

The splitting of cohomology of metacyclic p -groups

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Abstract

Let BP be the p -complete classifying space of a metacyclic p -group P . By using stable homotopy splitting of BP , we study the decomposition of $H^{even}(P; \mathbb{Z})/p$ and $CH^*(BP)/p$.

1 Introduction

Let P be a p -group and BP be its p -completed classifying space of P . We study the stable splitting and splitting of cohomology

$$(*) \quad BP \cong X_1 \vee \dots \vee X_i,$$

$$(**) \quad H^*(P) \cong H^*(X_1) \oplus \dots \oplus H^*(X_i) \quad (\text{for } * > 0)$$

where X_i are irreducible spaces in the stable homotopy category. Using the answer of the Segal conjecture by Carlsson, the splitting $(*)$ is given by only using modular representation theory by Nishida [Ni], Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr]. These theorems do not use splittings of cohomology

In particular, Dietz and Dietz-Priddy [Di], [Di-Pr] gave the stable splitting $(*)$ for groups P with $\text{rank}_p(P) = 2$ for $p \geq 5$. However it was not used splittings $(**)$ of the cohomology $H^*(P)$, and the cohomologies $H^*(X_i)$ were not given there.

In [Hi-Ya 1,2], we gives the cohomology of $H^*(X_i)$ (and hence $(**)$) for $P = (\mathbb{Z}/p)^2$ and $P = p_+^{1+2}$ the extraspecial p group of order p^3 and exponent p . Their cohomology $H^*(X_i)$ have very complicated but rich structures, in fact p_+^{1+2} is a p -Sylow subgroup of many interesting groups, e.g., $GL_3(\mathbb{F}_p)$ and many simple groups e.g. J_4 for $p = 3$.

In this paper, we give the decomposition of

$$H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0}) \quad (\text{and } H^{ev}(P) = H^{even}(P; \mathbb{Z})/p)$$

for metacyclic p -groups for odd primes p , while in most cases, $H^*(X_i)$ are easily got and seemed not to have so rich structure as p_+^{1+2} , because they are not p -Sylow subgroups of so interesting groups. Indeed, metacyclic p -groups P are

Swan groups, i.e. for all groups G which have a Sylow p -subgroup isomorphic to P , we have the isomorphism

$$H^*(G) \cong H^*(P)^W \quad \text{for some } W \subset \text{Out}(P).$$

However, we believe that it becomes quite clear the relations among splittings of different types of metacyclic p -groups. (We compute the coarse splitting of $H^*(X_i)$ at first, and next more fine splitting $H^*(X'_j)$, in the case $H^*(P) \cong H^*(P')$).

In the last section, we note the relation to the Chow ring $CH^*(BP)/p$ and $H^{\text{even}}(P; \mathbb{Z})/p$, and note that the Chow group of the direct summand X_i is represented by some motive.

2 The double Burnside algebra and stable splitting

Let us fix an odd prime p and $k = \mathbb{F}_p$. For finite groups G_1, G_2 , let $A_{\mathbb{Z}}(G_1, G_2)$ be the double Burnside group defined by the Grothendieck group generated by (G_1, G_2) -bisets. Each element Φ in $A_{\mathbb{Z}}(G_1, G_2)$ is generated by elements $[Q, \phi] = (G_1 \times_{(Q, \phi)} G_2)$ for some subgroup $Q \leq G_1$ and a homomorphism $\phi : Q \rightarrow G_2$. In this paper, we use the notation

$$[Q, \phi] = \Phi : G_1 \geq Q \xrightarrow{\phi} G_2.$$

For each element $\Phi = [Q, \phi] \in A_{\mathbb{Z}}(G_1, G_2)$, we can define a map from $H^*(G_2; k)$ to $H^*(G_1; k)$ by

$$x \cdot \Phi = x \cdot [Q, \phi] = \text{Tr}_Q^{G_1} \phi^*(x) \quad \text{for } x \in H^*(G_2; k).$$

When $G_1 = G_2$, the group $A_{\mathbb{Z}}(G_1, G_2)$ has the natural ring structure, and call it the (integral) double Burnside algebra. In particular, for a finite group G , we have an $A_{\mathbb{Z}}(G, G)$ -module structure on $H^*(G; k)$ (and $H^*(G; \mathbb{Z})/p$).

The following lemma is an easy consequence of Quillen's theorem such that the restriction map

$$H^*(G; \mathbb{Z}/p) \rightarrow \lim_{\substack{\longrightarrow \\ V}} H^*(V; \mathbb{Z}/p)$$

is an F-isomorphism (i.e. the kernel and cokernel are nilpotent) where V ranges elementary abelian p -subgroups of G .

Lemma 2.1. *Let $\sqrt{0}$ be the nilpotent ideal in $H^*(G; k)$ (or $H^*(G; \mathbb{Z})/p$). Then $\sqrt{0}$ itself is an $A_{\mathbb{Z}}(G, G)$ -module.*

In this paper we consider, at first, the cohomology modulo nilpotents elements, since it is not so complicated from the above Quillen's theorem. Hence we write it simply

$$H^*(G) = H^*(G; \mathbb{Z})/(p, \sqrt{0}).$$

However we also compute $H^{even}(G; \mathbb{Z})/p$ in §4 below.

By the preceding lemma, we see that $H^*(G)$ has the $A_{\mathbb{Z}}(G, G)$ -module structure. (Here note that $A_{\mathbb{Z}}(G, G)$ acts on unstable cohomology.) Throughout this paper, we assume that degree $*$ > 0 (or we consider $H^*(-)$ as the reduced theory $\tilde{H}^*(-)$). (We consider $H^*(G)$ as an element in $K_0(\text{Mod}(A_{\mathbb{Z}}(G, G)))$.)

Let $BG = BG_p$ be the p -completion of the classifying space of G . Recall that $\{BG, BG\}_p$ is the (p -completed) group generated by stable homotopy self maps. It is well known from the Segal conjecture (Carlsson's theorem) that this group is isomorphic to the double Burnside group $A_{\mathbb{Z}}(G_1, G_2)^\wedge$ completed by the augmentation ideal.

Since the transfer is represented as a stable homotopy map Tr , an element $\Phi = [Q, \phi] \in A(G_1, G_2)$ is represented as a map $\Phi \in \{BG_1, BG_2\}_p$

$$\Phi : BG_1 \xrightarrow{Tr} BQ \xrightarrow{B\phi} BG_2.$$

(Of course, the action for $x \in H^*(G_2)$ is given by $Tr_Q^{G_1} \phi^*(x)$ as stated.)

Let us write

$$A(G_1, G_2) = A_{\mathbb{Z}}(G_1, G_2) \otimes k \quad (k = \mathbb{Z}/p).$$

Hereafter we consider the cases $G_i = P$; p -groups. Given a primitive idempotents decomposition of the unity of $A(P, P)$

$$1 = e_1 + \dots + e_n,$$

we have an indecomposable stable splitting

$$BP \cong X_1 \vee \dots \vee X_n \quad \text{with } e_i BP = X_i.$$

In this paper, an isomorphism $X \cong Y$ for spaces means that it is a stable homotopy equivalence.

Recall that

$$M_i = A(P, P)e_i / (\text{rad}(A(P, P))e_i)$$

is a simple $A(P, P)$ -module where $\text{rad}(-)$ is the Jacobson ideal. By Wedderburn's theorem, the above decomposition is also written as

$$BP \cong \bigvee_j (\bigvee_k X_{jk}) = \bigvee_j m_j X_{j1} \quad \text{where } m_j = \dim(M_j)$$

for $A(P, P)e_{jk} / \text{rad}(A(P, P)e_{jk}) \cong M_j$. Therefore the stable splitting of BP is completely determined by the idempotent decomposition of the unity in the double Burnside algebra $A(P, P)$.

For a simple $A(P, P)$ -module M , define a stable summand $X(M)$ by

$$e_M = \sum_{M_i \cong M} e_i, \quad X(M) = \bigvee_{M_{jk} \cong M} X_{jk} = e_M BP.$$

Here $X(M)$ is only defined in the stable homotopy category. (So strictly, the cohomology ring $H^*(X(M))$ is not defined.) However we define $H^*(X(M))$ by

$$H^*(X(M)) = H^*(P) \cdot e_M \quad (= e_M^* H^*(P) \text{ stably})$$

where we think $e_M \in A(P, P)$ (rather than $e_M \in \{BP, BP\}$).

Lemma 2.2. *Given a simple $A(P, P)$ -module M , there is a filtration of $H^*(X(M))$ such that the associated graded ring $grH^*(X(M))$ is isomorphic to a sum of M , i.e., (for $s > 0$)*

$$grH^*(X(M)) \cong \bigoplus_{i=1}^s M[k_i], \quad 0 \leq k_1 \leq \dots \leq k_s \leq \dots$$

where $[k_s]$ is the operation ascending degree k_s .

From Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr], it is known that each simple $A(P, P)$ -module is written as

$$S(P, Q, V) \quad \text{for } Q \leq P, \text{ and } V : \text{simple } k[\text{Out}(Q)] - \text{module.}$$

(In fact $S(P, Q, V)$ is simple or zero.) Thus we have the main theorem of stable splitting of BP .

Theorem 2.3. *(Benson-Feshbach [Be-Fe], Martino-Priddy [Ma-Pr]) There are indecomposable stable spaces $X_{S(P, Q, V)}$ for $S(P, Q, V) \neq 0$ such that*

$$BP \cong \vee X(S(P, Q, V)) \cong \vee (\dim S(P, Q, V)) X_{S(P, Q, V)}.$$

3 Metacyclic groups for $p \geq 3$

In this section, we consider metacyclic p groups P for $p \geq 3$

$$0 \rightarrow \mathbb{Z}/p^m \rightarrow P \rightarrow \mathbb{Z}/p^n \rightarrow 0.$$

These groups are represented as

$$(*) \quad P = \langle a, b | a^{p^m} = 1, a^{p^{m'}} = b^{p^n}, [a, b] = a^{rp^\ell} \rangle \quad r \neq 0 \pmod{p}.$$

It is known by Thomas [Th], Huebuschmann [Hu] that $H^{even}(P; \mathbb{Z})$ is generated by Chern classes of complex representations. Let us write

$$\begin{cases} y = c_1(\rho), & \rho : P \rightarrow P/\langle a \rangle \rightarrow \mathbb{C}^* \\ v = c_{p^m-\ell}(\eta), & \eta = \text{Ind}_H^P(\xi), \quad \xi : H = \langle a, b^{p^{m-\ell}} \rangle \rightarrow \langle a \rangle \rightarrow \mathbb{C}^* \end{cases}$$

where ρ, ξ are nonzero linear representations. Then $H^{even}(P; \mathbb{Z})$ is generated by

$$y, c_1(\eta), c_2(\xi), \dots, c_{p^m-\ell}(\eta) = v.$$

(Lemma 3.5 and the explanation just before this lemma in [Ya1].) We can see

$$c_1(\eta) = 0, \dots, c_{p^m-\ell-1}(\eta) = 0 \quad \text{in } H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0}).$$

By using Quillen's theorem and the fact that P has just one conjugacy class of maximal abelian p -subgroups, we can prove

Theorem 3.1. (Theorem 5.45 in [Ya1]) For any metacyclic p -group P in $(*)$ with $p \geq 3$, we have a ring isomorphism

$$H^*(P) \cong k[y, v], \quad |v| = 2p^{m-\ell}.$$

We now consider the stable splitting.

(I) Non split cases. For a nonsplit metacyclic groups, it is proved that BP itself is irreducible [Di].

(II) Split cases with $(\ell, m, n) \neq (1, 2, 1)$. We consider a split metacyclic group. it is written as

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell} \rangle$$

for $m > \ell \geq \max(m - n, 1)$.

The outer automorphism is the semidirect product

$$\text{Out}(P) \cong (p - \text{group}) : \mathbb{Z}/(p - 1).$$

The p -group acts trivially on $H^*(P)$, and $j \in \mathbb{Z}/(p - 1)$ acts on $a \mapsto a^j$ and so acts on $H^*(P)$ as $j^* : v \mapsto jv$. There are $p - 1$ simple $\mathbb{Z}/(p - 1)$ -modules $S_i \cong k\{v^i\}$. We consider the decomposition by idempotens for $\text{Out}(P)$. Let us write $Y_i = e_{S_i}BP$ and

$$H_i^*(P) = H^*(S_i) \cong (\dim(S_i))H^*(Y_i) \subset H^*(P).$$

Hence we have the decomposition for $\text{Out}(P)$ -idempotents

$$H^*(Y_i) = H_i(P) \cong k[y, V]\{v^i\}, \quad V = v^{p-1}.$$

Here we used the notation such that $A\{a, b, \dots\}$ means the A -free module generated by $a, b, \dots$

We assume $P \neq M(1, 2, 1)$. By Dietz, we have splitting

$$(**) \quad BP \cong \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{i=0}^{p-2} L(1, i).$$

Here we write $X_i = e_{S(P, P, S_i)}BP$ identifying S_i as the $A(P, P)$ simple module (but not the simple $\text{Out}(P)$ -module).

The summand $L(1, i)$ is defined as follows. Recall that $H^*(\langle b \rangle) \cong k[y]$. The outer automorphism group is $\text{Out}(\langle b \rangle) \cong (\mathbb{Z}/p^n)^*$ and its simple k modules are $S'_i = k\{y^i\}$ for $0 \leq i \leq p - 2$. Hence we can decompose

$$B\langle b \rangle \cong \bigvee_{i=0}^{p-2} L(1, i), \quad H^*(L(1, i)) \cong k[Y]\{y^i\} \quad \text{with } Y = y^{p-1}.$$

Next we consider $L(1, i)$ as a split summand in BP as follows. (Consider the $A(P, P)$ -simple module $S(P, \langle b \rangle, S'_i)$.) Let $\Phi \in A(P, P)$ be the element defined by the map $\Phi : P \rightarrow \langle b \rangle \subset P$ which induced the isomorphism

$$H^*(P)\Phi \cong H^*(\bigvee_{i=0}^{p-2} L(1, i)) \cong k[y].$$

Thus we can show (since $k[y]$ is invariant under elements in $\text{Out}(P)$)

$$(***) \quad Y_i \cong \begin{cases} X_i & i \neq 0 \\ X_0 \vee \bigvee_{j=0}^{p-2} L(1, j) & i = 0. \end{cases}$$

Theorem 3.2. *Let P be a split metacyclic group with $(\ell, m, n) \neq (1, 2, 1)$. Then we have*

$$H^*(X_i) \cong \begin{cases} k[y, V]\{v^i\} & i \neq 0 \\ k[y, V]\{V\} & i = 0. \end{cases}$$

Proof. For $i \neq 0$, we have $H_i^*(P) = H^*(Y_i) \cong H^*(X_i)$. Let us use the notation that $A \oplus B = C$ means $A \cong B \oplus C$. Then we see

$$\begin{aligned} H^*(X_0) &\cong H^*(Y_0) \oplus H^*(\bigvee_{j=0}^{p-2} L(1, j)) \\ &\cong k[y, V] \oplus k[y] \cong k[y, V]\{V\}. \end{aligned}$$

□

(III) Split metacycle group with $(\ell, m, n) = (1, 2, 1)$.

This case $P = p_-^{1+2}$ and its cohomology is the same as (II). But the splitting is given

$$BP \cong \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{i=0}^{p-2} L(2, i) \vee \bigvee_{i=0}^{p-2} L(1, i).$$

Detailed explanation for $L(2, i)$ see [M-P], [Hi-Ya1]. Let $H = \langle b, a^p \rangle$ the maximal elementary abelian subgroup. The space $L(2, i)$ is the transfer ($Tr : BH \rightarrow BG$) image of the same named summand of BH . By using the double coset formula

$$Tr_H^P(u^{p-1})|_H = \sum_{i=0}^{p-1} (u + iy)^{p-1} = -y^{p-1}$$

taking the generator u in $H^*(\langle b, a^p \rangle) \cong k[y, u]$.

The group P has just one conjugacy class H of the maximal abelian p -groups. Hence by Quillen's theorem, we have

$$Tr_H^P(u^{p-1}) = -y^{p-1} \quad \text{in } H^*(P) = H^*(P; \mathbb{Z})/(p, \sqrt{0}).$$

We consider an element $\Phi \in A(P, P)$ defined by $\Phi : P \geq H \subset P$. Then we see

$$Im(Tr_H^P H^*(H)) \supset H^*(P)\Phi = H^*(\bigvee_{i=0}^{p-2} L(2, i)).$$

Thus we have the isomorphism

$$Y_i \cong \begin{cases} X_i \vee L(2, i) & i \neq 0 \\ X_0 \vee L(2, 0) \vee \bigvee_{j=0}^{p-2} L(1, j) & i = 0. \end{cases}$$

To compute cohomology of irreducible components X_i and $L(2, j)$, we recall the Dickson algebra

$$\mathbb{DA} = k[y, u]^{GL_2(\mathbb{Z}/p)} \cong k[D_1, D_2] \quad \text{with } D_1 = Y^p + V, \quad D_2 = YV.$$

We also write

$$\mathbb{CA} = k[Y, V] \cong \mathbb{DA}\{1, Y, \dots, Y^p\},$$

$$\mathbb{CB} = k[Y, D_2] \cong \mathbb{DA}\{1, Y, \dots, Y^{p-1}\}.$$

Hence $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$. Then it is known (see [Hi-Ya1] for details)

$$H^*(L(2, i)) \cong \begin{cases} \mathbb{CB}\{Y d_2^i\} & i \neq 0 \\ \mathbb{CB}\{Y D_2\} & i = 0. \end{cases}$$

Theorem 3.3. *Let $P = M(1, 2, 1) \cong p_-^{1+2}$. Then we have*

$$H^*(X_i) \cong \begin{cases} \mathbb{CA}\{1, \dots, \hat{y}^i, \dots, y^{p-2}\}\{v^i\} \oplus \mathbb{DA}\{d_2^i\} & i > 0 \\ \mathbb{CA}\{y, \dots, y^{p-2}\}\{V\} \oplus \mathbb{DA} & i = 0. \end{cases}$$

Proof. Let $i \neq 0$. We see

$$H^*(Y_i) \cong k[y, V]\{v^i\} \cong \mathbb{CA}\{1, y, \dots, y^{p-2}\}\{v^i\}.$$

The cohomology of the summand X_i is

$$\begin{aligned} H^*(X_i) &\cong H^*(Y_i) \ominus H^*(L(2, i)) \\ &\cong (\mathbb{DA} \oplus \mathbb{CB}\{Y\})\{v^i\}\{1, \dots, y^{p-2}\} \ominus \mathbb{CB}\{Y d_2^i\}. \end{aligned}$$

Here $v^i y^i = d_2^i$ we have the isomorphism in the theorem for $i \neq 0$.

Next we consider in the case $i = 0$. We have

$$\begin{aligned} H^*(X_0) &\cong H^*(Y_0) \ominus H^*(\vee_j L(1, j)) \ominus H^*(L(2, 0)) \\ &\cong \mathbb{CA}\{1, y, \dots, y^{p-2}\}\{V\} \ominus \mathbb{CB}\{Y D_2\} \cong \mathbb{CA}\{y, \dots, y^{p-2}\}\{V\} \oplus B \end{aligned}$$

where

$$B = \mathbb{CA}\{V\} \ominus \mathbb{CB}\{Y D_2\} \cong \mathbb{CA} \ominus H^*(L(1, 0)) \ominus H^*(L(2, 0)).$$

We can see $B \cong \mathbb{DA}$ by Lemma 3.4 below. □

Lemma 3.4. *Let $M(2) = L(2, 0) \vee L(1, 0)$ (as the usual notation of the homotopy theory). Then we have*

$$H^*(M(2)) \cong \mathbb{CB}\{Y\}, \quad \mathbb{CA} \cong \mathbb{DA} \oplus H^*(M(2)).$$

Proof. We can compute

$$\begin{aligned} H^*(M(2)) &\cong k[Y] \oplus \mathbb{CB}\{Y D_2\} \cong k[Y] \oplus k[Y, D_2]\{Y D_2\} \\ &\cong (k[Y] \oplus k[Y, D_2]\{D_2\})\{Y\} \cong \mathbb{CB}\{Y\} \quad (\text{assumed } * > 0). \end{aligned}$$

Since $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$, we have the last isomorphism in this lemma. □

4 Nilpotent elements

Let us write $H^{even}(X; \mathbb{Z})/p$ by simply $H^{ev}(X)$ so that

$$H^{ev}(G) = H^*(G) \oplus N(G)$$

where $N(G)$ is the nilpotent ideal in $H^{ev}(G)$.

Since BP is irreducible in nonsplit cases, we only consider in split cases,

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell} \rangle$$

for $m > \ell \geq \max(m - n, 1)$.

(I) Split metacyclic groups with $\ell > m - n$.

By Diethelm [Di], its mod p -cohomology is

$$H^*(P; \mathbb{Z}/p) \cong k[y, u] \otimes \Lambda(x, z) \quad |y| = |u| = 2, \quad |x| = |z| = 1.$$

Of course all elements in $H^*(P; \mathbb{Z})$ are (higher) p -torsion. The additive structure of $H^*(P; \mathbb{Z})/p$ is decided by that of $H^*(P; \mathbb{Z}/p)$ by the universal coefficient theorem. Hence we have additively (but not as rings)

$$H^*(P; \mathbb{Z})/p \cong H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong k[y, u]\{1, \beta(xz) = yz - ux\}.$$

Since $H^*(P)$ is multiplicatively generated by y and v with $|v| \geq 2p$ from Theorem 4.1, the element u is not integral class (i.e. $u \notin \text{Im}(\rho)$ for $\rho: H^*(P; \mathbb{Z}) \rightarrow H^*(P; \mathbb{Z}/p)$). Therefore xz is an integral class since

$$H^{even}(P; \mathbb{Z}/p) \cong k[y, u]\{1, xz\}.$$

In $H^4(P; \mathbb{Z}/p)$, the elements y^2 , yxz are integral but u^2 is not. Note that $\dim(H^4(P; \mathbb{Z}/p)) = 3$ and so xzu must be integral. Inductively, we see that

$$x_1 = xz, \quad x_2 = xzu, \quad \dots, \quad x_{p^{m-\ell}-1} = xzu^{p^{m-\ell}-2}$$

are integral classes.

The element $u \in H^2(P; \mathbb{Z}/p)$ is defined [Dim] using the spectral sequence

$$E_2^{*,*'} \cong H^*(P/\langle a \rangle; H^*(\langle a \rangle; \mathbb{Z}/p)) \implies H^*(P; \mathbb{Z}/p).$$

In fact $u = [u'] \in E_{\infty}^{0,2}$ identifying $H^2(\langle a \rangle; \mathbb{Z}/2) \cong k\{u'\}$. Hence $u|\langle a \rangle = u'$. On the other hand $v|\langle a \rangle = (u')^{p^{m-\ell}}$ because $v = c_{p^{m-\ell}}(\eta)$ and the total Chern class is

$$\sum c_i(\eta)|\langle a \rangle = (1 + u')^{p^{m-\ell}} = 1 + (u')^{p^{m-\ell}}.$$

Therefore we see $v = u^{p^{m-\ell}} \text{ mod } (y, xz)$ in $H^*(P; \mathbb{Z}/p)$. Thus we get

Theorem 4.1. *Let P be a split metacyclic group $M(\ell, m, n)$ with $\ell > m - n$. Then we have*

$$H^{ev}(P) \cong k[y, v]\{1, x_1, \dots, x_{p^{m-\ell}-1}\} \quad \text{with } x_i x_j = 0,$$

that is $N(P) \cong k[y, v]\{x_1, \dots, x_{p^{m-\ell}-1}\}$.

These x_i are also defined by Chern classes (from the arguments just before Theorem 4.1), and as $\text{Out}(P)$ modules, $x_i \cong S_j$ when $i = j \bmod(p-1)$. Therefore we have

Corollary 4.2. *Let P be a split metacyclic group $M(\ell, m, n)$ with $\ell > m - n$. Then*

$$H^{ev}(X_i) \cong H^*(X_i) \oplus k[y, V]\{v^r x_s | r + s = i \bmod(p-1)\}$$

where $1 \leq s \leq p^{m-\ell} - 1$.

(II) Split metacyclic groups $P = M(\ell, m, n)$ with $\ell = m - n$.

By also Diethelm, its mod p -cohomology is

$$H^*(P; \mathbb{Z}/p) \cong k[y, v'] \otimes \Lambda(a_1, \dots, a_{p-1}, b, w) / (a_i a_j = a_i y = a_i w = 0)$$

where $|a_i| = 2i - 1$, $|b| = 1$, $|y| = 2$, $|w| = 2p - 1$, $|v'| = 2p$. So we see

$$H^*(P; \mathbb{Z}/p) / \sqrt{0} \cong k[y, v'].$$

Note that additively $H^*(P; \mathbb{Z})/p \cong H^*(p_-^{1+2}; \mathbb{Z})/p$, which is well known. In particular, we get additively

$$H^{ev}(P) \cong (k[y] \oplus k\{x_1, \dots, x_{p-1}\}) \otimes k[v'] \quad (\text{with } x_i = a_i b)$$

$$\cong (k[y] \oplus k\{x_1, \dots, x_{p-1}\}) \otimes k[v]\{1, v', \dots, (v')^{p^{m-\ell-1}-1}\}.$$

Therefore $H^{ev}(P)$ is additively isomorphic to

$$H^{ev}(P) \cong \oplus_{i,j} k[v]\{a_i b (v')^j\} \oplus \oplus_j k[v, y]\{(v')^j\}$$

where $1 \leq i \leq p-1$ and $0 \leq j \leq p^{m-\ell-1} - 1$. Here $a_i b (v')^j$ is nilpotent and hence integral class and let $x_{jp+i} = a_i b (v')^j$. The element (v') is not nilpotent and we can take as the integral class wb of dimension $2p$. Let us write $x_{pj} = wb(v')^{j-1}$. Thus we have

Theorem 4.3. *Let P be a split metacyclic group $M(\ell, m, n)$ with $\ell = m - n$. Then*

$$H^{ev}(P) \cong k[y, v] \oplus k[y, v]\{x_i | i = 0 \bmod(p)\} \oplus k[v]\{x_i | i \neq 0 \bmod(p)\}$$

where i ranges $1 \leq i \leq p^{m-\ell} - 1$. Here the multiplications are given by $x_i x_j = 0$, $y x_k = 0$ for $k \neq 0 \bmod(p)$.

Hence we have

Corollary 4.4. *Let $P = M(\ell, m, n)$ for $\ell = m - n$. Then*

$$H^{ev}(X_i) = H^*(X_i) \oplus k[y, V]\{v^r x_s | s = 0 \bmod(p), r + s = i \bmod(p-1)\}$$

$$\oplus k[V]\{v^r x_s | s \neq 0 \bmod(p), r + s = i \bmod(p-1)\}.$$

Let $CH^*(BG)$ be the Chow ring of the classifying space BG (see §5 below for the definition). The following theorem is proved by Totaro, with the assumption $p \geq 5$ but without the assumption of transferred Euler classes (since it holds when $p \geq 5$).

Theorem 4.5. (*Theorem 14.3 in [To2]*) Suppose $\text{rank}_p P \leq 2$ and P has a faithful complex representation of the form $W \oplus X$ where $\dim(W) \leq p$ and X is a sum of 1 dimensional representation. Moreover $H^{ev}(P)$ is generated by transferred Euler classes. Then we have $CH^*(P)/p \cong H^{ev}(P)$.

Proof. (See page 179-180 in [To2].) First note the cycle map is surjective, since $H^{ev}(P)$ is generated by transferred Euler classes. Using the Riemann-Roch theorem without denominators, we can show

$$CH^*(BP)/p \cong H^{2*}(P; \mathbb{Z})/p \quad \text{for } * \leq p.$$

By the dimensional conditions of representations $W \oplus X$ and Theorem 12.7 in [To2], we see the following map

$$\begin{aligned} CH^*(BP)/p &\rightarrow \prod_V CH^*(BV) \otimes_{\mathbb{Z}/p} CH^{\leq p-1}(BC_P(V)) \\ &\rightarrow \prod_V H^*(V; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H^{\leq 2(p-1)}(C_G(V); \mathbb{Z}/p) \end{aligned}$$

is also injective. Here V ranges elementary abelian p -subgroups of P and $C_P(V)$ is the centralizer group of V in P . So we see that the cycle map is also injective. $\square$

Therefore we have

Corollary 4.6. Let P be the metacycle group $M(\ell, m, n)$ with $m - \ell = 1$. Then $CH^*(BP)/p \cong H^{ev}(BG)$.

Totaro computed $CH^*(BP)/p$ for split metacyclic groups with $m - \ell = 1$ in 13.12 in [To]. When P is the extraspecial p -groups of order p^3 , the above result is first proved in [Ya2].

For a cohomology theory $h^*(-)$, define the $h^*(-)$ -theory topological nilpotence degree $d_0(h^*(BG))$ to be the least nonnegative integer d such that the map

$$h^*(BG)/p \rightarrow \prod_V h^*(BG) \otimes h^{\leq d}(BC_G(V))/p$$

is injective. Note that $d_0(H^*(BG; \mathbb{Z})) \leq d_0(H^*(BG; \mathbb{Z}/p))$.

Totaro computed in the many cases of groups P with $\text{rank}_p P = 2$. In particular, if P is a split metacyclic p -group for $p \geq 3$, then $d_0(H^*(BP; \mathbb{Z}/p)) = 2$ and $d_0(CH^*(BP)) = 1$ when $m - \ell = 1$. Hence $d_0(H^*(P; \mathbb{Z})) = 2$ for these split metacyclic groups P (for $p \geq 3$).

This fact also show easily from Theorem 8.1 and 8.2. Consider the restriction map

$$H^{ev}(P) \rightarrow H^{ev}(V) \otimes H^2(P) \quad (\text{where } V = \langle a^{p^{m-1}} \rangle \subset Z(P) : \text{center})$$

induced the product map $V \times P \rightarrow P$. Then the element defined in Theorem 8.1, 8.3

$$c_j = xzu^{j-1} \rightarrow \sum_i xzu^i \otimes u^{j-i-1} \equiv u^{j-1} \otimes x_1 \neq 0 \in H^{ev}(V) \otimes H^2(P)$$

for $\ell > m - n$. For $\ell = m - n$ and $n = 1$, we also see that the nilpotent element x_j maps to $ab \otimes u^{j-1}$ (or $wb \otimes u^{j-p-1}$ for $j = 0 \bmod(p)$) in $H^{ev}(V) \otimes H^2(P)$. (From the proof of Theorem 2 in [Dim], we see $w|V = zu^{p-1}$.)

5 Motives and stable splitting

For a smooth projective algebraic variety X over $\mathbb{C}$, let $CH^*(X)$ be the Chow ring generated by algebraic cycles of codimension $*$ modulo rational equivalence. There is a natural (cycle) map

$$cl : CH^*(X) \rightarrow H^{2*}(X(\mathbb{C}); \mathbb{Z}).$$

where $X(\mathbb{C})$ is the complex manifold of $\mathbb{C}$ -rational points of X .

Let V_n be a $G - \mathbb{C}$ -vector space such that G acts freely on $V_n - S_n$, with $\text{codim}_{V_n} S_n = n$. Then it is known that $(V_n - S_n)/G$ is a smooth quasi-projective algebraic variety. Then Totaro define the Chow ring of BG ([To1]) by

$$CH^*(BG) = \lim_{n \rightarrow \infty} CH^*((V_n - S_n)/G).$$

(Note that $H^*(G, \mathbb{Z}) = \lim_{n \rightarrow \infty} H^*((V_n - S_n)/G)$ also.) Moreover we can approximate $\mathbb{P}^\infty \times BG$ by smooth projective varieties from Godeaux-Serre arguments ([To1]).

Let P be a p -group. By the Segal conjecture, the p -complete automorphism $\{BP, BP\}$ of stable homotopy groups is isomorphic to $A(P, P)_{\mathbb{Z}_p}$, which is generated by transfers and map induced from homomorphisms. Since $CH^*(BP)$ also has the transfer map, we see $CH^*(BP)$ is an $A(P, P)$ -module. For an $A(P, P)$ -simple module S , recall e_S is the corresponding idempotent element and $X_S = e_S BP$ the irreducible stable homotopy summand. Let us define

$$CH^*(X_S) = e_S CH^*(BP)$$

so that the following diagram commutes.

$$\begin{array}{ccc} CH^*(BP)_{(p)} & \xrightarrow{cl} & H^{2*}(BP; \mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow \\ CH^*(X_S)_{(p)} & \xrightarrow{cl} & H^{2*}(X_S; \mathbb{Z}_{(p)}). \end{array}$$

For smooth schemes X, Y over a field K , let $Cor(X, Y)$ be the group of finite correspondences from X to Y (which is a $\mathbb{Z}_p$ -module on the set of closed

subvarieties of $X \times_K Y$ which are finite and surjective over some connected component of X . Let $Cor(K, \mathbb{Z}_p)$ be the category of smooth schemes whose groups of morphisms $Hom(X, Y) = Cor(X, Y)$. Voevodsky constructs the triangulated category $DM = DM(K, \mathbb{Z}_p)$ which contains the category $Cor(K, \mathbb{Z}_p)$ (and *limit* of objects in $Cor(K, \mathbb{Z}_p)$).

Theorem 5.1. *Let S be a simple $A(P, P)$ -module. Then there is a motive $M_S \in DM(\mathbb{C}, \mathbb{Z}_p)$ such that*

$$CH^*(M_S) \cong CH^*(X_S) = e_S CH^*(BP).$$

Remark. Of course M_S is (in general) not irreducible, while X_S is irreducible.

The category $Chow^{eff}(K, \mathbb{Z}_p)$ of (effective) pure Chow motives is defined follows. An object is a pair (X, p) where X is a projective smooth variety over K and p is a projector, i.e. $p \in Mor(X, X)$ with $p^2 = p$. Here a morphism $f \in Mor(X, Y)$ is defined as an element $f \in CH^{dim(Y)}(X \times Y)_{\mathbb{Z}_p}$. We say that each $M = (X, p)$ is a (pure) motive and define the Chow ring $CH^*(M) = p^* CH^*(X)$, which is a direct summand of $CH^*(X)$. We identify that the motive $M(X)$ of X means $(X, id.)$. (The category $DM(K, \mathbb{Z}_p)$ contains the category $Chow^{eff}(K, \mathbb{Z}_p)$.)

It is known that we can approximate $\mathbb{P}^\infty \times BP$ by smooth projective varieties from Godeaux-Serre arguments ([To1]). Hence we can get the following lemma since

$$CH^*(X \times \mathbb{P}^\infty) \cong CH^*(X)[y] \quad |y| = 1.$$

Lemma 5.2. *Let S be a simple $A(P, P)$ -module. There are pure motives $M_S(i) \in Chow^{eff}(\mathbb{C}, \mathbb{Z}_p)$ such that*

$$\lim_{n \rightarrow \infty} CH^*(M_S(i)) \cong CH^*(X_S)[y], \quad deg(y) = 1.$$

Corollary 5.3. *Let P be a split metacycle p -group $M(\ell, m, n)$ with $m - \ell = 1$. Then for each simple $A(P, P)$ -module S , there is a motive $M_S \in DM(\mathbb{C}, \mathbb{Z}_p)$ with*

$$CH^*(M_S)/p \cong H^{ev}(X_S) = H^{even}(X_S; \mathbb{Z})/p.$$

References

- [Be-Fe] D. J. Benson and M. Feshbach, Stable splittings of classifying spaces of finite groups, *Topology* 31 (1992), 157-176.
- [Ca] G. Carlsson, Equivariant stable homotopy and Segal's Burnside ring conjecture, *Ann. Math.* 120 (1984), 189-224.
- [Dim] T. Diethelm, The mod p cohomology rings of the nonabelian split metacyclic p -groups. *Arch. Math* 44, (1985), 29-38. 93-103.

- [Di] J. Dietz, Stable splitting of classifying space of metacyclic p -groups, p odd. *J. Pure and Applied Algebra* 90 (1993) 115-136.
- [Di-Pr] J. Dietz and S. Priddy, The stable homotopy type of rank two p -groups, in: *Homotopy theory and its applications*, *Contemp. Math.* 188, Amer. Math. Soc., Providence, RI, (1995), 93-103.
- [Hi-Ya1] A. Hida and N. Yagita, Representation of the double Burnside algebra and cohomology of extraspecial p -group. *J. Algebra* 409 (2014), 265-319.
- [Hi-Ya2] A. Hida and N. Yagita, Representation of the double Burnside algebra and cohomology of extraspecial p -group II. Preprint. (2015).
- [Hu] J. Huebuschmann. Chernclasses for metacyclic groups. *Arch. Math.* 61 (1993), 124-136.
- [Ma-Pr] J. Martino and S. Priddy, The complete stable splitting for the classifying space of a finite group, *Topology* 31 (1992), 143-156.
- [Mi-Pr] S. Mitchell and S. Priddy, Stable splitting derived from the Steiberg module. *Topology* 22 (1983), 285-298.
- [Ni] G. Nishida, Stable homotopy type of classifying spaces of finite groups. *Algebraic and Topological theories ; to the memopry of Dr. Takehiko Miyata.* (1985) 391-404.
- [Qu] D. Quillen, The spectrum of an equivariant cohomology ring: I, *Ann. of Math.* 94 (1971), 549-572.
- [Th] C.B.Thomas. Characteristic classes and 2-modular representations for some sporadic groups. *Lecture note in Math. Vol. 1474* (1990), 371-381.
- [To1] B. Totaro. The Chow ring of classifying spaces. *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington,Seattle)* 67 (1999), 248-281.
- [To2] B. Totaro. Group cohomology and algebraic cycles. *Cambridge tracts in Math.* 204 (2014).
- [Ya1] N. Yagita. Cohomology for groups of $rank_p G = 2$ and Brown-Peterson cohomology *J. Math. Soc. Japan.* 45 (1993) 627-644.
- [Ya2] N. Yagita. Chow rings of nonabelian p -groups of order p^3 . *J. Math. Soc. Japan.* 64 (2012) 507-531.