

REGULARITY OF POWERS OF SOME IDEALS

YUJI KAMOI

INTRODUCTION

Let $A = K[x_1, \dots, x_d]$ be a polynomial ring over a field K and $\mathfrak{m} = (x_1, \dots, x_d)$. We regard A as a graded object with some positive degree $\deg(x_i) = w_i$ for $i = 1, \dots, d$. Let I be a graded ideal of A . In this note, we consider the regularity $\text{reg}(I^n)$ for all $n \geq 0$. For a graded A -module M , $\text{reg}(M)$ is define to be the following,

$$\text{reg}(M) = \max\{\text{reg}_i(M) \mid i \geq 0\}$$

where $\text{reg}_i(M) = \max\{a \mid [\text{Tor}_A^i(K, M)]_{a-i} \neq 0\}$. In other word, $\text{reg}(M)$ is a maximal degree shifts in a graded minimal A -free resolution of M .

In thier paper[1], Cutkosky-Herzog-Trung showed the following theorem.

Theorem. *Let I be a graded ideal of A and $i \geq 0$. Then there exist integrers $c_i(I)$ and $d_i(I)$ such that*

$$\text{reg}_i(I^n) = c_i(I)n + d_i(I)$$

for every sufficiently large n . Furthermore, $\text{reg}(I^n)$ is also linear and a leading coefficient coincides with $c_0(I)$.

It is natural to ask the following.

Question.

- (1) To describe the function $\text{reg}(I^n)$, precisely.
- (2) What is the smallest number $\text{reg}(I^n)$ to be linear.

We put $s = \min\{t \mid \text{reg}(I^n) \text{ is linear for all } n \geq t\}$. It is easy to see that the constant term of $\text{reg}(I^n)$ is $\text{reg}(I^s) - c_0(I)s$. Thus it is enough to decide $c_0(I)$ for describing $\text{reg}(I^n)$.

1. ABOUT $c_0(I)$

$c_0(I)$ is closely related to a reduction of I . We first define the following numbers arising from reduction ideals.

Definition 1.1. We set

$$rdeg(I) = \min\{\text{reg}_0(J) \mid J \subset I \text{ is a graded reduction ideal}\}.$$

An element $a \in I$ said to be reduced modulo $\mathfrak{m}I$, if each homogeneous components of a is nonzero in $I/\mathfrak{m}I$. Also, a sequence $a_1, \dots, a_t \in I$ is called reduced modulo $\mathfrak{m}I$, if every a_i is reduced modulo $\mathfrak{m}I$.

Now, we give an answer of Question (1) as follows.

Proposition 1.2. Let I be a graded ideal of A . Assume that I has a minimal reduction. Then $c_0(I) = rdeg(I)$. More precisely, if $a_1, \dots, a_l \in I$ is a minimal reduction which is reduced modulo $\mathfrak{m}I$, then $c_0(I) = rdeg(I) = deg(a_1, \dots, a_l)$, where $deg(a_1, \dots, a_l) = \max\{\deg(a_i) \mid i = 1, \dots, l\}$.

Proof. Let $a_1, \dots, a_l \in I$ be a minimal reduction of I and $c = deg(a_1, \dots, a_l)$. A reduction property does not depend on the difference of elements of $\mathfrak{m}I$. Hence, we may assume that $a_1, \dots, a_l$ is reduced modulo $\mathfrak{m}I$.

Let J' be a graded ideal generated by all homogeneous components of $a_1, \dots, a_l$. (Note that J' depends on the choice of minimal generators of J).

Then J' becomes a reduction of I . Indeed, we have the following inclusions for all $n \gg 0$

$$I^n = JJ^{n-1} \subset J'I^{n-1} \subset I^n.$$

This shows that $rdeg(I) \leq c$.

For a graded reduction $J \subset I$, if $I^{n+r} = J^n I^r$ for $n \geq 0$, then

$$\text{reg}_0(I^{n+r}) \leq \text{reg}_0(J^n) + \text{reg}_0(I^r) \leq \text{reg}_0(J)n + \text{reg}_0(I^r)$$

for all $n \leq 0$. Hence we have

$$c_0(I) = \lim_{n \rightarrow \infty} \frac{\text{reg}_0(I^n)}{n} \leq \text{reg}_0(J).$$

This implies that $c_0(I) \leq rdeg(I)$.

Finally, we will show that $c \leq c_0(I)$. We may assume that $deg(a_1) = c$ and denote by b a homogeneous component of a_1 in degree c . Since $a_1, \dots, a_l$ is analytically independent, b^n (a head term of a_1^n) is nonzero in $I^n/\mathfrak{m}I^n$ for all $n > 0$. In other words, $[I^n/\mathfrak{m}I^n]_{cn} \neq 0$. Thus

$$cn \leq \max\{t \mid [I^n/\mathfrak{m}I^n]_t \neq 0\} = \text{reg}_0(I^n).$$

This shows that $c \leq c_0(I)$ in the same way as above. Hence we have $c_0(I) = rdeg(I) = c$.

At this moment, we don't have enough tool solving Question (2). In the next section, we give a trivial answer for the simple situation.

2. REGULARITY FOR D-SEQUENCES

In this section, we prove the following.

Theorem 2.1. *Let $I \subset A$ be a ideal generated by monomial d-sequence. Then $\text{reg}(I^n) = \text{reg}_0(I)n + (\text{reg}(I) - \text{reg}_0(I))$.*

Recall that a sequence $a_1, \dots, a_r$ of elements of A is a *d-sequence* (cf. [3]), if it generates $(a_1, \dots, a_r)$ minimally and satisfies the following condition

$$(a_1, \dots, a_i) : a_{i+1}a_j = (a_1, \dots, a_i) : a_j$$

for every $1 \leq i < j \leq r$.

By results of [4], we can construct a free resolution of the Rees algebra of $(a_1, \dots, a_r)$. Such a resolution contains A -free resolutions of I^n . In our case, these A -free resolutions are reduced to be minimal. Thus we can compute $\text{reg}(I^n)$ for a monomial d-sequence.

In the following, we give a construction of resolutions.

Let $a_1, \dots, a_r$ be a d-sequence and $I = (a_1, \dots, a_r) \subset A$. We set $S = A[T_1, \dots, T_r]$ and $\deg(T_i) = 1$ for $i = 1, \dots, r$. (At this moment, we don't consider the grading on A . In fact, the following argument is possible for any ring. Thus we regard $\deg(a) = 0$ for $a \in A$ in the grading on S .)

We put $Z_i(I, S) = Z_i(I) \otimes_A S(-i)$ for $i = 0, \dots, r$ where $Z_\bullet(I)$ is a cycle of a Koszul complex of I . Then the Koszul complex $K_\bullet(T_1, \dots, T_r; S)$ induces

$$0 \rightarrow Z_r(I, S) \rightarrow \dots \rightarrow Z_2(I, S) \rightarrow Z_1(I, S) \rightarrow Z_0(I, S) \rightarrow 0$$

,so call Z -complex. By [4], if I is generated by a d-sequence, then $Z_\bullet(I, S)$ is acyclic with 0th homology isomorphic to the Rees algebra $R(I)$.

Let $P_\bullet^{(i)}(I)$ be a A -free resolution of $Z_i(I)$ ($i = 0, \dots, r$) and $P_{i,\bullet}(I, S) = P_\bullet^{(i)}(I) \otimes S(-i)$. Then the differentials $Z_i(I, S) \rightarrow Z_{i-1}(I, S)$ lifts to a chain map $\varphi : P_{i,\bullet}(I, S) \rightarrow P_{i-1,\bullet}(I, S)$ and $P_{\bullet,\bullet}(I, S)$ becomes a S -double complex.

By the stadard arguments of a spectral sequence and a cyclicity of $Z_\bullet(I, S)$, the associated total complex $\text{Tot}(P_{\bullet,\bullet}(I, S))$ gives a S -free resolution of the Rees algebra $R(I)$.

In this case, $\text{Tot}(P_{\bullet,\bullet}(I, S))$ is not only acyclic, but also it has some information about the differential φ . If we put $I' = (a_1, \dots, a_{r-1})$, then

$$0 \rightarrow Z_\bullet(I') \rightarrow Z_\bullet(I) \rightarrow Z_\bullet(I')[-1] \rightarrow 0$$

is exact.

Now, we consider the monomial case. Assume that $a_1, \dots, a_r$ is a monomial d-sequence.

For $F \subset [r]$ and $1 \leq i \leq r$, we set

$$a_F^{(i)} = \begin{cases} \text{LCM}(\prod_{j \in G} a_j \mid G \subset F, \#G = i), & (\#F \geq i) \\ 0, & (\#F < i). \end{cases}$$

Then we can choose that $P_{\bullet}^{(i)}(I) = \bigoplus_{F \in [n], |F| > i} Ae_F^{(i)}$ with a differential ∂

$$\partial(e_F^{(i)}) = \sum_{j \in F} \sigma(j, F) \frac{a_F^{(i)}}{a_{F \setminus \{j\}}^{(i)}} e_{F \setminus \{j\}}^{(i)}.$$

((2.3) in [5]) Furthermore, the lifting $\varphi : P_{i,\bullet}(I, S) \rightarrow P_{i,\bullet}(I, S)$ is give by

$$d(e_F^{(i)} \otimes 1) = (-1)^{|F|-i} \sum_{j \in F} \sigma(j, F) \frac{a_F^{(i)}}{a_{F \setminus \{j\}}^{(i-1)} a_j} T_j e_{F \setminus \{j\}}^{(i-1)} \otimes 1.$$

Then there is a exact sequence

$$0 \rightarrow P_{\bullet,\bullet}(I', S) \rightarrow P_{\bullet,\bullet}(I, S) \rightarrow P_{\bullet,\bullet}(I', S)[-1, 0] \rightarrow 0$$

of double complexes. Then, by induction on r , the following is also exact

$$C = \cdots \rightarrow P_{3,\bullet}(I, S) \rightarrow P_{2,\bullet}(I, S) \rightarrow \varphi(P_{2,\bullet}(I, S)) \rightarrow 0.$$

Finally, we have the exact sequence $Tot(P_{\bullet,\bullet}(I, S)/C) \cong P_{1,\bullet}(I, S)/\varphi(P_{2,\bullet}(I, S))$ and this is actually A -free . After the small Gröbner basis computation, this resolution is wriiten in the following form.

Proposition 2.2. *Let $\Sigma = \{(F, \alpha) \mid F \subset [r], \alpha \in \mathbb{N}, \max(F) \geq \max(\alpha)\}$. Here we denote $\max(F)$ is a maximal number in F and $\max(\alpha) = \max(\text{supp}(\alpha))$. Then I^n has a free resolution $P_{\bullet}$ of the form*

$$P_i = \bigoplus_{(F, \alpha) \in \Sigma} Ae_F^{(i)} \otimes T^{\alpha}.$$

Furthermore, if a_r does not divide $\text{LCM}(a_1, \dots, a_{r-1})$, then the above A -free resolution is minimal.

At last, it is easy to compute degrees of $e_F^{(i)} \otimes T^{\alpha}$, and then we have

$$\text{reg}(I^n) = \text{reg}_0(I)n + (\text{reg}(I) - \text{reg}_0(I))$$

for all $n > 0$.

REFERENCES

- [1] D. CUTKOSKY - J. HERZOG - N.V. TRUNG. Asymptotic behaviour of the Castelnuovo - Mumford regularity. Preprint 1997.
- [2] D. EISENBUD. *Commutative Algebra with a view to algebraic geometry*. Springer 1995.
- [3] C. HUNEKE. The theory of d -sequences and powers of ideals. *Adv. in Math.* **46** (1982), 249-279.
- [4] J. HERZOG -A. SIMIS - V. VASCONCELOS. Approximation complexes and blowing-up rings. *J. of Alg.* **74**(2), (1982), 466-493.
- [5] J. HERZOG -Y. KAMOI. Taylorcomplexes for Koszul boundaries, *manuscripta math.* **96**, (1998), 133-147.

Yuji Kamaoi
Department of Commerce
Meiji University
Eifuku 1-9-1, Suginami-ku
Tokyo 168-8555, Japan
03-5300-1264
03-5300-1203 (FAX)
kanoi@isc.meiji.ac.jp