

Integrals associated with 2-posets and applications to multiple zeta values

By

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Abstract

This is a survey article on the study of multiple zeta values based on a certain type of multiple integrals, which is a combinatorial generalization of the well-known iterated integral expression of multiple zeta values. Elementary computations of such integrals lead to a class of linear relations among multiple zeta values, called the integral-series identity. We discuss the relationships of the integral-series identity with several other classes of relations.

§ 1. Introduction

In the study of the multiple zeta values, the iterated integral expression is a source of various developments, including the geometric understanding of them. Even at an elementary level, the integral expression has important consequences such as the duality relation and the shuffle relation.

The author [17] introduced a combinatorial generalization of the iterated integral, the integral associated with a 2-poset. Since then, it has been used to study the multiple zeta values by several people, e.g. [9, 10, 14]. In particular, Kaneko and the author [9] obtained a new class of linear relations of the multiple zeta values, called the integral-series identity, and showed its equivalence (in a certain sense) with the regularized double shuffle relation.

In this article, we give an overview of these developments, mainly focusing on the results in [17, 9]. Roughly, §2 is based on [17] and §3 on [9].

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In §2, the notions of 2-posets and associated integrals are defined, and then several kinds of multiple zeta values which can be expressed by the 2-poset integrals are described. Moreover, the formula which we call the integral-series identity (Theorem 2.10) is presented as a simultaneous generalization of the integral expressions of the multiple zeta and zeta-star values. In §3, we regard the integral-series identity as a class of linear relations among the multiple zeta values, and discuss its relationships with several other classes of relations, namely, the shuffle and harmonic relations, the regularization theorems and the Kawashima relation. These results are summarized, in a rough style, as follows:

Theorem 1.1 (Theorems 3.6, 3.10, 3.12 and 3.18).

- (1) *Assuming the integral-series identity, the shuffle relation and the harmonic relation are equivalent.*
- (2) *Assuming the shuffle and harmonic relations, the regularization theorem, the star-regularization theorem and the integral-series identity are mutually equivalent.*
- (3) *The shuffle relation, the integral-series identity and the duality relation imply the Kawashima relation.*

§ 2. 2-posets and associated integrals

In this section, we introduce the integrals associated with 2-posets, the main object of this article. Then we explain how multiple zeta values of several types are expressed by these 2-poset integrals. From such identities of the form “an integral = a series”, one deduces linear relations among the multiple zeta values.

§ 2.1. Definition and basic properties

Definition 2.1. A 2-poset is a pair (X, δ) of a finite partially ordered set (poset) $X = (X, \leq)$ and a map $\delta: X \rightarrow \{0, 1\}$ (we often say “a 2-poset X ” omitting the map δ). A 2-poset is said *admissible* if $\delta(x) = 0$ for all maximal $x \in X$ and $\delta(x) = 1$ for all minimal $x \in X$. When $X = (X, \delta)$ is an admissible 2-poset, we define the associated integral

$$I(X) = \int_{\Delta_X} \prod_{x \in X} \omega_{\delta(x)}(t_x),$$

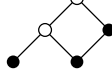
where

$$\Delta_X = \{(t_x) \in [0, 1]^X \mid t_x < t_y \text{ if } x < y\},$$

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}.$$

As a convention, we put $I(\emptyset) = 1$.

It is convenient to express a 2-poset by a colored Hasse diagram, where an element x with $\delta(x) = 0$ (resp. 1) is depicted by a vertex $\circ$ (resp. $\bullet$). For example, the diagram



expresses the 2-poset consisting of five elements $x_1, \dots, x_5$ with the order

$$x_1 < x_2 < x_3 > x_4 > x_5, \quad x_2 > x_5$$

and the map

$$\delta(x_1) = 1, \quad \delta(x_2) = 0, \quad \delta(x_3) = 0, \quad \delta(x_4) = 1, \quad \delta(x_5) = 1.$$

This 2-poset is admissible, and the associated integral is

$$I \left(\begin{array}{c} \circ \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right) = \int_{\Delta} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3} \frac{dt_4}{1-t_4} \frac{dt_5}{1-t_5},$$

where

$$\Delta = \{(t_1, \dots, t_5) \in [0, 1]^5 \mid t_1 < t_2 < t_3 > t_4 > t_5, t_2 > t_5\}.$$

Here we list basic constructions on 2-posets and elementary properties of the associated integrals.

Definition 2.2.

- (1) For 2-posets X and Y , the direct sum $X \amalg Y$ is defined obviously.
- (2) If two elements a, b of a 2-poset X are non-comparable (i.e., neither $a \leq b$ nor $b \leq a$ hold), we denote by X_a^b the 2-poset obtained by adjoining the new relation $a < b$.
- (3) For a 2-poset X , we denote by X^t the *transpose* of X , i.e., the 2-poset with the same underlying set as X , with the reversed order and the map $\delta_{X^t}(x) = 1 - \delta_X(x)$.

Proposition 2.3 ([17, Proposition 2.3]).

- (1) If X and Y are admissible 2-posets, then $X \amalg Y$ is also admissible and

$$I(X \amalg Y) = I(X) \cdot I(Y).$$

- (2) If X is an admissible 2-poset and $a, b \in X$ are non-comparable, then both X_a^b and X_b^a are admissible and

$$I(X) = I(X_a^b) + I(X_b^a).$$

(3) If X is an admissible 2-poset, then its transpose X^t is admissible and

$$I(X^t) = I(X).$$

§ 2.2. Multiple zeta values expressed by 2-poset integrals

We explain the relations of certain 2-poset integrals with various types of multiple zeta values.

2.2.1. Multiple zeta values Let us begin the well-known case of the *multiple zeta value* of Euler-Zagier type (MZV for short)

$$(2.1) \quad \zeta(\mathbf{k}) = \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}.$$

Here $\mathbf{k} = (k_1, \dots, k_r)$ is an *index*, i.e., a tuple of positive integers. Moreover, for the convergence of the series, we assume that $k_r \geq 2$ (such an index is said *admissible*). For an index $\mathbf{k} = (k_1, \dots, k_r)$, its length r and the sum $k_1 + \dots + k_r$ of the entries are called the *depth* and the *weight*, and denoted by $\text{dep}(\mathbf{k})$ and $\text{wt}(\mathbf{k})$, respectively. As a convention, we consider the empty index $\emptyset$ of depth 0, regard it as an admissible index and set $\zeta(\emptyset) = 1$.

The iterated integral expression of MZVs, first noticed by M. Kontsevich, is the following (cf. [19, p. 510]):

Proposition 2.4. *For an admissible index $\mathbf{k} = (k_1, \dots, k_r)$, we have*

$$(2.2) \quad \zeta(\mathbf{k}) = I \left(\begin{array}{c} \text{Diagram of a path with } r \text{ segments, each having } k_i \text{ vertices (black dots) and } k_i - 1 \text{ edges (white circles). The path starts at a black dot and ends at a white circle.} \end{array} \right).$$

Proof. This is obtained by expanding $\frac{1}{1-t} = 1 + t + t^2 + \dots$ for each $\bullet$ and

integrating term by term. For example, we have

$$\begin{aligned}
 I \left(\begin{array}{c} \circ \\ \bullet \bullet \bullet \end{array} \right) &= \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} \\
 &= \sum_{n_1, n_2 > 0} \int_{0 < t_1 < t_2 < t_3 < 1} t_1^{n_1-1} dt_1 t_2^{n_2-1} dt_2 t_3^{-1} dt_3 \\
 &= \sum_{n_1, n_2 > 0} \frac{1}{n_1(n_1+n_2)^2} = \zeta(1, 2).
 \end{aligned}$$

The general case follows the same pattern. $\square$

Corollary 2.5. *For any admissible 2-poset X , the associated integral $I(X)$ can be written as a finite sum of MZVs.*

Proof. This is done by induction on the number of non-comparable pairs in X : Proposition 2.3 (2) decreases the number. When the number is zero, which means that X is totally ordered, the associated integral is just like the right hand side of (2.2) and is equal to a single MZV. $\square$

The following relation of MZVs, called the *duality*, is an immediate consequence of the iterated integral expression (2.2).

Corollary 2.6. *We write an admissible index $\mathbf{k}$ as*

$$\mathbf{k} = (\underbrace{1, \dots, 1}_{a_1-1}, 1+b_1, \underbrace{1, \dots, 1}_{a_2-1}, 1+b_2, \dots, \underbrace{1, \dots, 1}_{a_s-1}, 1+b_s)$$

with positive integers $a_1, b_1, \dots, a_s, b_s$ and then define its dual index $\mathbf{k}^\dagger$ by

$$\mathbf{k}^\dagger = (\underbrace{1, \dots, 1}_{b_s-1}, 1+a_s, \dots, \underbrace{1, \dots, 1}_{b_2-1}, 1+a_2, \underbrace{1, \dots, 1}_{b_1-1}, 1+a_1).$$

Then we have

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\dagger).$$

Proof. Apply Proposition 2.3 (3) to (2.2). $\square$

2.2.2. Multiple zeta-star values The next example is the multiple zeta-star value (MZSV)

$$\zeta^\star(\mathbf{k}) = \sum_{0 < m_1 \leq \dots \leq m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

defined for an admissible $\mathbf{k} = (k_1, \dots, k_r)$. For the empty index, we put $\zeta^\star(\emptyset) = 1$ again.

Proposition 2.7 ([17, Corollary 1.3]). *For an admissible index $\mathbf{k} = (k_1, \dots, k_r)$, we have*

$$(2.3) \quad \zeta^*(\mathbf{k}) = I \left(\begin{array}{c} k_r \quad \quad \quad k_2 \quad k_1 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right).$$

Proof. This is obtained by expanding $\frac{1}{1-t} = 1 + t + t^2 + \dots$ for the leftmost $\bullet$, and then integrating term by term, using the expansion $\frac{1-t^m}{1-t} = 1 + t + \dots + t^{m-1}$ at other $\bullet$. For example, we have

$$\begin{aligned} I \left(\begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} \right) &= \int_{0 < t_1 < t_2 < 1, 0 < t_3 < t_2} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{dt_3}{1-t_3} \\ &= \sum_{m_1 > 0} \int_0^1 \frac{dt_3}{1-t_3} \int_{t_3}^1 \frac{dt_2}{t_2} \int_0^{t_2} t_1^{m_1-1} dt_1 \\ &= \sum_{m_1 > 0} \frac{1}{m_1} \int_0^1 \frac{dt_3}{1-t_3} \int_{t_3}^1 t_2^{m_1-1} dt_2 \\ &= \sum_{m_1 > 0} \frac{1}{m_1^2} \int_0^1 \frac{1-t_3^{m_1}}{1-t_3} dt_3 \\ &= \sum_{m_1 > 0} \frac{1}{m_1^2} \sum_{m_2=1}^{m_1} \int_0^1 t_3^{m_2-1} dt_3 \\ &= \sum_{0 < m_2 \leq m_1} \frac{1}{m_2 m_1^2} = \zeta^*(1, 2). \end{aligned}$$

The general case follows the same pattern. □

2.2.3. Mordell-Tornheim multiple zeta values The multiple zeta value of Mordell-Tornheim type (cf. [13, 16]) is defined by the series

$$\zeta_{\text{MT}}(k_1, \dots, k_r; k_{r+1}) = \sum_{m_1, \dots, m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r} (m_1 + \dots + m_r)^{k_{r+1}}},$$

for positive integers $k_1, \dots, k_{r+1}$.

Proposition 2.8 ([17, §3.2]). *For positive integers $k_1, \dots, k_{r+1}$, we have*

$$(2.4) \quad \zeta_{\text{MT}}(k_1, \dots, k_r; k_{r+1}) = I \left(\begin{array}{c} k_{r+1} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right).$$

In particular, $\zeta_{\text{MT}}(k_1, \dots, k_r; k_{r+1})$ is a finite sum of MZVs.

Proof. This is obtained similarly to Proposition 2.4. A typical example is

$$\begin{aligned}
 I\left(\begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}\right) &= \int_0^1 \frac{du}{u} \int_0^u \frac{dt_1}{1-t_1} \int_0^u \frac{dt_2}{1-t_2} \\
 &= \sum_{m_1, m_2 > 0} \int_0^1 \frac{du}{u} \int_0^u t_1^{m_1-1} dt_1 \int_0^u t_2^{m_2-1} dt_2 \\
 &= \sum_{m_1, m_2 > 0} \frac{1}{m_1 m_2} \int_0^1 u^{m_1+m_2-1} du \\
 &= \sum_{m_1, m_2 > 0} \frac{1}{m_1 m_2 (m_1 + m_2)} = \zeta_{\text{MT}}(1, 1; 1).
 \end{aligned}$$

In the general case, the pattern is the same. □

Remark. Bradley-Zhou [2] proved by a different method that $\zeta_{\text{MT}}(k_1, \dots, k_r; k_{r+1})$ belongs to the $\mathbb{Q}$ -vector space generated by MZVs. Moreover, Kamano [7] obtained an explicit formula

$$\zeta_{\text{MT}}(k_1, \dots, k_r; k_{r+1}) = \zeta((e_{k_1} \text{ III } \dots \text{ III } e_{k_r}) e_0^{k_{r+1}})$$

using the algebraic language which will be introduced in the next section. In fact, this formula is equivalent to (2.4) through the repeated use of Proposition 2.3 (2).

2.2.4. Arakawa-Kaneko multiple zeta values The Arakawa-Kaneko multiple zeta function (see [1]) is defined as

$$\xi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}_{\mathbf{k}}(1 - e^{-x})}{e^x - 1} x^{s-1} dx$$

for an index $\mathbf{k} = (k_1, \dots, k_r)$, where

$$\text{Li}_{\mathbf{k}}(t) = \sum_{0 < m_1 < \dots < m_r} \frac{t^{m_r}}{m_1^{k_1} \dots m_r^{k_r}}$$

is the multiple polylogarithm.

Proposition 2.9 ([17, §3.1]). *For an index $\mathbf{k} = (k_1, \dots, k_r)$ and a positive in-*

teger n , we have

$$(2.5) \quad \xi(\mathbf{k}; n) = I \left(\begin{array}{c} \text{Diagram: A path of nodes starting from a black dot at the bottom left, passing through white nodes labeled } k_1, k_{r-1}, k_r, \text{ and ending at a black dot labeled } n \text{ at the top right.} \end{array} \right).$$

Proof. The change of variable $1 - e^{-x} = t$ shows that

$$\xi(\mathbf{k}; n) = \frac{1}{(n-1)!} \int_0^1 \text{Li}_{\mathbf{k}}(t) (-\log(1-t))^{n-1} \frac{dt}{t}.$$

Then, by substituting the usual iterated integral expression of $\text{Li}_{\mathbf{k}}(t)$ (similar to (2.2), with the integral up to t instead of 1) and

$$\begin{aligned} \frac{1}{(n-1)!} (-\log(1-t))^{n-1} &= \frac{1}{(n-1)!} \left(\int_0^t \frac{du}{1-u} \right)^{n-1} \\ &= \int_{0 < u_1 < \dots < u_{n-1} < t} \frac{du_1}{1-u_1} \dots \frac{du_{n-1}}{1-u_{n-1}}, \end{aligned}$$

we obtain the equality (2.5). $\square$

Remark.

(1) In the depth one case $\mathbf{k} = (k)$, one sees that

$$\xi(k; n) = I \left(\begin{array}{c} \text{Diagram: A path of nodes starting from a black dot at the bottom left, passing through white nodes labeled } k \text{ and } n, \text{ and ending at a black dot at the top right.} \end{array} \right) = \zeta^*(\underbrace{1, \dots, 1}_{n-1}, k+1)$$

by (2.5) and (2.3). This formula was obtained by Ohno [15, Theorem 2] with a different proof.

(2) By applying Proposition 2.3 (3) to (2.5), one sees that

$$(2.6) \quad \xi(\mathbf{k}; n) = I \left(\begin{array}{c} \text{Diagram: A path of nodes starting from a white node at the top left, passing through black nodes labeled } k_1, k_{r-1}, k_r, \text{ and ending at a white node labeled } n \text{ at the bottom right.} \end{array} \right).$$

Then, by using Proposition 2.3 (2) repeatedly, one deduces

$$(2.7) \quad \xi(\mathbf{k}; n) = \sum_{\substack{\mathbf{j}=(j_1, \dots, j_s) \in \mathbb{Z}_{\geq 0}^s \\ j_1 + \dots + j_s = n-1}} b((\mathbf{k}_+)^{\dagger}; \mathbf{j}) \zeta((\mathbf{k}_+)^{\dagger} + \mathbf{j}).$$

Here $\mathbf{k}_+ = (k_1, \dots, k_{r-1}, k_r + 1)$ with the dual index $(\mathbf{k}_+)^{\dagger} = (l_1, \dots, l_s)$, and the coefficient $b((\mathbf{k}_+)^{\dagger}; \mathbf{j})$ is defined by

$$b((\mathbf{k}_+)^{\dagger}; \mathbf{j}) = \prod_{i=1}^s \binom{l_i + j_i - 1}{j_i}.$$

The formula (2.7) was first proved by Kaneko-Tsumura [8, Theorem 2.5] by a different method. The proof based on (2.6) was given by Kawasaki-Ohno [10].

§ 2.3. Integral-series identity

Now let us state a formula which we call the *integral-series identity*.

Theorem 2.10 ([9, Theorem 4.1]). *Let $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$ be non-empty indices (possibly non-admissible). Then we have*

$$(2.8) \quad I \left(\begin{array}{c} \text{Diagram: A path of nodes starting from a black dot labeled } k_1, \text{ passing through } k_r, \text{ and ending at } l_s. \text{ Nodes are connected by arcs. Some nodes are black dots, others are white circles. Labels } k_1, k_r, l_{s-1}, l_s \text{ are placed near specific nodes.} \end{array} \right) = \sum_{\substack{0 < m_1 < \dots < m_r \\ 0 < n_1 \leq \dots \leq n_s \\ m_r = n_s}} \frac{1}{m_1^{k_1} \dots m_r^{k_r} n_1^{l_1} \dots n_s^{l_s}}.$$

Note that this theorem is a simultaneous generalization of Proposition 2.4 and Proposition 2.7. In fact, these propositions are obtained by putting $\mathbf{l} = (1)$ or $\mathbf{k} = (1)$, respectively. Moreover, its proof is also an amalgam of the proofs of these propositions; that is, expansion $\frac{1}{1-t} = 1 + t + t^2 + \dots$ at each $\bullet$ in the k -part, and term-by-term integration from the left side using $\frac{1-t^m}{1-t} = 1 + t + \dots + t^{m-1}$ at each $\bullet$ in the l -part.

From Theorem 2.10, one can deduce various linear relations among MZVs. For example, for $\mathbf{k} = \mathbf{l} = (1, 1)$, we have

$$I \left(\begin{array}{c} \text{Diagram: A path of three nodes. The first and third nodes are black dots, the second is a white circle.} \end{array} \right) = \sum_{0 < m_1 < m_2 = n_2 \leq n_1 > 0} \frac{1}{m_1 m_2 n_1 n_2}.$$

By Proposition 2.3 (3), we see that

$$I \left(\begin{array}{c} \text{Diagram: A path of three nodes. The first and third nodes are black dots, the second is a white circle.} \end{array} \right) = 3I \left(\begin{array}{c} \text{Diagram: A path of three nodes. The first and second nodes are black dots, the third is a white circle.} \end{array} \right) = 3\zeta(1, 1, 2).$$

On the other hand, we also have

$$\begin{aligned}
& \sum_{0 < m_1 < m_2 = n_2 \geq n_1 > 0} \frac{1}{m_1 m_2 n_1 n_2} = \sum_{0 < m_1 < m_2 \geq n > 0} \frac{1}{m_1 m_2^2 n} \\
& = \left(\sum_{0 < m_1 < m_2 = n} + \sum_{0 < m_1 < n < m_2} + \sum_{0 < m_1 = n < m_2} + \sum_{0 < n < m_1 < m_2} \right) \frac{1}{m_1 m_2^2 n} \\
& = \zeta(1, 3) + \zeta(2, 2) + 2\zeta(1, 1, 2).
\end{aligned}$$

From these equalities, we deduce that

$$\zeta(1, 1, 2) = \zeta(1, 3) + \zeta(2, 2).$$

By generalizing this computation, we can prove the sum formula

$$\sum_{\text{wt}(\mathbf{k})=k, \text{dep}(\mathbf{k})=r} \zeta(\mathbf{k}) = \zeta(k),$$

and its generalization (the restricted sum formula) by Eie, Liaw and Ong [3] (see [9, Proposition 7.1]). For a further generalization, see also a recent paper by Murakami-Murahara [14].

Supported by such results and Theorem 3.10 in the next section, we conjecture that this method gives *all* linear relations among MZVs over $\mathbb{Q}$ ([9, Conjecture 4.3]). This is confirmed up to weight 17 by numerical computation.

§ 3. Relationship with other classes of relations

In the previous section, we observed that a class of linear relations of MZVs is deduced from the integral-series identity (2.8). In this section, we will consider implications among some classes of (linear or algebraic) relations of MZVs. To discuss such a relationship among relations, we first introduce a formal setup due to Hoffman [4]. Then we formulate in this formal setting several classes of relations including the integral-series identity, and state the results on the relationships among them.

§ 3.1. Formal setup of the integral-series identity

Let $\mathfrak{H} = \mathbb{Q}\langle e_0, e_1 \rangle$ be the non-commutative polynomial algebra over $\mathbb{Q}$ with two indeterminates e_0, e_1 . Thus, $\mathfrak{H}$ has a basis consisting of the words in the letters e_0 and e_1 , including the empty word denoted by 1. We define its subalgebras $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$ by

$$\mathfrak{H}^1 = \mathbb{Q} \oplus e_1 \mathfrak{H}, \quad \mathfrak{H}^0 = \mathbb{Q} \oplus e_1 \mathfrak{H} e_0.$$

In other words, $\mathfrak{H}^1$ (resp. $\mathfrak{H}^0$) is the subspace of $\mathfrak{H}$ spanned by the words in e_0, e_1 that begin with e_1 (resp. that begin with e_1 and end with e_0) together with 1. By setting $e_k = e_1 e_0^{k-1}$ for positive integers k , they are also written as

$$\mathfrak{H}^1 = \bigoplus_{\substack{(k_1, \dots, k_r) \\ \text{index}}} \mathbb{Q} e_{k_1} \cdots e_{k_r}, \quad \mathfrak{H}^0 = \bigoplus_{\substack{(k_1, \dots, k_r) \\ \text{admissible index}}} \mathbb{Q} e_{k_1} \cdots e_{k_r}.$$

The latter description of $\mathfrak{H}^0$ allows one to define $\mathbb{Q}$ -linear maps $\zeta, \zeta^*: \mathfrak{H}^0 \rightarrow \mathbb{R}$ by

$$\zeta(e_{k_1} \cdots e_{k_r}) = \zeta(k_1, \dots, k_r), \quad \zeta^*(e_{k_1} \cdots e_{k_r}) = \zeta^*(k_1, \dots, k_r)$$

for all admissible indices $(k_1, \dots, k_r)$. Moreover, there is a natural linear automorphism $(\cdot)^*: \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$ which preserves $\mathfrak{H}^0$ and satisfies that $\zeta^*(w) = \zeta(w^*)$. Up to depth 3, this is defined by

$$\begin{aligned} 1^* &= 1, \\ e_k^* &= e_k, \\ (e_k e_l)^* &= e_k e_l + e_{k+l}, \\ (e_k e_l e_m)^* &= e_k e_l e_m + e_{k+l} e_m + e_k e_{l+m} + e_{k+l+m}. \end{aligned}$$

In the following, by abuse of notation, we often identify the word $e_{k_1} \cdots e_{k_r}$ in $\mathfrak{H}^1$ with the index $(k_1, \dots, k_r)$. Hence we write, e.g., $(k, l)^* = (k, l) + (k + l)$, $(k) * (l) = (k, l) + (l, k) + (k + l)$, etc.

To treat the 2-poset integrals in the above setting, we consider the $\mathbb{Q}$ -vector space $\mathfrak{P}$ generated by all isomorphism classes of 2-posets. Just like $\mathfrak{H}$, this has the subspaces $\mathfrak{P}^0 \subset \mathfrak{P}^1 \subset \mathfrak{P}$: $\mathfrak{P}^1$ is spanned by the isomorphism classes of 2-posets (X, δ) such that $\delta(x) = 1$ for any minimal $x \in X$, and $\mathfrak{P}^0$ is spanned by the isomorphism classes of admissible 2-posets. Then, like ζ , the map I which assigns the integral $I(X)$ to an admissible 2-poset X is extended to a $\mathbb{Q}$ -linear map $I: \mathfrak{P}^0 \rightarrow \mathbb{R}$.

Corollary 2.5 says that the map $I: \mathfrak{P}^0 \rightarrow \mathbb{R}$ factors through the map $\zeta: \mathfrak{H}^0 \rightarrow \mathbb{R}$. More precisely, there is a $\mathbb{Q}$ -linear map $W: \mathfrak{P} \rightarrow \mathfrak{H}$, which transforms a 2-poset to a finite sum of words in e_0, e_1 , characterized by the conditions:

- For a totally ordered $X = \{x_1 < x_2 < \cdots < x_k\}$,

$$W(X) = e_{\delta(x_1)} e_{\delta(x_2)} \cdots e_{\delta(x_k)}.$$

- If a and b are non-comparable in X ,

$$W(X) = W(X_a^b) + W(X_b^a).$$

This W sends $\mathfrak{P}^1$ and $\mathfrak{P}^0$ onto $\mathfrak{H}^1$ and $\mathfrak{H}^0$ respectively, and satisfies

$$I = \zeta \circ W : \mathfrak{P}^0 \rightarrow \mathbb{R}.$$

In the following, we consider $\mathbb{Q}$ -linear maps $Z : \mathfrak{H}^0 \rightarrow R$, where R is any commutative $\mathbb{Q}$ -algebra, keeping $\zeta : \mathfrak{H}^0 \rightarrow \mathbb{R}$ in mind as a typical example. Then we may regard a class of relations among MZVs, e.g. the integral-series identity, as a condition on such Z . We note that, though Kaneko-Yamamoto [9] treat only the case $R = \mathbb{R}$, their arguments may be applied to any commutative $\mathbb{Q}$ -algebra R .

§ 3.2. Shuffle and harmonic relations

There are two natural ways to write a product of MZVs as a finite sum of MZVs, one of which is based on the iterated integral expression (2.2), and another on the multiple series expression (2.1) defining MZVs. They are formulated by two kinds of products on $\mathfrak{H}^0$.

Definition 3.1.

- (1) A $\mathbb{Q}$ -bilinear product $\mathfrak{m} : \mathfrak{H} \times \mathfrak{H} \rightarrow \mathfrak{H}$, called the *shuffle product*, is defined recursively by

$$\begin{aligned} 1 \mathfrak{m} w &= w \mathfrak{m} 1 = w, \\ va \mathfrak{m} wb &= (v \mathfrak{m} wb)a + (va \mathfrak{m} w)b, \end{aligned}$$

where v, w are any words in e_0, e_1 and $a, b \in \{e_0, e_1\}$.

- (2) A $\mathbb{Q}$ -bilinear product $*$: $\mathfrak{H}^1 \times \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$, called the *harmonic* (or *quasi-shuffle*) *product*, is defined recursively by

$$\begin{aligned} 1 * w &= w * 1 = w, \\ ve_k * we_l &= (v * we_l)e_k + (ve_k * w)e_l + (v * w)e_{k+l}, \end{aligned}$$

where v, w are any words in $e_1, e_2, \dots$ and k, l are any positive integers.

$\mathfrak{H}$ is a commutative $\mathbb{Q}$ -algebra with respect to the shuffle product, for which we write $\mathfrak{H}_{\mathfrak{m}}$. $\mathfrak{H}^1$ and $\mathfrak{H}^0$ form subalgebras of $\mathfrak{H}_{\mathfrak{m}}$, written $\mathfrak{H}_{\mathfrak{m}}^1$ and $\mathfrak{H}_{\mathfrak{m}}^0$ respectively. Similarly, $\mathfrak{H}^1$ is a commutative $\mathbb{Q}$ -algebra with respect to the harmonic product and $\mathfrak{H}^0$ is a subalgebra, written $\mathfrak{H}_*^1$ and $\mathfrak{H}_*^0$ respectively.

We equip the vector space $\mathfrak{P}$ generated by 2-posets with the product given by

$$[X] \cdot [Y] = [X \amalg Y]$$

for any 2-posets X and Y (here $[X]$ denotes the isomorphism class of X). Then $\mathfrak{P}$ becomes a commutative $\mathbb{Q}$ -algebra, and $\mathfrak{P}^1$ and $\mathfrak{P}^0$ are subalgebras. Moreover, the map $W: \mathfrak{P} \rightarrow \mathfrak{H}_{\text{III}}$ is an algebra homomorphism. Indeed, the definition of the shuffle product III is an inductive description of the way to compute $W(X \text{ II } Y)$ for totally ordered X and Y using the characterizing properties of W .

Definition 3.2. A $\mathbb{Q}$ -linear map Z from $\mathfrak{H}^0$ to a commutative $\mathbb{Q}$ -algebra R is said to satisfy the *shuffle relation* (resp. *harmonic relation*) if $Z: \mathfrak{H}_{\text{III}}^0 \rightarrow R$ (resp. $Z: \mathfrak{H}_*^0 \rightarrow R$) is a homomorphism, i.e.,

$$Z(v)Z(w) = Z(v \text{ III } w) \quad (\text{resp. } Z(v)Z(w) = Z(v * w))$$

holds for any $v, w \in \mathfrak{H}^0$. If Z satisfies both the shuffle and harmonic relations, we say that Z satisfies the *double shuffle relation*.

Proposition 3.3. The MZV map $\zeta: \mathfrak{H}^0 \rightarrow \mathbb{R}$ satisfies the double shuffle relation.

Proof. The harmonic relation follows from a computation using the definition (2.1) of MZVs. For example, one can compute as follows:

$$\begin{aligned} \zeta(e_k)\zeta(e_l) &= \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{n=1}^{\infty} \frac{1}{n^l} = \left(\sum_{0 < m < n} + \sum_{0 < n < m} + \sum_{0 < m = n} \right) \frac{1}{m^k n^l} \\ &= \zeta(k, l) + \zeta(l, k) + \zeta(k + l) = \zeta(e_k * e_l). \end{aligned}$$

The shuffle relation follows from a similar computation using the iterated integral expression (2.2). Alternatively, one may note that $W: \mathfrak{P}^0 \rightarrow \mathfrak{H}^0$ is a surjective homomorphism and $\zeta \circ W = I: \mathfrak{P}^0 \rightarrow \mathbb{R}$ is a homomorphism (Proposition 2.3 (1)). $\square$

We also reformulate the integral-series identity (2.8) as a condition for a linear map on $\mathfrak{H}^0$.

Definition 3.4. For non-empty indices $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$, we put

$$\mu(\mathbf{k}, \mathbf{l}) = W \left(\begin{array}{c} \text{Diagram of a path with vertices } k_1, \dots, k_r, l_s, \dots, l_1 \end{array} \right) \in \mathfrak{H}^0.$$

We also define a $\mathbb{Q}$ -bilinear product $\otimes$ on $e_1 \mathfrak{H}$ (the space spanned by the non-empty indices) by

$$ve_k \otimes we_l = (v * w)e_{k+l}.$$

With these symbols, we can write the identity (2.8) simply as

$$\zeta(\mu(\mathbf{k}, \mathbf{l})) = \zeta(\mathbf{k} \circledast \mathbf{l}^*).$$

Definition 3.5. A $\mathbb{Q}$ -linear map $Z: \mathfrak{H}^0 \rightarrow R$ is said to satisfy the *integral-series identity* if

$$Z(\mu(\mathbf{k}, \mathbf{l})) = Z(\mathbf{k} \circledast \mathbf{l}^*)$$

holds for any non-empty indices $\mathbf{k}$ and $\mathbf{l}$.

Now let us observe the following example. For integers $k, l \geq 2$, one can verify the equalities in $\mathfrak{H}^0$

$$(k-1) \circledast (l, 1)^* = (l, k) + (k+l) = (k) * (l) - (k, l)$$

and

$$\begin{aligned} \mu((k-1), (l, 1)) &= W \left(\begin{array}{c} k \quad l \\ \text{diagram} \end{array} \right) \\ &= W \left(\begin{array}{c} k \quad l \\ \text{diagram} \end{array} \right) - W \left(\begin{array}{c} l \\ \text{diagram} \end{array} \right) \\ &= (k) \amalg (l) - (k, l). \end{aligned}$$

Then it is easy to see that, among the three conditions

$$\begin{aligned} Z(k)Z(l) &= Z((k) * (l)), & Z(k)Z(l) &= Z((k) \amalg (l)), \\ Z(\mu((k-1), (l, 1))) &= Z((k-1) \circledast (l, 1)^*), \end{aligned}$$

any two implies the other. This simple observation suggests that there are some relationships among the harmonic relation, the shuffle relation and the integral-series identity. In fact, we have:

Theorem 3.6 ([9, Theorem 4.8]). *Assume that a $\mathbb{Q}$ -linear map $Z: \mathfrak{H}^0 \rightarrow R$ satisfies the integral-series identity. Then it satisfies the shuffle relation if and only if it satisfies the harmonic relation.*

Notice that, in the above example, the integral-series identity $\mu((k-1), (l, 1)) = (k-1) \circledast (l, 1)^*$ has a meaning even if $l = 1$, while the corresponding shuffle and harmonic relations are meaningless since $Z(1)$ is undefined. This suggest that the double shuffle relation is not enough to imply the integral-series identity. To fill this gap, we next introduce the regularizations, the method to extend certain maps on $\mathfrak{H}^0$ to those on $\mathfrak{H}^1$.

§ 3.3. Regularizations

Let $Z: \mathfrak{H}^0 \rightarrow R$ be a $\mathbb{Q}$ -linear map to a commutative $\mathbb{Q}$ -algebra R . In this subsection, we assume that Z satisfies the double shuffle relation.

Definition 3.7. The *shuffle regularization* of Z is the homomorphism $Z_{\text{sh}}: \mathfrak{H}_{\text{sh}}^1 \rightarrow R[T]$ of $\mathbb{Q}$ -algebras such that

$$Z_{\text{sh}}|_{\mathfrak{H}^0} = Z, \quad Z_{\text{sh}}(e_1) = T,$$

where T is an indeterminate. In fact, such Z_{sh} uniquely exists since $\mathfrak{H}_{\text{sh}}^1 \cong \mathfrak{H}_{\text{sh}}^0[e_1]$.

Similarly, the *harmonic regularization* of Z is the homomorphism $Z_*: \mathfrak{H}_*^1 \rightarrow R[T]$ of $\mathbb{Q}$ -algebras such that

$$Z_*|_{\mathfrak{H}^0} = Z, \quad Z_*(e_1) = T.$$

Such Z_* uniquely exists since $\mathfrak{H}_*^1 \cong \mathfrak{H}_*^0[e_1]$.

Applied to the MZV map $\zeta: \mathfrak{H}^0 \rightarrow \mathbb{R}$, the above constructions define the shuffle and harmonic regularizations $\zeta_{\text{sh}}, \zeta_*: \mathfrak{H}^1 \rightarrow \mathbb{R}[T]$. In general, they do not give the same element of $\mathbb{R}[T]$ for a non-admissible index, e.g.

$$\zeta_{\text{sh}}(1, 1) = \frac{T^2}{2}, \quad \zeta_*(1, 1) = \frac{T^2}{2} - \frac{\zeta(2)}{2}.$$

To state the precise relation of $\zeta_{\text{sh}}(w)$ and $\zeta_*(w)$ for general $w \in \mathfrak{H}^1$, we need an $\mathbb{R}$ -linear map $\rho: \mathbb{R}[T] \rightarrow \mathbb{R}[T]$ by

$$\rho(e^{Tu}) = A(u)r^{Tu},$$

which is an identity in the formal power series algebra $\mathbb{R}[T][[u]]$ on which ρ acts coefficientwise. Here

$$A(u) = e^{\gamma u} \Gamma(1+u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) \in \mathbb{R}[[u]].$$

Theorem 3.8 ([6, Theorem 1]). *For any $w \in \mathfrak{H}^1$, we have*

$$\zeta_{\text{sh}}(w) = \rho(\zeta_*(w)).$$

We formalize this theorem as a condition on Z .

Definition 3.9. Define a $\mathbb{Q}$ -linear map ρ_Z on $R[T]$ by

$$\rho_Z(e^{Tu}) = A_Z(u)e^{Tu},$$

where

$$(3.1) \quad A_Z(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} Z(n) u^n\right) \in R[[u]].$$

Then we say that Z satisfies the *regularization theorem* if

$$Z_{\text{in}}(w) = \rho_Z(Z_*(w))$$

holds for any $w \in \mathfrak{H}^1$.

Theorem 3.10 ([9, Theorem 4.6]). *Assume that $Z: \mathfrak{H}^0 \rightarrow R$ satisfies the double shuffle relation. Then Z satisfies the regularization theorem if and only if Z satisfies the integral-series identity.*

§ 3.4. Star-regularization theorem

Here we state the star-version of Theorem 3.10. We continue to assume that $Z: \mathfrak{H}^0 \rightarrow R$ satisfies the double shuffle relation.

Recall that there is a map $w \mapsto w^*$ on $\mathfrak{H}^1$ such that $\zeta^*(w) = \zeta(w^*)$ if $w \in \mathfrak{H}^0$. For general Z , we put

$$Z^*(\mathbf{k}) = Z(w^*) \quad \text{for } w \in \mathfrak{H}^0.$$

The star-harmonic regularization is defined by the same formula

$$Z_*^*(w) = Z_*(w^*) \quad \text{for } w \in \mathfrak{H}^1.$$

On the other hand, the star-shuffle regularization is defined by

$$Z_{\text{in}}^*(\mathbf{k}) = Z_{\text{in}} \circ W \left(\begin{array}{c} k_r \quad \quad \quad k_2 \quad k_1 \\ \text{diagram} \end{array} \right)$$

for (not necessarily admissible) index $\mathbf{k} = (k_1, \dots, k_r)$.

Definition 3.11. Define a $\mathbb{Q}$ -linear map ρ_Z^* on $R[T]$ by

$$\rho_Z^*(e^{Tu}) = A_Z(-u)^{-1} e^{Tu},$$

where $A_Z(u)$ is given in (3.1). Then we say that Z satisfies the *star-regularization theorem* if

$$Z_{\text{in}}^*(w) = \rho_Z^*(Z_*^*(w))$$

holds for any $w \in \mathfrak{H}^1$.

Theorem 3.12 ([9, Theorem 4.6]). *Assume that $Z: \mathfrak{H}^0 \rightarrow R$ satisfies the double shuffle relation. Then Z satisfies the star-regularization theorem if and only if Z satisfies the integral-series identity.*

In conclusion, under the assumption of the double shuffle relation, the three conditions 1) the regularization theorem, 2) the star-regularization theorem and 3) the integral-series identity, are equivalent to each other. In particular, through this equivalence, the regularization theorem (Theorem 3.8) and the star-regularization theorem (Corollary 3.13 below) for the map ζ are deduced from Theorem 2.10, which is proved by an elementary computation of the multiple integral.

Corollary 3.13 ([9, Corollary 4.7]). *For any $w \in \mathfrak{H}^1$, we have*

$$\zeta_{\text{III}}^*(w) = \rho^*(\zeta_*^*(w)),$$

where ρ^* denotes ρ_Z^* for $Z = \zeta$.

Remark. Corollary 3.13 is applied to a study of certain symmetric sums of regularized MZSVs by Machide [12].

§ 3.5. Kawashima relation

Lastly, we discuss a class of algebraic relations of MZVs proved by Kawashima [11].

Definition 3.14. For an index $\mathbf{k} = (k_1, \dots, k_r)$, define its *Hoffman dual index* $\mathbf{k}^\vee = (l_1, \dots, l_s)$ by the conditions

$$\begin{aligned} \text{wt}(\mathbf{k}) &= \text{wt}(\mathbf{k}^\vee), \\ \{1, 2, \dots, \text{wt}(\mathbf{k}) - 1\} &= \{k_1, k_1 + k_2, \dots, k_1 + \dots + k_{r-1}\} \\ &\quad \cup \{l_1, l_1 + l_2, \dots, l_1 + \dots + l_{s-1}\}. \end{aligned}$$

Recall that the dual $\mathbf{k}^\dagger$ of an index is used in Corollary 2.6 to describe the duality relation $\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\dagger)$. Similarly, the Hoffman dual $\mathbf{k}^\vee$ is used to describe the ‘duality’ of finite sums (see [5, 11])

$$(3.2) \quad \sum_{n=1}^N s^*(\mathbf{k}, n) = \sum_{n=1}^N (-1)^{n-1} s^*(\mathbf{k}^\vee, n) \binom{N}{n},$$

where

$$s^*((k_1, \dots, k_r), n) = \sum_{0 < m_1 \leq \dots \leq m_r \leq n} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}.$$

One of the motivations to consider the 2-poset integrals was to understand the equation (3.2) in terms of a change of variables in some integral; see [17, §1].

Kawashima [11] introduced a one-variable complex function

$$F_{\mathbf{k}}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} s^*(\mathbf{k}^\vee, n) \binom{z}{n}$$

and proved:

(1) For any non-empty index $\mathbf{k}$, $F_{\mathbf{k}}(z)$ has the Taylor expansion

$$F_{\mathbf{k}}(z) = \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(\underbrace{(1, \dots, 1)}_m \circledast (\mathbf{k}^{\vee})^{\star}) z^m.$$

(2) For any non-empty indices $\mathbf{k}$ and $\mathbf{l}$, the harmonic relation

$$F_{\mathbf{k}}(z)F_{\mathbf{l}}(z) = F_{\mathbf{k} \bar{*} \mathbf{l}}(z)$$

holds. Here $\bar{*}$ is the star-version of the harmonic product defined by

$$(\mathbf{k} \bar{*} \mathbf{l})^{\star} = \mathbf{k}^{\star} * \mathbf{l}^{\star},$$

and the map $\mathbf{k} \mapsto F_{\mathbf{k}}(z)$ is linearly extended.

(For further information on Kawashima's function $F_{\mathbf{k}}(z)$, see also [18]). By combining these properties, one obtains:

Theorem 3.15 ([11, Corollary 5.4]). *For any non-empty indices $\mathbf{k}$ and $\mathbf{l}$ and any integer $m \geq 1$, we have*

$$\sum_{\substack{p, q \geq 1 \\ p+q=m}} \zeta(\underbrace{(1, \dots, 1)}_p \circledast (\mathbf{k}^{\vee})^{\star}) \zeta(\underbrace{(1, \dots, 1)}_q \circledast (\mathbf{l}^{\vee})^{\star}) = -\zeta(\underbrace{(1, \dots, 1)}_m \circledast ((\mathbf{k} \bar{*} \mathbf{l})^{\vee})^{\star}).$$

Definition 3.16. We say that a $\mathbb{Q}$ -linear map $Z: \mathfrak{H}^0 \rightarrow R$ satisfies the *Kawashima relation* if

$$\sum_{\substack{p, q \geq 1 \\ p+q=m}} Z(\underbrace{(1, \dots, 1)}_p \circledast (\mathbf{k}^{\vee})^{\star}) Z(\underbrace{(1, \dots, 1)}_q \circledast (\mathbf{l}^{\vee})^{\star}) = -Z(\underbrace{(1, \dots, 1)}_m \circledast ((\mathbf{k} \bar{*} \mathbf{l})^{\vee})^{\star}).$$

holds for any non-empty indices $\mathbf{k}$ and $\mathbf{l}$ and any integer $m \geq 1$.

Kawashima also showed that his relation implies the duality:

Theorem 3.17 ([11, Corollary 7.2]). *If $Z: \mathfrak{H}^0 \rightarrow R$ satisfies the Kawashima relation, then it also satisfies the duality relation, i.e., $Z(w) = Z(w^{\dagger})$ holds for any $w \in \mathfrak{H}^0$.*

On the other hand, it is an open problem that whether the double shuffle relation with the regularization theorem implies the duality. Hence, so far, we have not been able to deduce the Kawashima relation from the double shuffle relation and the regularization theorem. Nevertheless, we can prove:

Theorem 3.18 ([9, Theorem 6.7]). *Let $Z: \mathfrak{H}^0 \rightarrow R$ be a $\mathbb{Q}$ -linear map satisfying the double shuffle relation and the regularization theorem (or, equivalently, the shuffle (or harmonic) relation and the integral-series identity). In addition, assume that Z satisfies the duality relation. Then Z satisfies the Kawashima relation.*

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