

A decomposition of the adjoint representation of $U_q(\mathfrak{sl}_2)$

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Quantum algebra. First, we introduce notation.

DEFINITION. Let $U_q^{(1)}$ be an associative algebra / $K = \mathbb{Q}(q)$ (q is an indeterminate.), defined by a system of generators; $e, f, k^{\frac{1}{2}}, k^{-\frac{1}{2}}$, and their relations:

$$\begin{aligned} k^{\frac{1}{2}} k^{-\frac{1}{2}} &= 1, \quad k^{-\frac{1}{2}} k^{\frac{1}{2}} = 1 \\ k^{\frac{1}{2}} e k^{-\frac{1}{2}} &= qe, \quad k^{\frac{1}{2}} f k^{-\frac{1}{2}} = q^{-1}f \\ ef - fe &= \frac{k^2 - k^{-2}}{q^2 - q^{-2}} \end{aligned}$$

As usual, we give U_q a Hopf algebra structure by equipping it with

$$\begin{aligned} \Delta : \quad e &\mapsto e \otimes k^{-1} + k \otimes e \\ f &\mapsto f \otimes k^{-1} + k \otimes f \\ k^{\frac{1}{2}} &\mapsto k^{\frac{1}{2}} \otimes k^{\frac{1}{2}} \\ S : \quad e &\mapsto -q^{-2}e, \quad f \mapsto -q^2f, \quad k^{\frac{1}{2}} \mapsto k^{-\frac{1}{2}} \\ \epsilon : \quad e &\mapsto 0, \quad f \mapsto 0, \quad k^{\frac{1}{2}} \mapsto 1 \end{aligned}$$

DEFINITION.

- (1) $U_q^{(m)}$ denotes the subalgebra of $U_q^{(1)}$ generated by e, f, k, k^{-1} .
- (2) $U_q^{(s)}$ denotes the subalgebra of $U_q^{(m)}$ generated by $E = ek, F = k^{-1}f, K = k^2, K^{-1}$.

REMARK. If we choose another system of generators; $E, F, k^{\frac{1}{2}}, k^{-\frac{1}{2}}$ for $U_q^{(1)}$, then Δ and S become

$$\begin{aligned} \Delta : \quad E &\mapsto E \otimes 1 + k^2 \otimes E \\ F &\mapsto F \otimes k^{-2} + 1 \otimes F \\ S : \quad E &\mapsto -k^{-2}E, \quad F \mapsto -Fk^2 \end{aligned}$$

We put, $C = fe + \frac{q^2k^2 + q^{-2}k^{-2}}{(q^2 - q^{-2})^2}$

Adjoint Representation.

DEFINITION. $U_q^{(1)}$ becomes a $U_q^{(1)}$ -module by

$$\begin{aligned} \text{Ad}(e)x &= exk - q^{-2}kxe \\ \text{Ad}(f)x &= fxk - q^2kxf \quad (x \in U_q^{(1)}) \\ \text{Ad}(k^{\frac{1}{2}})x &= k^{\frac{1}{2}}xk^{-\frac{1}{2}} \end{aligned}$$

We denote it by $(\text{Ad}, U_q^{\text{ad}})$, and we call it the adjoint representation.

DEFINITION. We define submodules of U_q^{ad} as follows:

$$\begin{aligned} V_{\alpha+\frac{1}{2}} &= \text{Ad}(U_q^{(1)})k^{\alpha+\frac{1}{2}} & (\alpha \in \mathbb{Z}) \\ V_{2\alpha+1} &= \text{Ad}(U_q^{(1)})k^{2\alpha+1} & (\alpha \in \mathbb{Z}) \\ V_{2\alpha} &= \text{Ad}(U_q^{(1)})C^{-\alpha}k^2 & (\alpha \in \mathbb{Z}_{\leq 0}) \\ V_{2\alpha} &= \text{Ad}(U_q^{(1)})k^{-\alpha+2}e^{\alpha} + \text{Ad}(U_q^{(1)})k^{-\alpha+2}f^{\alpha} & (\alpha \in \mathbb{Z}_{>0}) \\ V_{\text{loc}} &= \bigoplus_{n \geq 0} K[C] \text{Ad}(U_q^{(1)})k^{-n}e^n \end{aligned}$$

DEFINITION.

(1)

$$X(d) = U_q^{(1)}/U_q^{(1)}(k^{\frac{1}{2}}-q^d) \quad (d \in \mathbb{Z})$$

It has a naturally induced $U_q^{(1)}$ -module structure using the left regular representation of $U_q^{(1)}$.

We put,

$$v_d = 1 \bmod U_q^{(1)}(k^{\frac{1}{2}}-q^d)$$

(2) $L(d)$ is the irreducible module with the highest weight q^{2d} .

(3) τ is a K -algebra automorphism which sends $e, f, k^{\frac{1}{2}}$ to $f, e, k^{-\frac{1}{2}}$ respectively.

REMARK. τ induces an isomorphism between $X(d)$ and $X(-d)$.

Decomposition of the Adjoint Representation.

THEOREM.

(1)

$$U_q^{\text{ad}} = V_{\text{loc}} \oplus (\bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n)$$

(2)

$$U_q^{(m)} = V_{\text{loc}} \oplus (\bigoplus_{n \in \mathbb{Z}} V_n)$$

(3)

$$U_q^{(s)} = V_{loc} \oplus (\oplus_{n \in 2\mathbb{Z}} V_n)$$

(4) Any irreducible submodule of U_q^{ad} is contained in V_{loc} .(5) V_n ($n \in \frac{1}{2}\mathbb{Z}$)'s are indecomposable modules.(6) $V_{2\alpha}$ ($\alpha \in \mathbb{Z}_{>0}$) is isomorphic to

$$X(\alpha) \oplus X(-\alpha)/(-id \oplus \tau)(U_q^{(1)} f^\alpha v_\alpha)$$

 V_β ($\beta \notin 2\mathbb{Z}_{>0}$)'s are isomorphic to $X(0)$.(7) $X(0)$ and $V_{2\alpha}$'s ($\alpha \in \mathbb{Z}_{>0}$) are mutually non-isomorphic.(8) If a direct summand of U_q^{ad} is finitely generated and is indecomposable, then it is isomorphic to a $L(d)$, or a direct summand of $X(0)^{\oplus s} \oplus (\oplus_{j=1}^r V_{2j})$.

REFERENCES

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(The author is grateful to Dr. M. Noumi for showing him this problem.)