

Notes on liftable vector fields

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Abstract

We introduce the author's research about an estimate for the highest degrees of liftable vector fields and the module of liftable vector fields for non-singular mono-germs and function mono-germs of one variable.

1 Introduction

In this paper, we introduce the author's research about liftable vector fields. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this paper, suppose that all mappings are smooth (that is, of class C^∞ if $\mathbb{K} = \mathbb{R}$ or holomorphic if $\mathbb{K} = \mathbb{C}$).

The notion of liftable vector fields was introduced by Arnol'd [1] for studying bifurcations of wave front singularities. As results and applications of liftable vector fields, Bruce and West [2] obtained diffeomorphisms preserving a crosscap to classify functions on it, and Houston and Littlestone [4] obtained generators for the module of vector fields liftable over the generalized cross cap to find $\mathcal{A}_e$ -codimension 1 maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Houston and Atique [3] classified $\sqrt{\mathcal{K}}$ -codimension 2 maps on the generalized crosscap to apply to a classification of $\mathcal{A}_e$ -codimension 2 maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Nishimura [8] characterized the minimal number of generators for the module of vector fields liftable over a finitely determined multigerm of corank at most one satisfying a special condition when $n \leq p$.

In previous work [6], the author showed that we can find polynomial vector fields liftable over f if f is a polynomial multigerm and gave an estimate for the highest degrees of liftable vector fields. The highest degree of polynomial vector field ξ means maximum of that of component functions of ξ . Let $[x]$ be the greatest integer not exceeding x . $\text{Lift}(f)$ denotes the module of vector fields liftable over f . We proved the following theorem.

Theorem 1.1 ([6]). *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n < p$) be a polynomial multigerm. Then, there exists a non-zero polynomial vector field in $\text{Lift}(f)$ such that*

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the highest degree is at most

$$N = \left\lceil \sqrt[p-n]{(\alpha+1)(p-1)!} \right\rceil + 1,$$

where

$$\alpha = r \left(\frac{p \cdot 2^n - n}{n!} \right) (D + n - 1)^n, \quad r = |S|,$$

$$D = \max\{D_i \mid i \in \{1, 2, \dots, r\}\}, \quad D_i = \max\{\deg(X_j \circ f_i) \mid j \in \{1, 2, \dots, p\}\}.$$

The proof of Theorem 1.1 also gives a method to find a non-zero element of $\text{Lift}(f)$. However, we can usually take values of N that are much lower than those calculated in Theorem 1.1. Therefore, we needed to improve this estimate. In [7], a better estimate for the highest degrees of liftable vector fields was discovered when $n = 1$. It is the following result. Let $\lceil x \rceil$ be the smallest integer greater than or equal to x .

Theorem 1.2 ([7]). *Let $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^p, 0)$ ($p \geq 2$) be a polynomial multigerm which contains no branch of zero map. Then there exist a non-zero polynomial vector field of $\text{Lift}(f)$ such that the highest degree is at most*

$$N = \max \left\{ \left\lceil \sqrt[p-1]{(Ap - A)(p-1)!} - \sqrt[p]{p!} \right\rceil, 1 \right\},$$

and the highest degree of a corresponding lowerable vector field for f_i is at most

$$D_i N - D_i + 1,$$

where

$$A = \sum_{i=1}^r D_i, \quad D_i = \max\{\deg(X_j \circ f_i) \mid j \in \{1, \dots, p\}\}.$$

This paper is organized as follows. In Section 2, we explain various definitions, basic facts and examples implying difference of estimates between Theorem 1.1 and Theorem 1.2. In Section 3 the sketch of proof of Theorem 1.2 is described. In Section 4 and 5, topics about the module of liftable vector fields are given. The number of generators for the module of vector fields liftable over a non-singular mono-germ is identified in Section 4. Theorem 1.1 and Theorem 1.2 claims that there exist non-zero polynomial vector fields liftable over f when f is a polynomial. It is natural to ask whether there exist non-zero polynomial liftable vector fields when f is not a polynomial. In Section 5 we investigate the module of vector fields liftable over a function germ of one variable and also consider this problem.

2 Preliminary

Let S be a subset of $\mathbb{K}^n$. A map germ $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is called a *multigerm*. If S is a singleton, f is called a *mono-germ*. Let C_S (resp., C_0) be the set of

function germs $(\mathbb{K}^n, S) \rightarrow \mathbb{K}$ (resp., $(\mathbb{K}^p, 0) \rightarrow \mathbb{K}$), and let m_S (resp., m_0) be the subset of C_S (resp., C_0) consisting of function germs $(\mathbb{K}^n, S) \rightarrow (\mathbb{K}, 0)$ (resp., $(\mathbb{K}^p, 0) \rightarrow (\mathbb{K}, 0)$). The sets C_S and C_0 have natural $\mathbb{K}$ -algebra structures. A multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ can be defined by $(f_1, f_2, \dots, f_r)$, where $f_i : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$. Each f_i is called a *branch*. In this paper, for a multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ defined by $(f_1, f_2, \dots, f_r)$ with $f_i : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$, we consider S to be a set consisting of r distinct points.

For a map germ $f : (\mathbb{K}^n, S) \rightarrow \mathbb{K}^p$, let $\theta_S(f)$ be the set of germs of vector fields along f . The set $\theta_S(f)$ has a natural C_S -module structure and is identified with the direct sum of p copies of C_S . Put $\theta_S(n) = \theta_S(\text{id}_{(\mathbb{K}^n, S)})$ and $\theta_0(p) = \theta_{\{0\}}(\text{id}_{(\mathbb{K}^p, 0)})$, where $\text{id}_{(\mathbb{K}^n, S)}$ (resp., $\text{id}_{(\mathbb{K}^p, 0)}$) is the germ of the identity mapping of $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$). For a multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, following Mather [5], we define tf and ωf as

$$tf : \theta_S(n) \rightarrow \theta_S(f), \quad tf(\eta) = df \circ \eta,$$

$$\omega f : \theta_0(p) \rightarrow \theta_S(f), \quad \omega f(\xi) = \xi \circ f,$$

where df is the differential of f . Following Wall [9], we put $TR_e(f) = tf(\theta_S(n))$ and $TL_e(f) = \omega f(\theta_0(p))$.

For a multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, a vector field $\xi \in \theta_0(p)$ is said to be *liftable* over f if $\xi \circ f \in TR_e(f) \cap TL_e(f)$. The set of vector fields liftable over f is denoted by $\text{Lift}(f)$. Note that $\text{Lift}(f)$ has a natural C_0 -module structure. Let $(x_1, x_2, \dots, x_n)$ (resp., $(X_1, \dots, X_p)$) be the standard local coordinates of $\mathbb{K}^n$ (resp., $\mathbb{K}^p$) at the origin. Sometimes (x_1, x_2) (resp., (X_1, X_2)) is denoted by (x, y) (resp., (X, Y)) and (x_1, x_2, x_3) (resp., (X_1, X_2, X_3)) is denoted by (x, y, z) (resp., (X, Y, Z)). We see easily that

$$\xi = (\psi_1(X_1, X_2, \dots, X_p), \dots, \psi_p(X_1, X_2, \dots, X_p)) \in \text{Lift}(f),$$

where $\psi_j : (\mathbb{K}^p, 0) \rightarrow \mathbb{K}$ ($j = 1, 2, \dots, p$), if and only if for every $i \in \{1, \dots, r\}$ there exist a vector field

$$\eta_i = (\phi_{i,1}(x_1, x_2, \dots, x_n), \dots, \phi_{i,n}(x_1, x_2, \dots, x_n)),$$

where $\phi_{i,k} : (\mathbb{K}^n, 0) \rightarrow \mathbb{K}$ ($k = 1, 2, \dots, n$), such that $\xi \circ f_i = df_i \circ \eta_i$ i. e.

$$\begin{aligned} & \begin{pmatrix} \psi_1(X_1, X_2, \dots, X_p) \\ \vdots \\ \psi_p(X_1, X_2, \dots, X_p) \end{pmatrix} \circ f_i(x_1, x_2, \dots, x_n) \\ &= \begin{pmatrix} \frac{\partial(X_1 \circ f_i)}{\partial x_1} & \dots & \frac{\partial(X_1 \circ f_i)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(X_p \circ f_i)}{\partial x_1} & \dots & \frac{\partial(X_p \circ f_i)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \phi_{i,1}(x_1, x_2, \dots, x_n) \\ \vdots \\ \phi_{i,n}(x_1, x_2, \dots, x_n) \end{pmatrix}. \end{aligned}$$

We call this η_i a *lowerable* vector field for f_i corresponding to ξ .

Example 2.1. Let $f : (\mathbb{K}, 0) \rightarrow (\mathbb{K}^2, 0)$ be given by $f(x) = (x^2, x^3)$. Then, it can be seen easily that the following vector fields are liftable over f :

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix}, \begin{pmatrix} 2Y \\ 3X^2 \end{pmatrix}.$$

The forms of vector fields liftable over a polynomial multiterm are complicated generally.

Example 2.2. Let $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^2, 0)$ be given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (x^3, x^2)$. Then, it is known [8] that the following vector fields are liftable over f :

$$(-6X^2Y^2 + 6XY)\frac{\partial}{\partial X} + (-9XY^3 + 5X^3 + 4Y^2)\frac{\partial}{\partial Y},$$

In fact,

$$\begin{pmatrix} -6X^2Y^2 + 6XY \\ -9XY^3 + 5X^3 + 4Y^2 \end{pmatrix} \circ f_1 = \begin{pmatrix} -6x^{10} + 6x^5 \\ -9x^{11} + 9x^6 \end{pmatrix} = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix} (-3x^9 + 3x^4),$$

$$\begin{pmatrix} -6X^2Y^2 + 6XY \\ -9XY^3 + 5X^3 + 4Y^2 \end{pmatrix} \circ f_2 = \begin{pmatrix} -6x^{10} + 6x^5 \\ -4x^9 + 4x^4 \end{pmatrix} = \begin{pmatrix} 3x^2 \\ 2x \end{pmatrix} (-2x^8 + 2x^3)$$

holds.

The following property is very fundamental and important.

Proposition 2.3. We assume $g = t \circ f \circ s$, which t and s are diffeomorphisms (that is, f is $\mathcal{A}$ -equivalent to g). Then,

$$\xi \in \text{Lift}(f) \Rightarrow dt \circ \xi \circ t^{-1} \in \text{Lift}(g).$$

This means that only diffeomorphism of the target of f affects liftable vector fields of g . In addition, we will see that only that of the source affects lowerable vector fields in the proof.

Proof. There exists $\eta \in \theta_S(n)$ such that

$$\xi \circ f = df \circ \eta.$$

Then, the following holds:

$$\begin{aligned} (dt \circ \xi \circ t^{-1}) \circ g &= dt \circ \xi \circ t^{-1} \circ (t \circ f \circ s) \\ &= dt \circ (\xi \circ f) \circ s \\ &= dt \circ (df \circ \eta) \circ s \\ &= d(t \circ f) \circ \eta \circ s \\ &= d(g \circ s^{-1}) \circ \eta \circ s \\ &= dg \circ (ds^{-1} \circ \eta \circ s) \end{aligned}$$

Thus, $dt \circ \xi \circ t^{-1} \in \text{Lift}(g)$. □

We compare estimates of Theorem 1.1 and Theorem 1.2 using examples.

Example 2.4. When $n = 1$ and $p = 2$, by Theorem 1.1

$$N = 3rD + 2.$$

For $f(x) = (x^2, x^3)$, since $r = 1$ and $D = 3$, we get $N = 11$. On the other hand, when $p = 2$ by Theorem 1.2

$$N = \left\lceil A - \sqrt{2} \right\rceil,$$

Since $A = 3$, we get $N = 2$. In fact, the highest degree of the following liftable vector fields are 1 and 2 respectively;

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix}, \begin{pmatrix} 2Y \\ 3X^2 \end{pmatrix}.$$

Example 2.5. For $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^2, 0)$ given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (-x^3, x^2)$, $f_3(x) = (x^2 - x^3, x^2 + x^3)$, since $r = 3$ and $D = 3$, by Theorem 1.1 $N = 29$. On the other hand, by Theorem 1.2 since $A = 9$, we get $N = 8$. In fact, the following vector field are liftable over f and the highest degree is 6;

$$\begin{aligned} & (-15X^6 - 45X^5Y - 45X^4Y^2 + 19X^3Y^3 + 4X^2Y^4 \\ & - 4X^5 - 64X^4Y + 45X^3Y^2 + 41X^2Y^3 + 57XY^4 \\ & - 7Y^5 + 4X^4 - 12X^3Y - 8X^2Y^2 + 52XY^3 \\ & - 14Y^4 + 8X^3 - 16X^2Y) \frac{\partial}{\partial X} \\ & + (-10X^5Y - 30X^4Y^2 - 38X^3Y^3 + 18X^2Y^4 + 6XY^5 \\ & + 8X^5 - 8X^4Y - 46X^3Y^2 + 34X^2Y^3 + 24XY^4 \\ & + 56Y^5 - 4X^4 + 6X^3Y - 26X^2Y^2 - 10XY^3 \\ & + 28Y^4 + 12X^2Y - 20XY^2) \frac{\partial}{\partial Y}. \end{aligned}$$

Example 2.6. When $n = 1$ and $p = 3$, by Theorem 1.1

$$N = \left\lceil \sqrt{2(5rD + 1)} \right\rceil.$$

For $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^3, 0)$ given by $f_1(x) = (x^3, x^4, x^5)$, $f_2(x) = (x^4, x^5, x^3)$, $f_3(x) = (x^5, x^3, x^4)$, we get $r = 3$ and $D = 5$. Therefore, $N = 13$. On the other hand, when $p = 3$ by Theorem 1.2

$$N = \left\lceil 2\sqrt{A} - \sqrt[3]{6} \right\rceil.$$

Since $A = 15$, we get $N = 6$. In fact, the following vector field are liftable over f and the highest degree is 5;

$$\begin{aligned}
& (12X^3 + 18X^4 - 12X^2Y + 12X^3Y + 18X^4Y + 6X^2Y^2 - \\
& 6X^3Y^2 + 18XY^3 - 18X^2Y^3 + 6Y^4 + 18X^3Z - 6XYZ - \\
& 12X^2YZ - 6X^3YZ - 18XY^2Z - 18X^2Y^2Z + 12Y^3Z + \\
& 6XY^3Z + 18Y^4Z - 6X^2Z^2 - 18X^3Z^2 - 18XYZ^2 - \\
& 6X^2YZ^2 - 18Y^2Z^2 + 12XZ^3 + 6X^2Z^3 + 6YZ^3 - \\
& 12Y^2Z^3 - 6Z^4 + 12XZ^4) \frac{\partial}{\partial X} + \\
& (-8X^4 + 16X^2Y + 16X^3Y + 24X^4Y - 8XY^2 + 8X^2Y^2 + \\
& 16X^3Y^2 + 16XY^3 - 8Y^4 - 8X^2Z + 8XYZ + 16X^2YZ + 8X^3YZ - \\
& 8XY^2Z - 24X^2Y^2Z - 16Y^3Z - 16XY^3Z - 24X^3Z^2 + \\
& 8YZ^2 - 16X^2YZ^2 - 8Y^2Z^2 - 16XZ^3 - 8X^2Z^3 + 8YZ^3 + \\
& 32XYZ^3 - 8Z^4) \frac{\partial}{\partial Y} + \\
& (10X^4Y + 10XY^2 + 10X^2Y^2 + 30X^3Y^2 + 10XY^3 + \\
& 10X^2Y^3 + 20Y^4 + 10X^2Z + 20X^3Z - 20XYZ + 10X^2YZ + \\
& 10X^3YZ + 10XY^2Z - 30XY^3Z - 10Y^4Z - 10X^3Z^2 - \\
& 10YZ^2 - 20XYZ^2 - 40X^2YZ^2 - 20Y^2Z^2 - 30XY^2Z^2 - \\
& 10Y^3Z^2 - 10X^2Z^3 - 10YZ^3 + 30Y^2Z^3 + 20Z^4 + 10XZ^4) \frac{\partial}{\partial Z}.
\end{aligned}$$

3 The sketch of proof of Theorem 1.2

We want non-negative integers N and N'_i ($i = 1, 2, \dots, r$) such that we can find a coefficient vector

$$(a_1^{(0,0,\dots,0)}, a_1^{(1,0,\dots,0)}, \dots, a_p^{(0,0,\dots,N)}, a_{1,0}, a_{1,1}, \dots, a_{r,N'_r}) \neq 0$$

such that for every $i \in \{1, 2, \dots, r\}$, the following polynomial equation with respect to the variable x_1 holds:

$$\begin{aligned}
& \left(\begin{array}{c} \sum_{d=0}^N \left(\sum_{i_1+\dots+i_p=d} a_1^{(i_1,i_2,\dots,i_p)} \prod_{h=1}^p X_h^{i_h} \right) \\ \vdots \\ \sum_{d=0}^N \left(\sum_{i_1+\dots+i_p=d} a_p^{(i_1,i_2,\dots,i_p)} \prod_{h=1}^p X_h^{i_h} \right) \end{array} \right) \circ f_i \\
& = \left(\begin{array}{c} \frac{\partial(X_1 \circ f_i)}{\partial x_1} \\ \vdots \\ \frac{\partial(X_p \circ f_i)}{\partial x_1} \end{array} \right) (a_{i,0} + a_{i,1}x_1 + \dots + a_{i,N'_i}x_1^{N'_i}),
\end{aligned}$$

where $i_1, i_2, \dots, i_p$ are non-negative integers. Note that for every $i \in \{1, 2, \dots, r\}$, the highest degree of the left-hand side is at most $N \cdot D_i$ and that of the right-hand side is at most $N'_i + D_i - 1$. By comparing the coefficients of the terms on the both sides, a system of linear equations with respect to

$$a_1^{(0,0,\dots,0)}, a_1^{(1,0,\dots,0)}, \dots, a_p^{(0,0,\dots,N)}, a_{1,0}, a_{1,1}, \dots, a_{r,N'_r}$$

is obtained. Let the number of unknowns of the system of linear equations be denoted by U and the number of equations by E . We assume that $N \geq 1$ to put

$$N'_i = ND_i - D_i + 1.$$

The number of combinations of non-negative integers $i_1, \dots, i_p$ such that $i_1 + \dots + i_p = d$ is

$$\binom{d+p-1}{d} = \frac{(d+p-1)!}{d!(p-1)!} = \frac{(d+p-1) \cdots (d+1)}{(p-1)!}.$$

Thus, we get

$$U = p \sum_{d=0}^N \frac{(d+p-1) \cdots (d+1)}{(p-1)!} + \sum_{i=1}^r (ND_i - D_i + 2)$$

and

$$E \leq p \sum_{i=1}^r (ND_i + 1).$$

Here, the following formula is known.

Proposition 3.1. *For a non-negative integer k ,*

$$\sum_{d=1}^n d(d+1) \cdots (d+k) = \frac{n(n+1) \cdots (n+k+1)}{k+2}.$$

In addition, the following lemma holds.

Lemma 3.2. *For $p \in \mathbb{N}_{\geq 2}$ and $x \in \mathbb{R}_{>0}$,*

$$\left(x + \sqrt[p]{p!}\right)^p < (x+1)(x+2) \cdots (x+p)$$

Put

$$A = \sum_{i=1}^r D_i.$$

By Proposition 3.1 and Lemma 3.2, we get

$$U - E > \frac{1}{(p-1)!} \left(N + \sqrt[p]{p!}\right) \left\{ \left(N + \sqrt[p]{p!}\right)^{p-1} - A(p-1)(p-1)! \right\}.$$

Thus, if we put

$$N = \max \left\{ \left\lceil \sqrt[p-1]{(Ap - A)(p-1)!} - \sqrt[p]{p!} \right\rceil, 1 \right\}$$

and

$$N'_i = ND_i - D_i + 1,$$

then we can obtain $U - E > 0$. This completes the proof.

4 Lifiable vector fields for non-singular mono-germs

A mono-germ $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ is *singular* if $\text{rank} Jf(0) < \min\{n, p\}$ holds, where $Jf(0)$ is the Jacobian matrix of f at 0. A mono-germ $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ is *non-singular* if f is not singular.

We identify the number of generators for $\text{Lift}(f)$ for a non-singular mono-germ f .

Proposition 4.1. *Let $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ be non-singular. Then, the number of generators for $\text{Lift}(f)$ is*

$$\begin{cases} n + (p - n)(p - n) & (n < p) \\ p & (n \geq p) \end{cases}.$$

Proof. When $n \geq p$ we know that f is $\mathcal{A}$ -equivalent to the following form :

$$g(x_1, \dots, x_n) = (x_1, \dots, x_p).$$

Then, we can easily check that $\text{Lift}(g) = C_0$. Therefore, the number of generators for $\text{Lift}(f)$ is p .

When $n < p$ we know that f is $\mathcal{A}$ -equivalent to the following form :

$$h(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0).$$

Then, we can check that the following vector fields belong to $\text{Lift}(h)$:

$$\begin{aligned} & \frac{\partial}{\partial X_i} \quad (1 \leq i \leq n) \\ & X_{n+i} \frac{\partial}{\partial X_{n+j}} \quad (1 \leq i, j \leq p - n). \end{aligned}$$

Therefore, we know

$$\left\langle \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}, X_{n+1} \frac{\partial}{\partial X_{n+1}}, X_{n+2} \frac{\partial}{\partial X_{n+1}}, \dots, X_p \frac{\partial}{\partial X_p} \right\rangle_{C_0}$$

is contained in $\text{Lift}(h)$.

Conversely, for $\xi = (\psi_1(X_1, X_2, \dots, X_p), \dots, \psi_p(X_1, X_2, \dots, X_p)) \in \text{Lift}(h)$, since there exist smooth function germs $\phi_i(x_1, x_2, \dots, x_n) (i = 1, 2, \dots, n)$ such that

$$\psi_i(x_1, x_2, \dots, x_n, 0, 0, \dots, 0) = \begin{cases} \phi_i(x_1, x_2, \dots, x_n) & (1 \leq i \leq n) \\ 0 & (n + 1 \leq i \leq p) \end{cases},$$

when $n + 1 \leq i \leq p$ there exist smooth function germs $\tilde{\psi}_i(X_1, X_2, \dots, X_p) (i = 1, 2, \dots, p - n)$ such that

$$\begin{aligned} \psi_i(X_1, X_2, \dots, X_p) &= \psi_i(X_1, X_2, \dots, X_n, 0, 0, \dots, 0) + \tilde{\psi}_1 X_{n+1} + \dots + \tilde{\psi}_{p-n} X_p \\ &= \tilde{\psi}_1 X_{n+1} + \dots + \tilde{\psi}_{p-n} X_p. \end{aligned}$$

Therefore, $\xi = (\psi_1(X_1, X_2, \dots, X_p), \dots, \psi_p(X_1, X_2, \dots, X_p)) \in \text{Lift}(h)$ belongs to

$$\left\langle \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}, X_{n+1} \frac{\partial}{\partial X_{n+1}}, X_{n+2} \frac{\partial}{\partial X_{n+1}}, \dots, X_p \frac{\partial}{\partial X_p} \right\rangle_{C_0}.$$

Thus, the number of generators for $\text{Lift}(f)$ is $n + (p-n)(p-n)$. This completes the proof. $\square$

When f is singular, characterization of the number of generators for $\text{Lift}(f)$ is essentially difficult (see [8]).

5 Lifiable vector fields for function mono-germs of one variable

We investigate $\text{Lift}(f)$ for $n = p = 1$.

Proposition 5.1. *Let a smooth function $f : (\mathbb{K}, 0) \rightarrow (\mathbb{K}, 0)$ be denoted by $f(x) = \tilde{f}(x)x^n$, where $\tilde{f} : (\mathbb{K}, 0) \rightarrow \mathbb{K}$ satisfies $\tilde{f}(0) \neq 0$ and n is an integer greater than 1. Then, $\text{Lift}(f) = \langle X \rangle_{C_0}$.*

Proof. Since $f(x) = \tilde{f}(x)x^n$, we can see easily

$$f'(x) = \tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}.$$

At first we show $(\subset)$. Since $f(0) = f'(0) = 0$, for $\xi \in \text{Lift}(f)$ $\xi(0) = 0$ holds. Therefore, there exists a function $\psi(X)$ such that $\xi(X) = \psi(X)X$. This means $\text{Lift}(f) \subset \langle X \rangle_{C_0}$.

Next, we show $(\supset)$. It is sufficient to show $X \in \text{Lift}(f)$. In fact,

$$\tilde{f}(x)x^n = (\tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}) \left(\frac{\tilde{f}(x)x}{\tilde{f}'(x)x + n\tilde{f}(x)} \right)$$

holds. Thus, $\text{Lift}(f) \supset \langle X \rangle_{C_0}$. This completes the proof. $\square$

This implies that there exist some cases that we can take non-zero polynomial vector fields liftable over f even though f is not a polynomial. Proposition 5.1 does not hold generally for a flat function f . For example, if

$$f(x) = \begin{cases} x^2 e^{-1/x} & (x > 0) \\ 0 & (x = 0) \\ x^2 e^{1/x} & (x < 0) \end{cases}$$

then X is not liftable. We prove this statement. we show that

$$X \notin \text{Lift}(f).$$

At first, since

$$g(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

is a C^∞ function, so is $g(-x)$. Therefore,

$$h(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x = 0) \\ e^{1/x} & (x < 0) \end{cases}$$

is a C^∞ function. Thus, $f(x) = x^2 h(x)$ is a C^∞ function.

We assume that $X \in \text{Lift}(f)$. Then, a lowerable vector field $\phi(x)$ for a liftable vector field X must be

$$\phi(x) = \begin{cases} \frac{x^2}{2x+1} & (x > 0) \\ a & (x = 0) \\ \frac{x^2}{2x-1} & (x < 0) \end{cases}$$

($a \in \mathbb{R}$). However, $\phi(x)$ is not class of C^∞ . Thus, $X \notin \text{Lift}(f)$.

On the other hand, we can show that X^2 is liftable. In fact, we can give a lowerable vector field $\phi(x)$ as follows:

$$\phi(x) = \begin{cases} \frac{x^4 e^{-1/x}}{2x+1} & (x > 0) \\ 0 & (x = 0) \\ \frac{x^4 e^{1/x}}{2x-1} & (x < 0) \end{cases}.$$

It can be seen easily that $\phi(x)$ is class of C^∞ . Thus, $X^2 \in \text{Lift}(f)$.

The author does not know examples of f such that there exist no polynomial vector fields liftable over f .

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