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THE CENTER PROBLEM OF THE FIRST ORDER ODE AND GEOMETRY OF PLANE CURVES.

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ABSTRACT. The purpose of this note is to give a fresh insight into the problem of relations of two germs of holomorphic diffeomorhisms from the view point of the center problem of ordinary differential equations, and to lead to the geometry of plane curves, giving an alternative proof, at least in real analytic case, to the result by brudnyi and Yomdin that the center condition holds for analytic first order ODEs if all heigher moments vanish in [1, 23]. The results in §1-4 are based on the forthcoming paper [22].

1. RELATION OF GERMS OF HOLOMORPHIC DIFFEOMORPHISMS

A *relation of length* ℓ of $f=f_1$, $g=f_2$ in $\mathrm{Diff}(\mathbb{C},0)$, the group of germs of holomorphic diffeomorphisms, is an equality of a word

$$W(f,g)=f_{i_1}^{(m_1)}\circ\cdots\circ f_{i_\ell}^{(m_\ell)}=\mathrm{id}, \tag{1.1}$$

where $i_k\in\{1,2\}$ and $m_k\in\{-1,1\}$ for all k, $f^{(m)}$ stands for the m-fold iteration of f, and the id stands for the identity of $\mathbb{C}$. We assume W is *reduced* i.e. W contains neither $f_i\circ f_i^{(-1)}$ nor $f_i^{(-1)}\circ f_i$, $i=1,2$.

In general all solvable sugroups of the fornmal group $\widehat{\mathrm{Diff}}(\mathbb{C},0)$ as well as $\mathrm{Diff}(\mathbb{C},0)$ are know to be meta abelian (solvable of length 2)[20].

Definition 1.1. *A pair of formal diffeomorphisms* (f,g) *is elementary if it generates a solvable group. It is commuting* if f,g commute. An *elementary relation* is a relation of an elementary pair.

For instanse, the group of fractional linear transformations of $\mathbb{C}$ fixing 0 is meta abelian. Therefore all pairs of fractional llinear transformations are elementary, and fulfill the following universal (elementary) identity holds for any words W_i

$$\{\{W_1(f,g),W_2(f,g)\},\{W_3(f,g),W_4(f,g)\}\}=\mathrm{id}\ , \tag{1.2}$$

where the *commutator* is given by $\{f,g\}=f^{(-1)}\circ g^{(-1)}\circ f\circ g$. This phenomenon can be explained by the fact that the winding number of the Cayley diagram, defined later on, is trivial (the word id zero homologous). A pair of diffeomorphisms (f_1,f_2) is *conjugate* (respectively *formally conjugate*) to a (g_1,g_2) if there exists a diffeomorphism (resp. formal diffeomorphism) ϕ such that $f_i=\phi^{(-1)}\circ g_i\circ\phi$ for $i=1,2$. Every non commuting elementary pair is formally conjugate to a pair of *ramified fractional linear transformations of an equal degree* k, which are of the form

$$\frac{\alpha_1 z}{\sqrt[k]{1-\frac{k\alpha_{k+1}}{\alpha_1}z^k}}=\alpha_1 z+\alpha_{k+1}z^{k+1}+\frac{{\alpha_{k+1}}^2(1+k)}{2\alpha_1}z^{2k+1}+\cdots,. \tag{1.3}$$

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For a word $W(f_1, f_2)$ in (1.1), define a connected piecewise linear diagram in the x_1x_2-plane $\mathbb{R}^2$ with base point at the origin by replacing the letter f_i (respectively $f_i^{(-1)}$) in W with a directed edge H_i (resp. H_i^{-1}) of length 1 in the positive (resp. negative) x_i-direction with velocity 1. The resulting diagram, denoted $\gamma^\vee$, and its *dual diagram* (transpose) called the *Cayley diagram* and denoted γ are respectively

$$\gamma^\vee = H_{i_1}^{m_1} * \cdots * H_{i_\ell}^{m_\ell}, \qquad \gamma = H_{i_\ell}^{m_\ell} * \cdots * H_{i_1}^{m_1}, \tag{1.4}$$

where $*$ stands for the concatenation of paths. A word W_γ is *closed* if its Cayley diagram γ is closed. Now let $x_1 = x$, $x_2 = y$ and $H_1 = V, H_2 = H$.

Let us consider, for instance, the word

$$W_{\gamma_1}(f,g) = \{f,g\}^{(p)} \circ \{f,\{f,g\}\}^{(q)} = \mathrm{id}\,. \tag{1.5}$$

The Cayley diagram γ_1 is

$$\gamma_1 = \{H^{-1},\{H^{-1},V^{-1}\}\}^{-q} * \{H^{-1},V^{-1}\}^{-p}$$

by definition, which is reduced to the following diagram. (The diagram may be drawn in the Lie algebra of formal vector fields without constant term $\hat{\chi}(\mathbb{C},0)$ in the manner of §2).

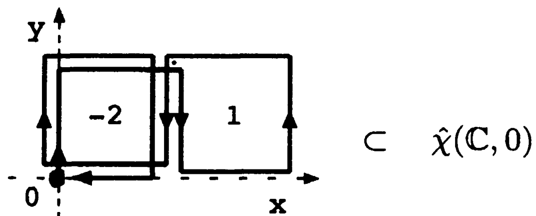

FIGURE 1. Reduced form of the Cayley diagram γ_1 for $p = q = 1$ and its winding number

Definition 1.2. *For a closed diagram $\gamma \subset \mathbb{R}^2$, define the Laurant characteristic polynomial of two commuting indeterminates x, y by*

$$P_\gamma(x,y) = \sum_{(i,j)\in\mathbf{Z}^2} \rho_{ij}\, x^i y^j,$$

where ρ_{ij} denotes the number of γ winding around the domain $(i,i+1)\times(j,j+1)$. For instance, $P_\gamma = 1$ for the commutator $\gamma = \{H^{-1},V^{-1}\} = H * V * H^{-1} * V^{-1}$.

For instance, the characteristic polynomial of the above diagram γ_1 is $P_{\gamma_1}(x,y) = qx - (p+q)$ by the definition.

A word W_γ is naturally regarded as an element of the free group F_2 of rank 2 generated by non commuting indeterminates X, Y, substituting f, g with X, Y respectively. Then W_γ is closed if and only if it is an element of the commutator subgroup $\{F_2, F_2\}$. The winding number $\rho = \{\rho_{ij}\}$ is defined for elements of $\{F_2, F_2\}$ in the same manner, and it gives an isomorphic of the first homology group of the commutator subgroup

$$H_1(\{F_2,F_2\}) = \{F_2,F_2\}/\{\{F_2,F_2\},\{F_2,F_2\}\},$$

onto the additive group of winding numbers with compact support. A closed Cayley diagram γ is *atomic* if P_γ is a monomial or a non zero integer, and we say γ is *zero homologous* if $P_\gamma = 0$, i.e. $W_\gamma \in \{\{F_2,F_2\},\{F_2,F_2\}\}$ such as (1.2). Define the *j-th characteristic curves* $C_\gamma^j \subset \mathbb{C}^2, j \geq 1$, by $P_\gamma(x^j, y^j) = 0$ for non zero homologous γ. We denote C_γ^1 by C_γ, and call it the *characteristic curve.* Now fix the Cayley diagram γ and let us seek pairs of formal power series (f,g) which fulfill the relation $W_\gamma(f,g)$ id. The following theorem suggests the geometry of Cayley diagram, in particular its winding number (or the homologu class of W_γ dominates the relations of holomorphic diffeomorphisms.

Theorem 1.1. [21, 22] *Let $\gamma \subset \mathbb{R}^2$ be a non zero homologous closed diagram and $\alpha, \beta \in \mathbb{C}^*$. Assume either $\alpha^k \neq 1, k \geq 1$ or $\beta^k \neq 1, k \geq 1$. Then all formal solutions with multiplier $(f'(0), g'(0)) = (\alpha, \beta)$ of $W_\gamma(f,g) = \mathrm{id}$ are as follows:*

(1) *If $(\alpha,\beta) \notin C_\gamma^k$ for all $k \geq 1$, the solutions are commuting.*
(2) *If $(\alpha,\beta) \in C_\gamma^\ell$ for an ℓ and $(\alpha,\beta) \notin C_\gamma^k$ for all $k \neq \ell$, then the set of formal solutions with multiplier (α,β) consists of*
- *all commuting pairs with multiplier (α,β),*
- *all non commuting elementary pairs of degree ℓ with multiplier (α,β).*

(3) *Assume (α,β) is contained in a finite intersection of C_γ^k for some distinct k:*

$$(\alpha,\beta) \in \bigcap_{\substack{k=\ell_1,\dots,\ell_n \\ 1\leq \ell_1 < \dots < \ell_n}} C_\gamma^k \setminus \bigcup_{k \neq \ell_1,\dots,\ell_n} C_\gamma^k, \qquad n \geq 2.$$

Then the set of formal solutions with multiplier (α,β) consists of
- *all commuting pairs with multiplier (α,β),*
- *all non commuting elementary pairs of degree ℓ_i with multiplier (α,β) for $i = 1,2,\dots, n$,*
- *some non elementary pairs with ℓ_i-jets of commuting pairs and ℓ_j-jets of elementary pairs for some $i < j$.*

For the detailed description of the formal solutions, and also other various results for the case $(\alpha,\beta) = (0,0)$, consult the paper [22].

Finally let us consider the relation $W_{\gamma_1}(f,g) = \mathrm{id}$ in (1.5). Let p, q be non zero integers such that $p+q \neq 0$. By Theorem 1.1 (2), every non commuting solution (f,g) of the relation has the multiplier $(f'(0), g'(0)) = (\alpha,\beta) = (\sqrt[\ell]{(p+q)/q}, \beta)$ for some $\ell \geq 1$, $\beta \in \mathbb{C}^*$ being arbitrary. The k-th characteristic curves $C_{\gamma_1}^k : x = \sqrt[k]{(p+q)/q}$, $k \geq 1$ do not intersect mutually. Therefore all the solutions are elementary by the theorem. We may formally linearlize f and assume $(f,g) = (\alpha z, h)$. Then h is a ramified fractional linear transformation of degree ℓ by an argument with straight forward formal calculation.

In order construct non elementary relations, consider, for instance, the polynomial $P_\gamma(x,y) = (x-2)(x-4)$. □

2. The center problem in the Lie algebra $\hat{\chi}(\mathbb{C},0)$ of holomorphic vector fields

All germs of holomorphic diffeomorphisms $f = \alpha_1 z + \alpha_2 z^2 + \cdots, \alpha_1 \neq 0$, are time-1 maps of some formal vector fields or composites of two time-1 maps. So assume $f_i = \exp a_i\partial_z, i = 1,2$, with holomorphic vector fields $a_i\partial_z$ and let us consider the piecewise constant non linear ordinary differential equation

$$\frac{dz}{dt} = \lambda(t,z) = \quad m_{i_{\ell-j+1}}\, a_{i_{\ell-j+1}}(z) \quad \text{if} \quad t \in [\, j-1, j)\ , \ z \in \mathbb{C},\ 0 \leq t \leq \ell, \tag{2.1}$$

where $m_{i_{\ell-j+1}}$ is the exponent of the j-th letter $f_{i_{\ell-j+1}}$ from the right hand side in $W_\gamma(f,g)$ in (1.1). The word W_γ is then the *time-ℓ transport map h_ℓ* (or *the product integral in* [8]) of the equation, which assigns the value $h_\ell(y) = z(\ell)$ of a solution $z(t)$ at $t = \ell$ to its initial value $y = z(0)$. More in general for a differential equation

$$\frac{dz}{dt} = \lambda(t, z), \; z \in \mathbb{C}, \; 0 \le t \le \ell, \tag{2.2}$$

define the transport map h_ℓ similarly.

Definition 2.1. *The center condition* for the equation (2.2) is the condition $h_\ell = \mathrm{id}$,

which is, in our case of (2.1), equivalent to the relation $W_\gamma(f, g) = \mathrm{id}$. The Cayley diagram γ may be drawn in the Lie algebra of formal vector fields $\hat{\chi}(\mathbb{C}, 0)$ without constant terms in the following manner: to a time-1 map $\exp a(z)\partial_z$ of $a(z)\partial_z \in \hat{\chi}(\mathbb{C}, 0)$ corresponds a segment of an γ, which is a parallel displacement of the directed segment $\overline{0\ a(z)\partial_z}$ in $\hat{\chi}(\mathbb{C}, 0)$. Now we generalize the notion of words of diffeomorphisms to piecewise smooth curves $\gamma(t) \in \hat{\chi}(\mathbb{C}, 0)$.

From now on let $\gamma(0) = 0$, $\gamma(t) \in \hat{\chi}(\mathbb{C}, 0)$, $0 \le t \le \ell$, be a piecewise smooth and continuous family, and $\dot{\gamma}(t) = \lambda(t, z)\partial_z$ its velocity. Define the *tautological* $\hat{\chi}(\mathbb{C}, 0)$*-valued 1-form* ω on $\hat{\chi}(\mathbb{C}, 0)$ by $\omega(X) = X$, identifying the tangent space $T_*\hat{\chi}(\mathbb{C}, 0)$ naturally with $\hat{\chi}(\mathbb{C}, 0)$. Let us consider the equation of formal 1-forms valued in $T_*\mathbb{C}$

$$\nabla : \partial_z \; dz = \omega$$

on the trivial $(\mathbb{C}, 0)$-bundle over $\hat{\chi}(\mathbb{C}, 0)$, where z is the coordinate of the fiber. The equation ∇ restricts on the curve $\gamma \subset \hat{\chi}(\mathbb{C}, 0)$ to give the equation $\partial_z \frac{dz}{dt} dt = \gamma^* \omega$. Dividing both sides by $\partial_z dt$ formally, we obtain the equation $\frac{dz}{dt} = \lambda(t, z)$. The ∇ possesses the *universal property* that any piecewise continuous 1-st order differential equation is obtained with an appropriate piecewise smooth curve γ. The equation ∇ may be also regarded as a $\hat{\chi}(\mathbb{C}, 0)$-valued connection on the trivial $(\mathbb{C}, 0)$-bundle over $\hat{\chi}(\mathbb{C}, 0)$ with the curvature 2-form $\Omega(X, Y) = [X, Y]$ on $\hat{\chi}(\mathbb{C}, 0)$. Define the *dual* (or *transpose*) $\gamma^\vee$ of a piecewise smooth curve $\gamma(t)$ by $(\dot{\gamma^\vee})(t) = \dot{\gamma}(\ell - t), \gamma^\vee(0) = 0, 0 \le t \le \ell$. The dual curve defined the duql differential equation

$$\frac{dz}{dt} = \lambda(\ell - t, z), \qquad 0 \le t \le \ell \tag{2.3}$$

Let h_γ (respectively $h_{\gamma^\vee}$) denote the time-ℓ transport map of the equations (2.2), (2.3).

Brudnyi and Yomdin [1, 23] proved that if the λ is real analytic, in other words γ is real analytic, and γ is confined in a complex two plane, the center condition holds if all generalized moment (integration of all coordinate monomial 1-forms on the plane) vanish.

3. Lie integral and Picard iteration

Let $G \subset \mathrm{GL}(\nu, \mathbb{C})$ be a Lie subgroup and $\mathcal{G}$ its Lie algebra, and let us consider the linear differential equation

$$\frac{dv(t)}{dt} = X(t)\; v(t), \qquad v \in \mathbb{C}^\nu, \; t \in \mathbb{R}, \tag{3.1}$$

where $X(t)$ is a $\mathcal{G}$-valued piecewise continuous function of t. A solution $v(t)$ is a continuous, piecewise smooth and defined for all $t \in \mathbb{R}$, and uniquely presented as

$$v(t) = \exp Z(t) \cdot v(0), \quad Z(0) = 0$$

with a piecewise smooth continuous curve $Z(t) \in \mathcal{G}$ for small t. The matrix $Z(t)$ is called the *Lie integral* (in [3]) or *Lie transport* (in [14]) of $X(t)$ (or the matrix valued one form $X(t)dt$) and denoted by $Z(t) = L\int_0^t X(u)du$ in the manner of [3]. Its exponential $\exp Z(t) \in G$ is called the *product integral* of $X(t)$ (in [8]) or the *time-t transport map* and denoted by $T_{X(u)du}(\gamma)$ in Chen's notation, where γ stands for the oriented interval $[0,t]$.

Let us recall the Picard iteration to construct the solution of (3.1). Put $v_0 \in \mathbb{C}^\nu$ and define

$$v_{n+1}(t) = v_n(0) + \int_0^t X(u)v_n(u)\ du = v_0 + \int_0^t X(u)v_n(u)\ du\ .$$

for $n = 1, 2, \ldots$. Then

$$v_n(t) = \{I + \int_0^t X(u)\ du + \cdots + \int_{0\le u_1 \le u_2 \le \cdots \le u_n \le t} X(u_n)\cdots X(u_1)du_1\cdots du_n\}\ v_0,$$

which is convergent to the product integral, and arrives at our goal

(3.2)

$$\exp Z(t) = I + \int_0^t X(u)v_n(u)\ du + \cdots + \int_{0\le u_1 \le u_2 \le \cdots \le u_n \le t} X(u_n)\cdots X(u_1)du_1\cdots du_n + \cdots,$$

which is called Volterra expansion. Transposing the matries the formula may be presented in the form

$$\text{(3.3)} \qquad {}^t\exp Z(t) = \int_{[0,t]} I + \omega + \omega^2 + \cdots + \cdots,$$

where $\omega = {}^tX(u)du$ and the integral stands for the iterated path integral defined in §5.

In the manner of Riemannian integral as in the papers [3], the product integral, for a $t > 0$, may be approximated by a finite product

$$\exp X(\xi_N)\Delta_N \cdot\ \cdots\ \cdot \exp X(\xi_2)\Delta_2 \cdot \exp X(\xi_1)\Delta_1$$

with a division $\Delta : 0 = t_0 < t_1 < \cdots < t_N = t$ of the interval $[0,t]$, $\Delta_i = t_i - t_{i-1}$ and $t_{i-1} \le \xi_i \le t_i$. And as $\|\Delta\| = \max \Delta_i$ tends to 0, the product is convergent to the product integral. Following the idea due to Dyson [9], we may substitute the above product with

$$(\ I + \int_{t_{N-1}}^t X(u)du\ )\ (\ I + \int_{t_{N-2}}^{t_{N-1}} X(u)du\ )\ \cdots\ (\ I + \int_0^{t_1} X(u)du\ ),$$

without changing the limit. Simply by opening the brackets and taking limit as $N \to \infty$ and $\|\Delta\| \to 0$, we arrive at the Volterra expansion (3.3). By taking the *logarithm* we obtain

Theorem 3.1 (Theorem 8 in p.242, [3]). *There exists a positive constant $\delta = \delta(\nu)$ such that if $X(t)$ is Riemann integrable and $\int_0^1 \|X(u)\|du \le \delta$, then the Taylor expansion of $L\int_0^t \epsilon X(u)du$ in ϵ,*

$$L\int_0^t \epsilon X(u)du = \epsilon H_1(t) + \epsilon^2 H_2(t) + \cdots$$

is absolutely convergent for $0 \le t, \epsilon \le 1$. Here $H_1(t) = \int_0^t X(u)du$, and $H_n(t)$ is defined by

$$(n+1)H_{n+1} = T_n + \sum_{r=1}^n \left(\frac{1}{2}[H_r, T_{n-r}] + \sum_{\substack{p\ge 1\\ 2p\le r}} k_{2p} \sum_{\substack{m_i>0\\ m_1+\cdots+m_{2p}=r}} [H_{m_1}, [\ldots, [H_{m_{2p}}, T_{n-r}]\cdots]]\right)$$

for $n \geq 1$, *and for* $\ell > 0$

$$T_\ell = \int_{0 \leq u_{\ell+1} \leq \cdots \leq u_1 \leq t} \cdots \int du_1 \cdots du_{\ell+1} [\cdots [[X(u_1), X(u_2)], \ldots, X(u_\ell)], X(u_{\ell+1})],$$

and $T_0 = H_1$, *where* $(2p)! k_{2p} = B_{2p}$ *is Bernoulli's number defined as the coefficients which occur in the power series expansion* $\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{p=1}^{\infty} B_{2p} \frac{t^{2p}}{(2p)!}$, *and the integral stands for the multiple integral.*

Generalizing the theorem we have

Theorem 3.2. *For* $X(t)$ *in the above theorem,*

$$\begin{aligned} L\int_0^t X(u)du &= H_1(t) + H_2(t) + H_3(t) + H_4(t) + \cdots, \quad 0 \leq t \leq 1 \\ L\int_0^t X(t-u)du &= H_1(t) - H_2(t) + H_3(t) - H_4(t) + \cdots, \quad 0 \leq t \leq 1. \end{aligned}$$

4. Lie integral of non linear ordinary differential equations

Let us consider the non linear holomorphic (formal) ordinary differential equation

$$\frac{dz}{dt} = \lambda(t, z) = \lambda_1(t)z + \lambda_2(t)z^2 + \lambda_3(t)z^3 + \cdots, \qquad z \in \mathbb{C},\ t \in \mathbb{R},$$

where the coefficients $\lambda_n(t)$, $n = 1, 2, \ldots$, are simultaneously piecewise continuous. In the case λ is holomorphic in z, we assume $\lambda(t, *), t \in \mathbb{R}$, are defined on a common neighborhood of $0 \in \mathbb{C}$, and let $h_t(z_0)$ denote the value at t of the solution $z = z(t)$ with the initial value z_0 at $t = 0$. The $h_t(z)$ is a germ of holomorphic diffeomorphism of z respecting $z = 0$, in other words, h_t is the *time-t map* of the time depending vector field $X_t = \lambda(t, z)\partial_z$. In the case λ is not convergent in z, the coefficients of the *time-t transport map*

$$h_t = C_1(t)z + C_2(t)z^2 + C_3(t)z^3 + \cdots \tag{4.1}$$

are determined by a certain recursive differential equation. Let h_t^* be the pull back of h_t acting on the ring $\mathbb{C}[[z]]_0$ of formal power series of z without constant term by variable change $Q \longrightarrow Q \circ h$. By a straightforward calculation we obtain

$$\frac{d}{dt} h_t^* \, Q = \frac{d}{dt}(Q \circ h_t) = (X_t \, Q) \circ h_t = h_t^* \, (X_t \, Q), \qquad Q \in \mathbb{C}[[z]]_0$$

From the formura we obtain the linear differential equation of infinite dimension

$$\frac{d}{dt}(h_t^{*-1}) = -X_t \, h_t^{*-1}. \tag{4.2}$$

$$(h_t{}^*)^{-1} = \exp\, L\int_0^t -X_u du,$$

from which

$$(h_t{}^*) = \exp\, L\int_t^0 -X_u du. \tag{4.3}$$

Applying both sides to the coordinate function z of $\mathbb{C}$, the k-jet of $h_t(z)$ in (4.1) is represented by the polynomial

$$h_\gamma(z) = C_1 z + C_2 z^2 + \cdots + C_k z^k = (\exp\ L\int_t^0 -X_u^{[k]} du)\ z\ . \tag{4.4}$$

One may present the transport map h_γ in Volterra expantion.. Here we present the formula for the Lie integral, i.e. the logarithm of the transport map.

Theorem 4.1. [22] *Let $h_\gamma^* : \mathbb{C}[[z]]_0 \to \mathbb{C}[[z]]_0$ be the pull back of the transport map h_γ. Assume γ is closed and the projection of γ to the k-jet space has sufficiently small velocity. Then the Lie integral of the velocity of the projection is well defined and*

$$\begin{aligned} \log\ h_\gamma^* &= L\int_{\gamma^\vee} \omega = H_1 - H_2 + H_3 - H_4 + \cdots \\ \log\ h_{\gamma^\vee}^* &= L\int_\gamma \omega = H_1 + H_2 + H_3 + H_4 + \cdots \end{aligned}$$

where the logarithm and the alternative sum in the right hand side extend to the whole space of piecewise smooth closed curves γ.

Corollary 4.1. *If* $\log\ h_\gamma^* = 0$*, then the center condition*$h_\gamma = \mathrm{id}$ *holds.*

The logarithm $\log h_\gamma^* \in \hat{\chi}(\mathbb{C}$ iscalled the *generating vector field* of h_γ. Determination of the taylor coefficients of the generating vector field needs the calculation of the moment of γ in the next section.

5. Signature and Moment

Let $\gamma_t \subset V = \mathbb{R}^d, t \in [0,1]$ be a piecewise smooth curve. Let $w_1, \ldots, w_s$ be 1-forms on V. Write $\gamma^* w_i = f_i(t)dt$ and define the *iterated path integral* $\int_X w_1 \cdots w_s$ of $w_1 \cdots w_s$ over γ by the integration of $f_1(t_1)\cdots f_s(t_s)dt_1 \wedge \cdots \wedge dt_s$ over the s-simplex $\Delta_s = \{0 \le t_1 \le \cdots \le t_s \le 1\}$ with the canonical orientation. This is independent of parameterization of γ respecting the orientation (see [15]). Moreover, this is determined up to by the homotopy type of γ with respect to the extremities if ω_i are all closed and $\omega_i \wedge \omega_{i+1} = 0$ for $i = 1, \ldots s-1$.

Lemma 5.1 (Shuffle formula [7]). *Let $\omega_i, i = 1, \ldots, s+t$, γ be as above. Then*

$$\int_\gamma w_1 \cdots w_s \times \int_\gamma w_{s+1} \cdots w_{s+t} = \sum_\sigma \int_\gamma w_{\sigma(1)} \cdots w_{\sigma(s+t)},$$

where σ runs over the set of all permutations of $s+t$ elements satisfying

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(s), \qquad \sigma^{-1}(s+1) < \cdots < \sigma^{-1}(s+t).$$

Lemma 5.2 (Product formula [7]). *Let γ_t, δ_t be piecewise smooth curves in V. Then*

$$\int_{\gamma*\delta} \omega_1 \cdots \omega_k = \sum_{i=0}^k \int_\gamma w_1 \cdots w_i \times \int_\delta w_{i+1} \cdots w_k$$

where $\int_\gamma 1, \int_\delta 1$ are define to be 1 and $\gamma * \delta$ stands for the concatenation of γ and 4delta by a suitable translation.

The *siguniture* $\Theta_\gamma^\otimes$ of a $\gamma \subset V$ is defined by the iterated path integral

$$\Theta_\gamma^\otimes = 1 + \int_{0\le u_1 \le 1} dX_{u_1} + \int_{0\le u_1\le\cdots\le u_k\le 1} dX_{u_1} \otimes \cdots \otimes dX_{u_k} + \cdots$$

in the closure $T^\otimes(V)$ of the tensor algebr $\oplus_{k=0}^\infty \overset{k}{\otimes} V$ in a suitable topology. By the product formula, we obtain the product formula

$$\Theta_{\gamma*\delta}^\otimes = \Theta_\gamma^\otimes \otimes \Theta_\delta^\otimes$$

Let $x_1, \dots, x_d$ be the canonical basis of $\mathbb{R}^d$. The tensor product $\overset{k}{\otimes} V$ is spanned, over $\mathbb{R}$, by

$$x^{\otimes I} = x_{i_1} \otimes \cdots \otimes x_{i_k}$$

for a k-tuple $I = (i_1, \dots, i_k)$ of integers $1 \le i_1, i_2, \dots, i_k \le d$. The *ordered basis* are those such that

$$i_1 \le \cdots \le i_k \; .$$

In Physics the (respectively inertia) *moment* of a distribution $f(x)$ on the x-line of one dimension is the integration of $f(x)x\, dx$ (resp. $f(x)x^2\, dx$) over the line. Also in statistics, the j-th moment of a random variable X is defined to be the expectation of the j-th power X^j. A piecewise smooth closed curve γ in $\mathbb{R}^2$ defines the winding number function ρ on the plane (ρ is defined to be 0 on γ). The (i,j)-*moment* of the ρ is defined by the left hand side of the equation

$$\int_{\mathbb{R}^2} \rho x^i y^j \, d\text{Area} = \frac{(-1)^j}{i!\; j!} \int_\gamma dx^{i+1} dy^{j+1}. \tag{5.1}$$

This equation holds for a quite large class of continuous curve (c.f. Theorem 5.1).

Let $T^\circ(V)$ denotes the closure of the symmetric tensor algebra $\oplus_{k=0}^\infty \overset{k}{\bigcirc} V$, which is isomorphic to the algebra $\mathbb{C}[[V]]$. Identifying $x^{\ I} = x_{i_1} \circ \cdots \circ x_{i_k}$ with the ordered $x^{\otimes I} = x_{i_1} \otimes \cdots \otimes x_{i_k}$, $T^\circ(V)$ is naturally embedded into $T^\otimes(V)$. Define the *moment* $\Theta_\gamma^\circ \in T^\circ(V)$ by the iterated path integral

$$\Theta_\gamma^\circ = 1 + \sum_{\text{ordered } I} \Big(\int_{0\le u_1\le\cdots\le u_k\le 1} dx_{i_1} \cdots dx_{i_k} \Big) \quad x_{i_1} \circ \cdots \circ x_{i_k}$$

An element of $T^\otimes(V)$ is presented by the basis $x^{\otimes I}$ as

$$P = \sum a_I x^{\otimes I}.$$

The projection of P to the ordered part in $T^\circ(V)$ is defined to be the sub-polynoial

$$\pi(P) = \sum_{I \text{ is ordered}} a_I x^{\otimes I}.$$

By the product formula, (Lemma 5.2) an ordered iterated path integral over a composite $\gamma * \delta$ decomposes into a sum of ordered integrals as follows.

$$\int_{\gamma*\delta} dx_{i_1} \cdots dx_{i_d} = \sum_{k=0}^{d} \int_\gamma dx_{i_1} \cdots dx_{i_k} \times \int_\gamma dx_{i_{k+1}} \cdots dx_{i_d}$$

Therefore we obtain $\Theta^{\circ}_{\gamma*\delta} = \pi(\Theta^{\circ}_{\gamma} \otimes \Theta^{\circ}_{\delta})$. Define the product $\bigcirc$ on $T^{\circ}(V)$ by $P \bigcirc Q = \pi(P \otimes Q)$. Then we obtain the product formula of the moment

$$\Theta^{\circ}_{\gamma*\delta} = \Theta^{\circ}_{\gamma} \bigcirc \Theta^{\circ}_{\delta}$$

A *monomial base* of $T^{\circ}(V)$ is two dimensional if it consists of a monomial of two coordinates, and a *two dimensional element of* $T^{\circ}(V)$ a (possibly infinite) sum of two dimensional monomials. The space of two dimensionlal elements $T^2(V)$ is naturally embedded into $T^{\otimes}(V)$ in the above manner but the image is not closed under the product $\otimes$. Let $\pi^2 : T^{\otimes} \to T^2(V)$ denote the projection to the 2-dimensional parts. And define the 2-dimensional moment of γ by the projection

$$\Theta^2_{\gamma} = \pi^2(\Theta^{\otimes}_{\gamma})$$

and define the product $\bigcirc$ of 2-dimensiona elements $P, Q \in T^2(V)$ by $P \bigcirc Q = \pi^2(P \bigcirc Q)$. Then we obtain the product formula of 2-dimensional elements.

$$\Theta^2_{\gamma*\delta} = \Theta^2_{\gamma} \bigcirc \Theta^2_{\delta}$$

By these product formula we obtain

Proposition 5.1. *Let γ^{-1} denote the inverse of γ. Then*

$$\Theta^{\otimes}_{\gamma} \otimes \Theta^{\otimes}_{\gamma^{-1}} = 1$$
$$\Theta^{\circ}_{\gamma} \otimes \Theta^{\circ}_{\gamma^{-1}} = 1$$
$$\Theta^{2}_{\gamma} \otimes \Theta^{2}_{\gamma^{-1}} = 1$$

Corollary 5.1. *The signature and moments*

$$\Theta^{\otimes}_{\gamma^{-1}}, \quad \Theta^{\otimes}_{\circ^{-1}}, \quad \Theta^{2}_{\gamma^{-1}}$$

are determined by those for γ respectively.

Proof. We prove the statement for 2-diensional moments. Let us consider an ordered iterated path integral $\int_{\gamma*\gamma^{-1}} dx^{\otimes I} = \int_{\gamma*\gamma^{-1}} dx^s_i dx^t_j$, $i < j$. By the product formula we obtain

$$\int_{\gamma*\gamma^{-1}} dx^s_i dx^t_j = \int_{\gamma} dx^s_i dx^t_j + \int_{\gamma} dx^s_i dx^{t-1}_j \cdot \int_{\gamma^{-1}} dx_j + \int_{\gamma} dx^s_i dx^{t-2}_j \cdot \int_{\gamma^{-1}} dx^2_j + \cdots + \int_{\gamma^{-1}} dx^s_i dx^t_j = 0$$

By way of the induction on the degree of I, assume that 2-dimensional moment $\int_{\gamma^{-1}} dx^{\otimes J}$ of degree smaller than $s + t$ are determined by those moments of γ. Then all but the last terms in the middle of the equation are determined by those moment of γ, and the last term is also determined by those of γ of degree $\leq s + t$. □

6. Determination by Signature and Moment

Two subsets $A, B \in V$ are *equivalent:* $A \sim B$, if B is a translation of A. Following the paper of Chen [6], we say a piecewise smooth path of the form $\alpha * \gamma * \gamma^{-1} * \beta$ is *reduced to* $\alpha * \beta$. The *irreducible path* of a path γ is the finite reduction of γ which is no longer reducible. The equivalence relation $\sim$ of piecewise smooth paths in V is generated by piecewise smooth re-parametrization, translation in V and reduction procedure. For paths of bounded variation, the definition is too much complicated to state in this brief note (see [18]).

It is rather obvious fact that relatively compact open subsets of the plane are faithfully coded into the collections of integrations of monomial 2-forms $x^i y^j d\text{Area}$ over those sets, i.e. the (i,j)-moments. To see this, it is enough to notice that for each point (x_0, y_0) on the plane, there exists a sequence of polynomials $f_n(x,y)$ such that $\int f_n(x,y)d\text{Area} = 1$ for all n and $f_n(x,y) \to 0$ for all $(x,y) \neq (x_0, y_0)$, so by integrating f_n over an open subset, one can detect whether (x_0, y_0) is contained in the subset or not. This fact may be generalized to weighted disconnected open subsets.

Definition 6.1. *Two closed curves in the plane are homologous* if their winding numbers are equivalent by translation

Lemma 6.1. *Two rectifiable closed curves $\gamma, \delta \subset \mathbb{R}^2$ are homologous if and only if $\Theta^2_\gamma = \Theta^2_\delta$.*

Proof. — We prove only the "if" part. As γ, δ are closed and rectifiable, the 2-dimensional moments determine the (i,j)-moments of their winding numbers for all (i,j) by Theorem 6.1 and the formula 5.1, hence their winding numbers. □

Theorem 6.1. (Green's theorem for rectifiable curves [2, 19]) *Let γ be a rectifiable closed continuous curve in an open subset $U \subset \mathbb{R}^2$. Let $\rho(x,y)$ denote the winding number of γ with respect to the point (x,y) and $\rho = 0$ on γ. Then the winding number ρ is Lebesgue measurable. Let $f, g \in C^1(U)$, and assume the winding number ρ is zero off U. Then*

$$\int_\gamma f dx + g dy = \iint \rho\ (g_x - f_y)\ dx \wedge dy\ ,$$

where the integral in the left hand side is the Lebesgue-Stieltjes integral and the integral in the right hand side is the Lebesgue integral over the plane $\mathbb{R}^2$ and ρ is the winding number function of γ: $\rho = 0$ on γ.

Theorem 6.2. (Chen) [18] *Let $\gamma, \delta \in V$ be piecewise smooth. Then $\gamma \sim \delta$ if and only if $\Theta^\otimes_\gamma = \Theta^\otimes_\delta$*

Corollary 6.1. *Assume γ is a piecewise smooth curve in $\mathbb{R}^2$ with the trivial signature. Then γ is tree like.*

This theorem is generalized by B.Hambly and T.Lyons [18] for curves of bounded variation. It tells that the piecewise smooth or bounded variation curves are coded by the non commutative formal series.

Question 6.1. *(posed in the paper* [18]*) What are the images*

$$\Theta^\otimes(\{\textit{bounded variation paths}\}).$$

Howcan one reconstruct the curve from a signature?

This is non commutative analogue of the so-called Hausdorff moment problem. The image is of course bounded by the apparent restriction by the shuffle relation in Lemma5.1. The subset in $T^\otimes(V)$ defined by the shuffle relation is a subgroup of $T^\otimes(V)$. An interest problem to pose here is

Question 6.2. *What are the following images?*

$$\Theta^\otimes(\{\textit{piecewise smooth paths}\}), \qquad \Theta^\otimes(\{\textit{preal analytic paths}\}).$$

Are there any interesting subgroups of Im $\Theta^\otimes$, of which all elements are realized by continuous path of some regularity class?

The next theorem tells us that there exist many hidden relations for most of regularity classes.

Theorem 6.3. *Two real analytic paths $\gamma, \delta \subset \mathbb{R}^2$ are euivalent if and only if $\Theta^2_\gamma = \Theta^2_\delta$. In other words, there is a bijection*

$$\{\textit{Real analytic paths}\} \;/\!\sim\; \overset{1:1}{\Longleftrightarrow} \{\textit{Im}\ \Theta^2\}$$

Proof of the theorem. — We prove the theorem for the case $d = 2$ for simplicity. Assume $\Theta^2_\gamma = \Theta^2_\delta$. Then $\Theta^2_{\gamma^{-1}} = \Theta^2_{\delta^{-1}}$ by Corollary 5.1, and

$$\Theta^2_{\gamma*\delta^{-1}} = \Theta^2_\gamma \bigcirc \Theta^2_{\delta^{-1}} = \Theta^2_\gamma \bigcirc \Theta^2_{\gamma^{-1}} = \Theta^2_{\gamma*\gamma^{-1}} = 1.$$

This implies that $\gamma * \delta^{-1}$ is closed and bounds no area, in other words, it winds 0-times any point in its complement by Lemma 6.1.

The image Imγ is a piecewise smooth finite topological graph, of which each edge is directed and weighted by a positive integer, by the γ_* image of the generator of $H_1([0,1],\{o,1\}) = \mathbb{Z}$. Let $\tilde{\gamma}$ denote the subgraph of the image consisting of edges with non trivial weights. The path is reduced to $\tilde{\gamma}$, and $\tilde{\gamma}$ is a topologically immersed circle S^1 or the interval $[0,1]$. Since $\gamma * \delta^{-1}$ is zero homologous, it follows $\tilde{\gamma} + \tilde{\delta}^{-1} = 0$, from which $\tilde{\gamma} \sim \tilde{\delta}$, hence $\gamma \sim \delta$. □

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