates all* $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$*-instances of* S.

Proof. We only give here a proof sketch here. Let $\mathcal{F} = (S, (R_e)_{e \in \mathcal{E}}, (R_\alpha)_{\alpha \in \mathcal{A}})$ and $\mathcal{M} = (\mathcal{F}, V)$ be the canonical Kripke frame and model for $\mathsf{K} \oplus \mathsf{S}$. An $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$-instance of a schema $\varphi = \varphi(p_1, \ldots, p_n)$ in S can be written as $\varphi(\psi_1, \ldots, \psi_n)$, where ψ_i is a formula in $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$ $(i = 1, \ldots, n)$.

We prove that $\mathcal{M}$, regarded as a model for $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$, validates $\varphi(\psi_1, \ldots, \psi_n)$. Define a valuation $V' : \mathcal{P} \to \wp(S)$ by

$$V'(p_i) = \{w \in S \mid (\mathcal{F}, V), w \Vdash \psi_i\}.$$

Then, we see that

$$(\mathcal{F}, V) \Vdash \varphi(\psi_1, \ldots, \psi_n) \text{ iff } (\mathcal{F}, V') \Vdash \varphi(p_1, \ldots, p_n).$$

[8] For example, D_{t} is *closed under substitution*, whilst P is not.

Therefore, the canonical model $\mathcal{M} = (\mathcal{F}, V)$ validates the instance $\varphi(\psi_1, \ldots, \psi_n)$, since $(\mathcal{F}, V')$ validates the formula $\varphi(p_1, \ldots, p_n)$ by virtue of the canonicity of the schema φ. □

We obtain the desired proposition as an immediate consequence:

Proposition 4.4.2. *Let* S *be a set of canonical schemata written in* $\mathcal{L}(\mathcal{E}, \mathcal{A})$*. Then,* $\mathsf{K_g} \oplus \mathsf{S}$ *in* $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$ *is conservative over* $\mathsf{K} \oplus \mathsf{S}$ *in* $\mathcal{L}(\mathcal{E}, \mathcal{A})$*.*

This proposition can be applied to many interesting pure epistemic axioms. For example, $\mathsf{S5}$ for $\mathcal{E}$ is canonical, and thus we see that $\mathsf{K_g} \oplus \mathsf{S5}$ is conservative over $\mathsf{K} \oplus \mathsf{S5}$.

However, this general method is not sufficiently powerful to prove the conservativity of sublogics of PALg and EAg. Let us take the schema $\mathsf{R_N}$ to illustrate this point. This schema is certainly canonical in $\mathcal{L}_{\mathsf{PAL}}$. However, we cannot apply the proposition above to the sublogics $\mathsf{K} \oplus \mathsf{R_N}$ and $\mathsf{K_g} \oplus \mathsf{R_N}$, since the former $\mathsf{R_N}$ is given for $\mathcal{A}_{\mathsf{PAL}}$ whilst the latter is given for $\mathcal{A}_{\mathsf{PALg}}$. In what follows, we look at individual cases of sublogics of PALg and EAg.

Conservativity of sublogics of PALg and EAg Let us first consider sublogics of PALg:

Proposition 4.4.3. *Conservativity holds for each of the following pairs of a logic in* $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$ *and one in* $\mathcal{L}(\mathcal{E}, \mathcal{A})$*:*

1.	$\mathsf{K_g} \oplus \mathsf{G}$	*and*	K	*2.*	$\mathsf{K_g} \oplus \mathsf{P}$	*and*	$\mathsf{K} \oplus \mathsf{P}$
3.	$\mathsf{K_g} \oplus \mathsf{D_t}$	*and*	$\mathsf{K} \oplus \mathsf{D_t}$	*4.*	$\mathsf{K_g} \oplus \mathsf{R_N R^L_E}$	*and*	$\mathsf{K} \oplus \mathsf{R_N R^L_E}$
5.	$\mathsf{K_g} \oplus \mathsf{R^R_E}$	*and*	$\mathsf{K} \oplus \mathsf{R^R_E}$	*6.*	$\mathsf{K_g} \oplus \mathsf{R_N R^L_G}$	*and*	$\mathsf{K} \oplus \mathsf{R_N}$

Proof. We only prove (1) and (3). Other cases are proved in similar ways. (1) The canonical Kripke model $\mathcal{M}_{\mathsf{K}}$ trivially validates G. (3) Note that schema $\mathsf{D_t}$ is canonical in $\mathcal{L}(\mathcal{E}, \mathcal{A})$. Therefore, $\mathcal{M}_{\mathsf{K} \oplus \mathsf{D_t}}$ validates $\mathsf{D_t}$ for $\alpha \in \mathcal{A}$ by Lemma 4.4.1. $\mathsf{D_t}$ for each $\beta \in \mathcal{B} - \mathcal{A}$ is trivially valid, since there is no β-transition. □

It is unclear so far whether $\mathsf{K_g} \oplus \mathsf{R^R_G}$ is conservative over K.

Similarly, for sublogics of EAg, we have:

Proposition 4.4.4. *Conservativity holds for the following pairs of a logic in* $\mathcal{L}_{\mathsf{EAg}}$ *and one in* $\mathcal{L}_{\mathsf{EA}}$*:*

1.	$\mathsf{K_g} \oplus \mathsf{R_N R^L_{E(EA)}}$	*and*	$\mathsf{K} \oplus \mathsf{R_N R^L_{E(EA)}}$	*2.*	$\mathsf{K_g} \oplus \mathsf{R^R_{E(EA)}}$	*and*	$\mathsf{K} \oplus \mathsf{R^R_{E(EA)}}$
3.	$\mathsf{K_g} \oplus \mathsf{R_N R^L_{G(EA)}}$	*and*	$\mathsf{K} \oplus \mathsf{R_N}$				

It is not clear whether conservativity holds for $\mathsf{K_g} \oplus \mathsf{R}^{\mathsf{R}}_{\mathsf{G(EA)}}$ and $\mathsf{K_g}$.

Note that the conservativity results in Propositions 4.4.2, 4.4.3 and 4.4.4 can be freely combined in appropriate languages; for example, logic $\mathsf{K_g} \oplus \mathsf{R_N R_E}$ in $\mathcal{L}_{\mathsf{g}}(\mathcal{E}, \mathcal{B})$ is conservative over $\mathsf{K} \oplus \mathsf{R_N R_E}$ in $\mathcal{L}(\mathcal{E}, \mathcal{A})$.

Conservativity of LEg and LSPg It is possible to consider the E-free counterparts of LEg and LSPg. Namely $\mathsf{LE} := \mathsf{K} \oplus \mathsf{R_P R_N R_E}$ and $\mathsf{LSP} := \mathsf{K} \oplus \mathsf{R_P R_N R_{E(EA)}}$. For these two extensions, conservativity obtains as well:

Proposition 4.4.5.

1. LEg *is conservative over* LE.
2. LSPg *is conservative over* LSP.

Proof. We give a proof sketch. (1) It is easy to see the conservativity of LEg. Let us first consider a proof of the completeness of LE. The canonical Kripke model for the logic LE can be regarded as an f-MTS $\mathsf{M_{LE}}$ consisting of one inner Kripke model. By using the construction in Proposition 4.3.8. We obtain an eliminative f-MTS $\mathsf{N_{LE}}$ and a 2-bounded morphism $\mathsf{f_{LE}}$:

$$\mathsf{f_{LE}} : \mathsf{N_{LE}} \to \vec{\mathsf{M}}_{\mathsf{LE}}.$$

Therefore, LE is (sound) and strongly complete with respect to the class of eliminative f-MTSs. This implies the conservativity of LEg.

(2) Similarly, we can prove the strong completeness of LSP for the class of subproduct f-MTSs. Note that, this time, the set of actions for LSP is $\mathcal{A}_{\mathsf{EA}}$, not $\mathcal{A}_{\mathsf{EAg}}$. However, this is easy to overcome. A subproduct f-MTS $\mathsf{M} = (I, M, \Phi, R)$ for $\mathcal{A}_{\mathsf{EA}}$ can be extended to one $\mathsf{M}' = (I \cup \{*\}, M', \Phi', R')$ for $\mathcal{A}_{\mathsf{EAg}}$ in a similar way to the M^+-construction. This extension preserves the validity of any formula in $\mathcal{L}_{\mathsf{EA}}$. Therefore, the conservativity obtains. Incidentally, the soundness of LSP for the class of subproduct f-MTSs for $\mathcal{A}_{\mathsf{EA}}$ is immediate from this extension. □

Conservativity of PALg and EAg The conservativity for $\mathsf{R_T}$ and $\mathsf{R_{T(EA)}}$ is not clear; however, it holds for the pairs of full systems:

Proposition 4.4.6.

- PALg *is conservative over* PAL*;*
- EAg *is conservative over* EA.

Proof. PALg and EAg are sound with respect to the class of Kripke models with Plaza's and Baltag *et al.*'s semantics. Therefore, if a formula $\varphi \in \mathcal{L}_{\mathsf{PAL}}$ is provable in PALg, it is valid in all Kripke models and thus provable in PAL. The case of EAg is similar. □

Remark 4.4.1. Based on their dynamic semantics, TPALg and TEAg are also conservative over TPAL and TEA, respectively. ⊣

Chapter 5

Conclusion

Summary In Chapter 3, we introduced a general framework for DELs. It consists of a generic language and model transition systems (MTSs). As we have seen, an MTS is a static expression of model transformations and has the following two features: (i) the structure of a two-layered Kripke model, so that several notions for Kripke models are extended to MTSs; a typical example is that of a two-layered bisimulation (2-bisimulation); (ii) expressivity and reflexivity, such that MTSs can express other static large-scale models wherein Kripke models are transformed: e.g. extended Kripke models and PAL/EA-generated ETL models.

In Chapter 4, we added the global operator to the language. This enabled us to introduce the natural notions of a canonical MTS and canonicity of a DEL formula for a property of MTSs. These notions provide a framework to formally express correspondences between axiom schemata and properties of actions (see Section 4.3.4 for the correspondence results). In passing, we introduced two logics of independent interest, LEg (sublogic of PALg) and LSPg (sublogic of EAg), and proved their completeness with respect to the classes of eliminative f-MTSs and subproduct f-MTSs, respectively.

Future work We shall address the following issues in our future work:

- *'Dynamic' Sahlqvist theory.* MTSs and canonicity considerations have revealed many semantic meanings of axiom schemata (Section 4.3). However, these results are mostly concerned with the axioms of PALg and EAg. It would be interesting to develop a more general theory, which may well be called a 'dynamic' Sahlqvist theory.
- *Converse operator.* The outer frame (I, Φ) of an MTS $\mathsf{M} = (I, M, \Phi, R)$ expresses a temporal structure and each inner Kripke model M_i expresses the epistemic

situation at time point i. Therefore, MTSs readily admit converse operators $\langle\langle\alpha^{-1}\rangle\rangle\varphi$ (its dual $[[\alpha^{-1}]]\varphi$ means that φ held before action α). Converse operators are studied by Aucher and Herzig [1], Greco, Kurz and Palmigiano [6] and Sack [12, 13]. We believe that the use of MTSs will yield further results.

- *Algebraic semantics.* In algebraic semantics, a boolean algebra with operators (BAO) and an algebraic model (AM), a BAO equipped with an assignment, are used instead of a Kripke frame and a Kripke model, respectively. Accordingly, it would be possible to construct a 'two-layered algebraic model' by combining a BAO and AMs corresponding to the outer frame and the inner Kripke models of an MTS. We think that this will give a general framework for dynamic algebraic semantics of DELs which encompasses that of PAL [9] and EA [8].

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