

Spherical nonspreadingness and perturbations for resolvents of convex functions on geodesic spaces 測地距離空間における凸関数のリゾルベントの 球面的非伸長性と摂動関数

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Abstract

In this paper, we consider resolvent operators of convex functions with various perturbation functions. We prove that there are many resolvent operators which are spherically nonspreading of sum-type in a geodesic space with curvature bounded above by one.

1 Introduction and preliminaries

For an admissible CAT(1) space X , a mapping T from X into itself is said to be *spherically nonspreading of sum-type* [3] if an inequality

$$2 \cos d(Tx, Ty) \geq \cos d(x, Ty) + \cos d(y, Tx)$$

holds for any two points x and y in X . This concept is a special case of vicinal mappings with ψ defined by Kajimura and Kimura on an admissible CAT(κ) space.

Definition 1.1 (Kajimura–Kimura [2])

Let X be an admissible CAT(1) space and $\psi: [0, \pi/2[\rightarrow]0, \infty[$ a right-continuous function at 0. Then a mapping T from X into itself is said to be *vicinal with ψ* (for $\kappa = 1$) if

$$\begin{aligned} (\psi(d(x, Tx)) + \psi(d(y, Ty))) \cos d(Tx, Ty) \\ \geq \psi(d(x, Tx)) \cos d(x, Ty) + \psi(d(y, Ty)) \cos d(y, Tx) \end{aligned}$$

for any $x, y \in X$.

Note that its original definition in [2] is in an admissible $\text{CAT}(\kappa)$ space for every $\kappa \in \mathbb{R}$; however, this paper only considers the case where $\kappa = 1$. Let $\mathbb{1}: [0, \pi/2[\rightarrow \mathbb{R}$ be a constant function defined by $\mathbb{1}(t) = 1$ for any $t \in [0, \pi/2[$. From Definition 1.1, we know that a mapping T is spherically nonspreading of sum-type if and only if it is vicinal with the constant function $\mathbb{1}$, or equivalently, it is vicinal with any positive constant function.

We say that a sequence $\{y_n\}$ on an admissible $\text{CAT}(1)$ space X is *spherically bounded* if there exists $u \in X$ such that $\sup_{n \in \mathbb{N}} d(y_n, u) < \pi/2$. Now we introduce the following theorem.

Theorem 1.2 (Kajimura [1, Theorem 5.15])

Let X be an admissible complete $\text{CAT}(1)$ space and $T: X \rightarrow X$ a vicinal mapping with ψ . Then T has a fixed point if and only if there exists $x \in X$ such that $\{T^n x\}_{n=1}^\infty$ is spherically bounded and $\sup_{n \in \mathbb{N}} \psi(d(T_{n+1}x, T_nx)) < \infty$.

In Theorem 1.2, the condition $\sup_{n \in \mathbb{N}} \psi(d(T_{n+1}x, T_nx)) < \infty$ follows automatically whenever T is vicinal with $\mathbb{1}$. Hence, the following holds:

Corollary 1.3

Let X be an admissible complete $\text{CAT}(1)$ space. Then a spherically nonspreading mapping T of sum-type has a fixed point if and only if $\{T^n x\}_{n=1}^\infty$ is spherically bounded for some $x \in X$.

Consequently, to check whether a spherically nonspreading mapping T of sum-type has a fixed point, we only need to examine the spherical boundedness of $\{T^n x\}$ for $x \in X$.

Let X be an admissible $\text{CAT}(1)$ space, and f a proper convex function from X into $]-\infty, \infty]$. We call a mapping J_f a resolvent operator of f if it is a single-valued mapping from X into itself and its fixed point set $F(J_f)$ coincides with the set $\text{argmin } f$ of all minimizers of f . From previous studies, such as [3, 5, 6, 7], we know that a set-valued mapping $J_f: X \rightarrow 2^X$ defined by the formula

$$\begin{aligned} J_f x &= \underset{y \in X}{\text{argmin}} (f(y) + \Phi(d(x, y))) \\ &= \left\{ z \in X \mid \inf_{y \in X} (f(y) + \Phi(d(x, y))) \geq f(z) + \Phi(d(x, z)) \right\} \end{aligned}$$

for $x \in X$ becomes a resolvent operator of f on X by using a perturbation function $\Phi: [0, \pi/2[\rightarrow \mathbb{R}$ under certain assumptions. Now we consider set-valued mappings Q_f , R_f , and S_f from X into 2^X defined by

$$\begin{aligned} Q_f x &= \underset{y \in X}{\text{argmin}} (f(y) - \log \cos d(x, y)); & R_f x &= \underset{y \in X}{\text{argmin}} (f(y) + 1 - \cos d(x, y)); \\ S_f x &= \underset{y \in X}{\text{argmin}} (f(y) + \tan d(x, y) \sin d(x, y)) \end{aligned}$$

for $x \in X$, respectively. If X is complete and f is lower semicontinuous, then Q_f and S_f can be defined as a single-valued mapping on X ; see [3, Theorem 3.2] and [5,

Theorem 4.2], respectively. Moreover, if X is complete, f is lower semicontinuous, and f has at least one minimizer, then R_f becomes a single-valued mapping, see [7, Lemma 5.2.1]. Furthermore, we know that Q_f and R_f are spherically nonspreading of sum-type; however, we do not know whether S_f is so or not.

In this paper, we show that there exist resolvent operators other than Q_f and R_f that satisfy the spherical nonspreadingness of sum-type. Specifically, we prove that the following resolvents $T_f^{\lambda,n}$ and U_f are spherically nonspreading of sum-type:

$$T_f^{\lambda,n}x = \operatorname{argmin}_{y \in X} \left(f(y) + \int_0^{1-\cos d(x,y)} \left(1 + \lambda \cdot \frac{s^n}{1-s} \right) ds \right);$$

$$U_fx = \operatorname{argmin}_{y \in X} \left(f(y) + \sqrt{(1 - \cos d(x,y))(-\log \cos d(x,y))} \right)$$

for $x \in X$, $\lambda \in [0, 1]$, and $n \in [1, \infty[$.

The following is a crucial notion that describes a behavior of resolvent operators:

Definition 1.4 (Kajimura–Kimura [2])

Let X be an admissible CAT(1) space and T a mapping from X into itself. Let $\psi: [0, \pi/2[\rightarrow]0, \infty[$ be a function which is right-continuous at 0. Then, T is said to be *firmly vicinal with ψ* (for $\kappa = 1$) if

$$\begin{aligned} & (\psi(d(x, Tx)) \cos d(x, Tx) + \psi(d(y, Ty)) \cos d(y, Ty)) \cos d(Tx, Ty) \\ & \geq \psi(d(x, Tx)) \cos d(x, Ty) + \psi(d(y, Ty)) \cos d(y, Tx) \end{aligned}$$

for any $x, y \in X$.

In Definition 1.4, since X is admissible, we know that every firmly vicinal mapping with ψ is vicinal with the same ψ .

Lemma 1.5 (Kajimura–Kimura [2, Theorem 4.1])

Let X be an admissible CAT(1) space and f a proper convex function from X into $] -\infty, \infty]$. Let $\overline{\varphi}:]0, 1] \rightarrow [0, \infty[$ be a nonincreasing and differentiable function such that $\overline{\varphi}'$ is continuous on $]0, 1]$. Suppose that a set-valued mapping $J_f: X \rightarrow 2^X$ defined by

$$J_fx = \operatorname{argmin}_{y \in X} (f(y) + \overline{\varphi}(\cos(d(x, y))))$$

for $x \in X$ is well-defined as a single-valued mapping on X . Then, J_f is firmly vicinal with $-\overline{\varphi}' \circ \cos$.

Lemma 1.6 ([6, Theorem 6.5])

The mapping J_f defined in Lemma 1.5 satisfies $F(J_f) = \operatorname{argmin} f$.

Lemma 1.7 ([6, Theorem 6.8, Theorem 5.30])

Let X be an admissible complete CAT(1) space and $f: X \rightarrow] -\infty, \infty]$ a proper lower

semicontinuous convex function. Let $\bar{\varphi}:]0, 1] \rightarrow [0, \infty[$ be a strictly decreasing and differentiable function such that $\bar{\varphi}'$ is nondecreasing and continuous on $]0, 1]$. Suppose that $\lim_{t \searrow 0} \bar{\varphi}(t) = \infty$. Define a set-valued mapping $J_f: X \rightarrow 2^X$ by

$$J_f x = \operatorname{argmin}_{y \in X} (f(y) + \bar{\varphi}(\cos d(x, y)))$$

for $x \in X$. Then the following hold:

- (i) J_f is well-defined as a single-valued mapping on X ;
- (ii) J_f is firmly vicinal with $-\bar{\varphi}' \circ \cos: [0, \pi/2[\rightarrow [0, \infty[$ (by Lemma 1.5);
- (iii) $F(J_f) = \operatorname{argmin} f$ (by Lemma 1.6).

2 Main results

In this section, we investigate the nature of several resolvents. Let X be an admissible CAT(1) space, $f: X \rightarrow]-\infty, \infty]$ a proper convex function, $\lambda \in [0, 1]$, and $n \in [1, \infty[$. Let us consider mappings $Q_f, R_f, S_f, T_f^{\lambda, n}$, and U_f defined in Section 1. Define functions $\bar{\varphi}_1^{\lambda, n}, \bar{\varphi}_2:]0, 1] \rightarrow [0, \infty[$ by

$$\bar{\varphi}_1^{\lambda, n}(t) = \int_0^{1-t} \left(1 + \lambda \cdot \frac{s^n}{1-s}\right) ds \quad \text{and} \quad \bar{\varphi}_2(t) = \sqrt{(1-t)(-\log t)}$$

for $t \in]0, 1]$, respectively. Then, $T_f^{\lambda, n}x$ and $U_f x$ are figured by

$$\operatorname{argmin}_{y \in X} (f(y) + \bar{\varphi}_1^{\lambda, n}(\cos d(x, y))) \quad \text{and} \quad \operatorname{argmin}_{y \in X} (f(y) + \bar{\varphi}_2(\cos d(x, y)))$$

for $x \in X$, respectively. Consider the case where X is complete, f is lower semicontinuous, and $\lambda \neq 0$. Then Lemma 1.7 ensures that each mapping $Q_f, S_f, T_f^{\lambda, n}, U_f: X \rightarrow 2^X$ becomes a single-valued mapping on X . In contrast, Lemma 1.7 does not guarantee the well-definedness of R_f as a single-valued mapping. However, as mentioned before, Sudo [7, Lemma 5.2.1] showed that R_f is well-defined if $\operatorname{argmin} f \neq \emptyset$.

Now we prove the following result, which provides a sufficient condition for a firmly vicinal mapping T with ψ to be spherically nonspreading of sum-type.

Theorem 2.1

Let $\chi:]0, 1] \rightarrow [0, \infty[$ be a left-continuous function at 1. Let X be an admissible CAT(1) space and $T: X \rightarrow X$ a firmly vicinal mapping with $\psi = \chi \circ \cos$. Suppose that there exists $k > 0$ such that

$$1 \leq \frac{1}{k} \chi(t) \leq \frac{1}{t} \tag{2.1}$$

for any $t \in]0, 1[$. Then, T is spherically nonspreading of sum-type.

Proof. Suppose that $T: X \rightarrow X$ is firmly vicinal with $\psi = \chi \circ \cos$ and that (2.1) holds for any $t \in]0, 1]$. Then we have $\psi(d) \cos d \leq k \leq \psi(d)$ for any $d \in [0, \pi/2[$. Hence,

putting $K_x = d(x, Tx)$ and $K_y = d(y, Ty)$, we get

$$\begin{aligned} 2k \cos d(Tx, Ty) &\geq (\psi(K_x) \cos d(x, Tx) + \psi(K_y) \cos d(y, Ty)) \cos d(Tx, Ty) \\ &\geq \psi(K_x) \cos d(x, Ty) + \psi(K_y) \cos d(y, Tx) \\ &\geq k \cos d(x, Ty) + k \cos d(y, Tx) \end{aligned}$$

for any $x, y \in X$, which is the desired result. $\square$

From the following result, we can construct many functions $\bar{\varphi}$ which make a resolvent defined by the perturbation $\bar{\varphi} \circ \cos$ spherically nonspreading of sum-type.

Theorem 2.2

Let X be an admissible CAT(1) space and f a proper convex function from X into $]-\infty, \infty]$. Let $\chi_1:]0, 1] \rightarrow \mathbb{R}$ be a continuous function such that

$$1 \leq \chi_1(t) \leq \frac{1}{t}$$

for any $t \in]0, 1]$. For arbitrarily fixed $k > 0$, define $\bar{\varphi}:]0, 1] \rightarrow [0, \infty[$ by

$$\bar{\varphi}(t) = \int_t^1 k \chi_1(s) ds \quad (2.2)$$

for $t \in]0, 1]$, and $J_f: X \rightarrow 2^X$ by

$$J_f x = \operatorname{argmin}_{y \in X} (f(y) + \bar{\varphi}(\cos d(x, y))) \quad (2.3)$$

for $x \in X$. Suppose that J_f is well-defined as a single-valued mapping on X . Then the following hold:

- (i) J_f is firmly vicinal with $\chi_1 \circ \cos: [0, \pi/2[\rightarrow]0, \infty[$;
- (ii) $F(J_f) = \operatorname{argmin} f$;
- (iii) J_f is spherically nonspreading of sum-type.

Theorem 2.3

Let X , f , χ_1 , k , and $\bar{\varphi}$ be the same as in Theorem 2.2. We additionally suppose that the following conditions hold:

- X is complete;
- f is lower semicontinuous;
- χ_1 is nondecreasing;
- $\int_0^1 \chi_1(s) ds = \infty$.

Then, $J_f: X \rightarrow 2^X$ defined by (2.3) becomes a single-valued mapping on X .

Proof of Theorem 2.2. Let $\chi := -\bar{\varphi}' = k\chi_1$. Then Theorem 1.5 implies that J_f is firmly vicinal with $\chi \circ \cos$, and hence (i) holds. Moreover, (ii) is proved by Lemma 1.6.

We also deduce that $1 \leq (1/k)\chi(t) \leq 1/t$ for any $t \in]0, 1[$, and therefore Theorem 2.1 implies (iii). $\square$

Proof of Theorem 2.3. We can see that $\bar{\varphi}$ is strictly increasing, $\bar{\varphi}'$ is nondecreasing, and $\lim_{t \searrow 0} \bar{\varphi}(t) = \infty$. Therefore, from Lemma 1.7, we obtain the conclusion. $\square$

We also obtain the following:

Corollary 2.4

Let X and f be the same as in Theorem 2.2. Let $M:]0, 1] \rightarrow [0, 1]$ be a continuous function, and define $\chi_1:]0, 1] \rightarrow \mathbb{R}$ by

$$\chi_1(t) = 1 + \frac{1-t}{t}M(t)$$

for $t \in]0, 1]$. For arbitrarily fixed $k > 0$, define $\bar{\varphi}$ and J_f by formulas (2.2) and (2.3). Then the following hold:

- (I) *If J_f is well-defined as a single-valued mapping on X , then the same statements (i)–(iii) as in Theorem 2.2 hold;*
- (II) *if the same four conditions as in Theorem 2.3 hold, then J_f becomes a single-valued mapping on X .*

Proof. Since $1 \leq \chi_1(t) \leq 1/t$ for any $t \in]0, 1[$, we obtain the conclusion. $\square$

In what follows, we always assume that X is an admissible CAT(1) space and $f: X \rightarrow]-\infty, \infty]$ is a proper convex function. Consider the resolvents $T_f^{\lambda, n}$, Q_f , and R_f defined in Section 1, and we assume that these are single-valued mappings on X . Using Corollary 2.4, we can show that these resolvents are spherically nonspreading of sum-type.

Theorem 2.5

For fixed real numbers $\lambda \in [0, 1]$ and $n \in [1, \infty[$, the resolvent $T_f^{\lambda, n}$ is spherically nonspreading of sum-type.

Proof. Put $M(t) = \lambda(1-t)^{n-1}$ if $n > 1$, and $M(t) = \lambda$ if $n = 1$. Moreover, let

$$\chi(t) = 1 + \frac{1-t}{t}M(t) = 1 + \lambda \cdot \frac{(1-t)^n}{t}$$

for $t \in]0, 1]$. Then we can see that $M(t) \in [0, 1]$ for any $t \in]0, 1]$, and

$$\bar{\varphi}_1^{\lambda, n}(t) = \int_0^{1-t} \left(1 + \lambda \cdot \frac{s^n}{1-s}\right) ds = \int_t^1 \chi(s) ds$$

for any $t \in]0, 1]$. Hence, Corollary 2.4 implies the desired result. $\square$

Theorem 2.6 (Kajimura–Kimura [3, Theorem 3.5])

The resolvent Q_f is spherically nonspreading of sum-type.

Proof. Consider the case where $k = 1$ and $M(t) = 1$ for $t \in]0, 1]$ in Corollary 2.4. $\square$

Theorem 2.7 (Sudo [7, Remark 11, Remark 16])

The resolvent R_f is spherically nonspreading of sum-type.

Proof. Consider the case where $k = 1$ and $M(t) = 0$ for $t \in]0, 1]$ in Corollary 2.4. $\square$

For $\lambda \in [0, 1]$ and $n \in [1, \infty[$, consider the perturbation $\Phi^{\lambda, n} := \overline{\varphi}_1^{\lambda, n} \circ \cos$ of $T_f^{\lambda, n}$, which is represented by

$$\Phi^{\lambda, n}(d) = \int_0^{1-\cos d} \left(1 + \lambda \cdot \frac{s^n}{1-s}\right) ds$$

for each $d \in [0, \pi/2[$. Then we see that $\Phi^{1,1}(d) = -\log \cos d$ for any $d \in [0, \pi/2[$. Moreover, for fixed $n \in [1, \infty[$, we obtain $\Phi^{0,n}(d) = 1 - \cos d$. Furthermore, for fixed $\lambda \in [0, 1]$ and $d \in [0, \pi/2[$, it follows that $\lim_{n \rightarrow \infty} \Phi^{\lambda, n}(d) = 1 - \cos d$. These mean that the perturbation of the resolvent $T_f^{\lambda, n}$ connects smoothly those of Q_f and R_f .

Recently, Kajimura et al. [4] showed that the resolvent $V_f: X \rightarrow X$ defined by

$$V_f x = \operatorname{argmin}_{y \in X} (f(y) + 1 - \cos d(x, y) - \log \cos d(x, y))$$

for $x \in X$ is spherically nonspreading of sum-type. It is obtained by considering the case where $k = 2$ and $M(t) = 1/2$ for $t \in]0, 1]$ in Corollary 2.4. Moreover, we also know that V_f is ideltical to $T_{(1/2)f}^{1/2,1}$.

Finally, we show the resolvent operator U_f is spherically nonspreading of sum-type. Before that, we introduce the following lemma.

Lemma 2.8

Let $f, g:]0, 1[\rightarrow \mathbb{R}$ be differentiable functions having the following properties:

- $0 < f(t) \leq g(t)$ for any $t \in]0, 1[$;
- $f'(t) \leq 0, g'(t) \leq 0$ for any $t \in]0, 1[$;
- $f'(t)g(t) - f(t)g'(t) \geq 0$ for any $t \in]0, 1[$.

Let $h(t) = \sqrt{f(t)g(t)}$ for any $t \in]0, 1[$. Then, $g'(t) \leq h'(t) \leq f'(t)$ for any $t \in]0, 1[$.

Proof. Fix $t \in]0, 1[$. Then we have $0 < f(t) \leq h(t) \leq g(t)$, which yields

$$2h(t)g'(t) \leq 2f(t)g'(t) \leq f'(t)g(t) + f(t)g'(t) = (f(t)g(t))'.$$

Similarly, it follows that

$$2h(t)f'(t) \geq 2g(t)f'(t) \geq f'(t)g(t) + f(t)g'(t) = (f(t)g(t))'.$$

Therefore we obtain

$$g'(t) \leq \frac{(f(t)g(t))'}{2h(t)} \leq f'(t).$$

Since

$$\frac{(f(t)g(t))'}{2h(t)} = \frac{(f(t)g(t))'}{2\sqrt{f(t)g(t)}} = h'(t),$$

we get the conclusion. $\square$

Assume that U_f is well-defined as a single-valued mapping on X .

Theorem 2.9

The resolvent U_f is spherically nonspreading of sum-type.

Proof. Define $\chi:]0, 1] \rightarrow [1, \infty[$ by

$$\chi(t) = -\bar{\varphi}'_2(t) = \frac{1 - t - t \log t}{2t\sqrt{(1-t)(-\log t)}}$$

for $t \in]0, 1[$, and $\chi(1) = \lim_{t \nearrow 1} \chi(t) = 1$. Then U_f is firmly vicinal with $\chi \circ \cos$. Let $f(t) = 1 - t$ and $g(t) = -\log t$ for $t \in]0, 1]$. Then $\bar{\varphi}_2(t) = \sqrt{f(t)g(t)}$ for any $t \in]0, 1]$. Therefore, from Lemma 2.8, we conclude that $g'(t) \leq \bar{\varphi}'_2(t) \leq f'(t)$ for any $t \in]0, 1[$, from which it follows that

$$1 \leq \chi(t) \leq \frac{1}{t}$$

for any $t \in]0, 1[$. Consequently, Theorem 2.1 implies the desired result. $\square$

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