

Notes on new class for certain analytic functions

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [1]. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$(1.1) \quad p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$), and such that $p(z) = q(w(z))$ ($z \in \mathbb{U}$). From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$(1.2) \quad p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}).$$

Remark 1.1 Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. If $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.1) is equivalent to the condition (1.2).

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We define new class for certain analytic functions. Let $\mathcal{S}(\alpha, \beta)$ be the class of functions $f(z) \in \mathcal{A}$ which satisfy the inequality

$$(1.3) \quad \alpha < \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \mathbb{U})$$

for some real number α ($\alpha < 1$) and some real number β ($\beta > 1$).

Remark 1.2 Let $f(z) \in \mathcal{S}(\alpha, \beta)$. If $\alpha \geq 0$, then $f(z)$ is starlike of order α in $\mathbb{U}$, which implies that $f(z)$ is univalent in $\mathbb{U}$.

Lemma 1.3 Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathcal{S}(\alpha, \beta)$ if and only if

$$(1.4) \quad \frac{zf'(z)}{f(z)} < 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right) \quad (z \in \mathbb{U}),$$

where $\alpha < 1$ and $\beta > 1$.

Proof. Let us consider the function $F(z)$ by

$$(1.5) \quad F(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right) \quad (z \in \mathbb{U})$$

with $\alpha < 1$ and $\beta > 1$. Then, it is easy to see that the function $F(z)$ is analytic and univalent in $\mathbb{U}$ with $F(0) = 1$. Furthermore, noting that

$$1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right) = \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{\pi} i \log \left(\frac{ie^{-\pi i \frac{1-\alpha}{\beta-\alpha}} - ie^{\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right),$$

a simple check gives us that $F(z)$ maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$. Thus, it follows from Remark 1.1 that the subordination (1.4) is equivalent to the inequality (1.3), which proves the assertion of Lemma 1.3. $\square$

We give some example for $f(z) \in \mathcal{S}(\alpha, \beta)$ as follows.

Example 1.4 Let us consider the function $f(z)$ given by

$$(1.6) \quad \begin{aligned} f(z) &= z \exp \left\{ \frac{\beta - \alpha}{\pi} i \int_0^z \frac{1}{t} \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} t}{1 - t} \right) dt \right\} \\ &= z + \frac{\beta - \alpha}{\pi} i \left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \right) z^2 + \dots \quad (z \in \mathbb{U}) \end{aligned}$$

with $\alpha < 1$ and $\beta > 1$. Then, we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right) \quad (z \in \mathbb{U}).$$

According to the proof of Lemma 1.3, it is clear that the function $f(z)$ given by (1.6) satisfies the inequality (1.3), which implies that $f(z) \in \mathcal{S}(\alpha, \beta)$.

2 Main results

Rogosinski [2] proved some coefficient estimates for subordinate functions.

Lemma 2.1 *Let $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and univalent in $\mathbb{U}$, and suppose that $q(z)$ maps $\mathbb{U}$ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies the following subordination*

$$p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

then

$$|A_n| \leq |B_1| \quad (n = 1, 2, \dots).$$

Applying Lemma 2.1, we deduced some coefficient estimates for $f(z) \in \mathcal{S}(\alpha, \beta)$ bellow.

Theorem 2.1 *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \beta)$, then*

$$|a_n| \leq \prod_{k=2}^n \frac{k-2 + \frac{2(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}}{(n-1)!} \quad (n = 2, 3, \dots).$$

Proof. According to the assertion of Lemma 1.3, the function $f(z)$ satisfies the subordination (1.4). Let us define $p(z)$ and $q(z)$ by

$$(2.1) \quad p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U})$$

and

$$(2.2) \quad q(z) = 1 + \frac{\beta-\alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z} \right) \quad (z \in \mathbb{U}).$$

Then, the subordination (1.4) can be written as follows :

$$(2.3) \quad p(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Note that the function $q(z)$ defined by (2.2) is convex in $\mathbb{U}$, and has the form

$$q(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta-\alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}} \right) \quad (n = 1, 2, \dots).$$

If we let

$$p(z) = 1 + \sum_{n=1}^{\infty} A_n z^n,$$

then by Lemma 2.1, we see that the subordination (2.3) implies that

$$(2.4) \quad |A_n| \leq |B_1| \quad (n = 1, 2, \dots),$$

where

$$(2.5) \quad |B_1| = \left| \frac{\beta - \alpha}{\pi} \left| 1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}} \right| \right| = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1 - \alpha)}{\beta - \alpha}.$$

Now, the equality (2.1) implies that

$$zf'(z) = p(z)f(z).$$

Then, the coefficients of z^n in both sides lead to

$$a_n = \frac{1}{n-1} (A_{n-1} + A_{n-2}a_2 + \dots + A_1a_{n-1}).$$

A simple calculation combined with the inequality (2.4) yields that

$$\begin{aligned} |a_n| &= \frac{1}{n-1} |A_{n-1} + A_{n-2}a_2 + \dots + A_1a_{n-1}| \\ &\leq \frac{1}{n-1} (|A_{n-1}| + |A_{n-2}||a_2| + \dots + |A_1||a_{n-1}|) \\ &\leq \frac{|B_1|}{n-1} \sum_{k=2}^n |a_{k-1}| \quad (|a_1| = 1), \end{aligned}$$

where B_1 is given in (2.5). To prove the assertion of the theorem, we need show that

$$(2.6) \quad |a_n| \leq \frac{|B_1|}{n-1} \sum_{k=2}^n |a_{k-1}| \leq \prod_{k=2}^n \frac{k-2+|B_1|}{(n-1)!}.$$

We now use the mathematical induction for the proof of the theorem.
Since

$$|a_2| \leq |B_1||a_1| = |B_1|,$$

it is clear that the assertion is holds true for $n = 2$.

We assume that the proposition is true for $n = m$. Then, some calculation gives us that

$$\begin{aligned} |a_{m+1}| &\leq \frac{|B_1|}{(m+1)-1} \sum_{k=2}^{m+1} |a_{k-1}| = \frac{|B_1|}{m} \left(\sum_{k=2}^m |a_{k-1}| + |a_m| \right) \\ &\leq \frac{|B_1|}{m} \left(1 + \frac{|B_1|}{m-1} \right) \sum_{k=2}^m |a_{k-1}| = \frac{m-1+|B_1|}{m} \frac{|B_1|}{m-1} \sum_{k=2}^m |a_{k-1}| \\ &\leq \frac{m-1+|B_1|}{m} \prod_{k=2}^m \frac{k-2+|B_1|}{(m-1)!} = \prod_{k=2}^{m+1} \frac{k-2+|B_1|}{((m+1)-1)!}, \end{aligned}$$

which implies that the inequality (2.6) is true for $n = m + 1$.
 By the mathematical induction, we prove that

$$|a_n| \leq \prod_{k=2}^n \frac{k-2+|B_1|}{(n-1)!} \quad (n = 2, 3, \dots),$$

where B_1 is given in (2.5). This completes the proof of Theorem 2.2. □

References

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