数理解析研究所講究録
第 1641 巻 2009 年 68-73

Convex bodies passing through holes

Hiroshi Maehara and Norihide Tokushige
College of Education, Ryukyu University, Nishihara, Okinawa, 903-0213 Japan

1. INTRODUCTION

For a given convex body, find a "small" wall hole through which the convex body can pass. This type of problems goes back to Zindler [14] in 1920, who considered a convex polytope which can pass through a fairly small circular holes. A related topic known as Prince Rupert's problem can be found in [2]. Here we concentrate on the case when the convex body is a regular tetrahedron or a regular n-simplex.

For a compact convex body $K\subset\mathbb{R}^n$, let $\operatorname{diam}(K)$ and $\operatorname{width}(K)$ denote the diameter and width of K, respectively. For $d>0$ let dK denote the convex body with diameter d and homothetic to K. Let S_n, Q_n, and B_n denote the n-dimensional regular simplex, the n-dimensional hypercube, and the n-dimensional ball, respectively. Thus, $1S_n$ has side length 1, $1Q_n$ has side length $1/\sqrt{n}$, and $1B_n$ has radius $1/2$.

Let $H\subset\mathbb{R}^{n-1}$ be a convex body, which we will call a hole. Let Π be the hyperplane containing H, which divides $\mathbb{R}^n$ into Π and two (open) half spaces Π^+ and Π^-. We want to push $1S_n$ from Π^+ to Π^- through H. In this situation, we are interested in two types of "small" holes, namely,

$$\gamma(n,H):=\min\{d:1S_n \text{ can pass through the hole of } dH \text{ in } \mathbb{R}^n\},$$

and

$$\Gamma(n,H):=\min\{d:1S_n\subset(dH)\times\mathbb{R}\}.$$

Notice that $\gamma(n,H)$ and $\Gamma(n,H)$ do not depend on $\operatorname{diam}(H)$. For given H, we resize H so that $1S_n$ can pass through the hole H. We will try to find a hole homethetic to K with minimum diameter, which will give γ or Γ. (Recall that dH is homothetic to H and $\operatorname{diam}(dH)=d$.) By definition, $1S_n$ can pass through a hole H by translation perpendicular to the hyperplane containing the hole iff $\operatorname{diam}(H)\geq\Gamma(n,H)$. Thus we have $\gamma(n,H)\leq\Gamma(n,H)$.

We have $\operatorname{width}(1Q_n)=1/\sqrt{n}$ and $\operatorname{width}(1B_n)=1$. Steinhagen [12] determined the width of S_n as follows.

$$\operatorname{width}(1S_n)=\begin{cases}\sqrt{\frac{2}{n+1}} & \text{if } n \text{ is odd,}\\ \sqrt{\frac{2n+2}{n(n+2)}} & \text{if } n \text{ is even.}\end{cases} \tag{1}$$

If $1S_n$ can pass through a hole dH by translation, then

$$\operatorname{width}(dH)\geq\operatorname{width}(1S_n)=(\sqrt{2}-o(1))/\sqrt{n}. \tag{2}$$

Let $n\geq 3$. If $1S_n$ can pass through a hole dH, then $d\geq\operatorname{width}(1S_2)=\sqrt{3}/2$. This gives $\gamma(n,H)\geq\sqrt{3}/2$.

Brandenberg and Theobald [1] proved the following.

$$\Gamma(n,B_{n-1}) = \begin{cases} \sqrt{\frac{2(n-1)}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{\sqrt{2n(n+1)}} & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

2. In the 3-space

Itoh, Tanoue, and Zamfirescu [6] proved

$$\gamma(3,Q_2) = \Gamma(3,Q_2) = 1, \quad \gamma(3,B_2) = 2r = 0.8956..., \tag{4}$$

where $r \in (0,1)$ is a unique root of the equation $216x^6 - 9x^4 + 38x^2 - 9 = 0$. We note that $\gamma(3,B_2) < \Gamma(3,B_2) = 1$.

In [9], the following is proved.

$$\gamma(3,S_2) = \Gamma(3,S_2) = \frac{1+\sqrt{2}}{\sqrt{6}} = 0.9855...$$

Zamfirescu [13] proved that most convex bodies can be held by a circular frame. Using (4), one can show that a square frame of diagonal length d can hold $1S_3$ iff $1/\sqrt{2} < d < 1$, and a circular frame of diameter d can hold $1S_3$ iff $1/\sqrt{2} < d < \gamma(3,B_2)$, see [6].

On the other hand, it is shown in [9] that

$$\text{no triangular frame can hold a convex body.} \tag{5}$$

This is a special property for triangular frames, and in fact, we have the following.

Theorem 1. [9] *Every non-triangular frame holds some tetrahedron in* $\mathbb{R}^3$.

Debrunner and Mani-Levitska [3] proved that any section of a right cylinder by a plane contains a congruent copy of the base, see also [7]. This together with (5) implies the following: if a convex body, not necessarily smooth, can pass through a triangular hole, then the convex body can pass through the hole by translation perpendicular to the wall, see [9].

Itoh and Zamfirescu [5] found a hole $H \subset \mathbb{R}^2$ with $\mathrm{diam}(H) = \mathrm{width}(1S_2) = \sqrt{3}/2$ and $\mathrm{width}(H) = \mathrm{width}(1S_3) = \sqrt{2}/2$, such that $1S_3$ can pass through H.

3. Higher dimensions

3.1. **The hole S_{n-1}.** Recall that any plane section of a right triangular prism contains a congruent copy of a base of the prism [3, 7]. The situation in higher dimension is different. In [3], it is proved that if $n > 3$, then for any right cylinder with convex polytope base, one can find a hyperplane section which does not contain a congruent copy of the base. Nevertheless, we have the following.

Theorem 2. [9] *Let $K \subset \mathbb{R}^n$ be a compact convex body, and let Δ_{n-1} be a general $(n-1)$-simplex. If K can pass through the hole Δ_{n-1}, then this can be done by translation only.*

Problem 1. *Is it possible to take the translation in Theorem 2 perpendicular to the wall? Or equivalently, do $\gamma(n,S_{n-1})$ and $\Gamma(n,S_{n-1})$ coincide?*

Theorem 3.

$$\gamma(n,S_{n-1}) \geq \begin{cases} \sqrt{1-\frac{1}{n}} & \text{if } n \text{ is odd,} \\ \sqrt{1-\frac{1}{n+2}} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Suppose that $1S_n$ can pass through the hole of dS_{n-1}. By Theorem 2, this can be done by translation only. Thus we can apply (2) with (1), which implies the desired inequality. □

The above result together with $\gamma(n,S_{n-1}) \leq \Gamma(n,S_{n-1}) \leq 1$ gives

$$\lim_{n\to\infty} \gamma(n,S_{n-1}) = \lim_{n\to\infty} \Gamma(n,S_{n-1}) = 1.$$

If the simplex does pass through a hole, then in particular the volume of some central hyperplane section of that simplex is no bigger than the volume of the hole. After the RIMS workshop, Jiří Matoušek suggested showing $\gamma(n,S_{n-1}) \to 1$ by using this simple observation. He also told us the information from Keith Ball: it is conjectured that the smallest central hyperplane section of S_n is obtained by a hyperplane parallel to a facet of the simplex. According to Keith Ball's suggestion, we asked Matthieu Fradelizi about the volume of central slices of a simplex. Then, Fradelizi told us that a result in [4] implies that the volume of the smallest central hyperplane section of S_n is more than $\mathrm{vol}(S_{n-1})/(2\sqrt{3})$, and this is enough for proving $\gamma(n,S_{n-1}) \to 1$.

Since the diameter of circumsphere of $1S_n$ is $\sqrt{2(n-1)/n}$, we have

$$\Gamma(n,S_{n-1})\sqrt{\tfrac{2(n-1)}{n}} \geq \Gamma(n,B_{n-1}).$$

This together with (3) implies

$$\Gamma(n,S_{n-1}) \geq \sqrt{1-\tfrac{1}{n+1}}$$

for n odd. (For n even, Theorem 3 gives a better lower bound for $\Gamma(n,S_{n-1})$.)

Actually S_n can pass through a hole smaller than its facet.

Theorem 4. $\Gamma(n,S_{n-1}) < 1$ *for all* $n \geq 2$.

Let us try the case $n=3$ to get a feel. Let $S_2 = A_0A_1A_2, A_0 = (0,1/2), A_1 = (0,-1/2), A_2 = (\sqrt{3}/2, 0)$, and let $\mathcal{P}$ be the right triangular prism with base

$A_0A_1A_2$. We put the unit regular tetrahedron $S_3 = B_0B_1B_2B_3$ in the prism, namely, we set

$$B_0 = (0,1/2,0), B_1 = (0,-1/2,0), B_2 = (1/\sqrt{2},0,1/2), B_3 = (1/\sqrt{2},0,-1/2).$$

Now we move the tetrahedron very slightly keeping it inside $\mathscr{P}$ so that all vertices are off the faces of $\mathscr{P}$. This can be done by rotating the tetrahedron along the x-axis, and push it in the direction of x-axis. This gives $\Gamma(3,S_2) < 1$.

3.2. **The hole Q_{n-1}.** In [8] the following is proved: for every $\varepsilon > 0$ there is an N such that for every $n > N$ one has

$$1S_n \subset (2+\varepsilon)Q_n.$$

This gives

$$\lim_{n\to\infty} \Gamma(n, Q_{n-1}) \le 2.$$

Clearly we have $\Gamma(n,Q_{n-1}) \ge \Gamma(n,B_{n-1})$, and we get a lower bound for $\Gamma(n,Q_{n-1})$ from (3). Here we include a simple proof of the following slightly weaker bound.

Theorem 5. *We have*

$$\Gamma(n,Q_{n-1}) \ge \sqrt{\frac{2(n-1)}{n+1}}, \tag{6}$$

with equality holding iff there exists an Hadamard matrix of order $n+1$.

Proof. Let $d = \Gamma(n,Q_{n-1})$. Then $1S_n$ can pass through a hole of dQ_{n-1} by translation. So (2) and (1) imply

$$\text{width}(dQ_{n-1}) = \frac{d}{\sqrt{n-1}} \ge \text{width}(1S_n) \ge \sqrt{\frac{2}{n+1}},$$

which gives (6). Moreover, if $1S_n \subset \ell Q_n$, then we have

$$\ell \ge \frac{\sqrt{n}}{\sqrt{n-1}} \Gamma(n,Q_{n-1}) \ge \sqrt{\frac{2n}{n+1}}.$$

It is known that $\ell = \sqrt{(2n)/(n+1)}$ iff there exists an Hadamard matrix of order $n+1$, see e.g., [11]. $\square$

Problem 2.

$$\gamma(n,Q_{n-1}) = \Gamma(n,Q_{n-1}) = \sqrt{2} - o(1)?$$

3.3. **The hole B_{n-1}.** We have $\Gamma(n, B_{n-1}) \to \sqrt{2}$ by (3). On the other hand, the following result shows $\gamma(n, B_{n-1}) \to 3/(2\sqrt{2})$. Namely, "rotation" does help for escaping from the ball hole.

Theorem 6. [10]

(i) *For n even,*

$$\gamma(n, B_{n-1}) = \frac{3}{2\sqrt{2}}\left(1+\frac{1}{n}\right)^{-1/2} = \frac{3}{2\sqrt{2}}\left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{5}{16n^3} + O(n^{-4})\right).$$

(ii) *Let r^2 be a unique real root of the cubic equation*

$$8(n+1)n^3X^3 + a_2X^2 + a_1X + a_0 = 0,$$

where

$$\begin{aligned} a_0 &= -(9/256)(n^2-1)^2(n^4-4n^3+2n^2+4n+13), \\ a_1 &= (1/16)(n^2-1)(2n^6-6n^5-15n^4+38n^3+42n^2+48n-29), \\ a_2 &= (1/4)(8n^6-8n^5-41n^4-28n^3-10n^2+36n+27). \end{aligned}$$

Then, for n odd,

$$\gamma(n, B_{n-1}) = 2r = \frac{3}{2\sqrt{2}}\left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{13}{16n^3} + O(n^{-4})\right).$$

3.4. **Hole having minimum volume.** In [5], the following problem is posed.

Problem 3. *Find the minimum $(n-1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that $1S_n$ can pass through it.*

The following variation seems to be easier.

Problem 4. *Find the minimum $(n-1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that $1S_n$ can pass through it by translation perpendicular to the hyperplane.*

We list possible candidates. Put $\sqrt{2}S_n$ in $\mathbb{R}^{n+1}$ so that the vertices are $e_1, \ldots, e_{n+1}$, where e_i is the i-th standard base of $\mathbb{R}^{n+1}$.

Project the $\sqrt{2}S_n$ in the direction of

$$(1, -1, \overbrace{0, \ldots, 0}^{n-1}).$$

Then the hole created by the shadow has volume

$$\frac{1}{(n-1)!}\sqrt{\frac{n+1}{2}}. \tag{7}$$

Next suppose that n is odd and write $n = 2k+1$. Project the $\sqrt{2}S_n$ in the direction of

$$(\overbrace{1, \ldots, 1}^{k+1}, \overbrace{-1, \ldots, -1}^{k+1}).$$

Then the corresponding hole has volume

$$\frac{2}{(n-1)!}. \tag{8}$$

Finally suppose that n is even and write $n = 2k$. Project the $\sqrt{2}S_n$ in the direction of

$$(\overbrace{k+1,\dots,k+1}^{k},\overbrace{-k,\dots,-k}^{k+1}).$$

In this case, the volume of the hole is

$$\frac{2}{(n-1)!}\sqrt{\frac{n}{n+2}}. \tag{9}$$

Among the above examples, the smallest one is (7) for $n \le 5$. For $n = 7$, (7) and (8) coincide. For the other cases, (8) and (9) give the smallest one.

Acknowledgment

The authors would like to thank Jiří Matoušek, Matthieu Fradelizi, and Keith Ball for helpful suggestions and comments.

References

[1] R. Brandenberg, T. Theobald. Radii of simplices and some applications to geometric inequalities. *Beiträge Algebra Geom.* 45 (2004) 581–594.

[2] H. T. Croft, K. J. Falconer, R. K. Guy. *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991.

[3] H. E. Debrunner, P. Mani-Levitska. Can you cover your shadows? *Discrete Comput. Geom.* 1 (1986) 45–58.

[4] M. Fradelizi. Hyperplane sections of convex bodies in isotropic position. *Beiträge Algebra Geom.* 40 (1999) 163–183.

[5] J. I. Itoh, T. Zamfirescu. Simplices passing through a hole, *J. Geom.* 83 (2005) 65–70.

[6] J. Itoh, Y. Tanoue, T. Zamfirescu. Tetrahedra passing through a circular or square hole, *Rend. Circ. Mat. Palermo (2) Suppl. No. 77* (2006) 349–354.

[7] G. Kós, J. Törőcsik. Convex disks can cover their shadow. *Discrete Comput. Geom.* 5 (1990) 529–531.

[8] H. Maehara, I. Ruzsa, N. Tokushige. Large regular simplices contained in a hypercube. *Period. Math. Hungarica*, 58 (2009) 121–126.

[9] H. Maehara, N. Tokushige. A regular tetrahedron passes through a hole smaller than its face. *submitted.*

[10] H. Maehara, N. Tokushige. in preparation.

[11] I. J. Schoenberg. Regular simplices and quadratic forms. *Journal of the London Mathematical Society* 12 (1937) 48-55.

[12] P. Steinhagen. Über die grösste Kugel in einer konvexen Punktmenge. *Abh. Math. Sem. Hamburg*, 1 (1921) 15–26.

[13] T. Zamfirescu. How to hold a convex body, *Geometrae Dedicata* 54 (1995) 313–316.

[14] K. Zindler. Über konvexe Gebilde, *Monatsh. Math. Physik* 30 (1920) 87–102.