

ON L^p BOUNDEDNESS OF A CLASS OF PSEUDODIFFERENTIAL OPERATORS

MICHIHIRO NAGASE (長瀬 道弘)

ABSTRACT. Let $p(x, \xi)$ be a symbol in Hörmander class $S_{1,\delta}^0$. Then it is known that the pseudodifferential operator $p(X, D_x)$ is $L^p(\mathbb{R}^n)$ bounded. In the present paper we give a class of pseudodifferential operators and study the $L^p(\mathbb{R}^n)$ boundedness of the operators. The class of operators is closely related to the Schrödinger operators with magnetic potentials.

Keywords: pseudodifferential operators, BMO, interpolation

1. INTRODUCTION

Let $S_{\rho,\delta}^m$ be the set of Hörmander class symbols, that is,

$$S_{\rho,\delta}^m = \{p(x, \xi) : |p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \text{ for any } \alpha \text{ and } \beta\}$$

Here we use that for any multiintegers $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$

$$p_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_x^\beta p(x, \xi)$$

and

$$\begin{aligned} \partial_\xi^\alpha &= \left(\frac{\partial}{\partial \xi} \right)^\alpha = \left(\frac{\partial}{\partial \xi_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_n} \right)^{\alpha_n} \\ D_x^\beta &= \left(\frac{\partial}{i \partial x} \right)^\beta = \left(\frac{\partial}{i \partial x_1} \right)^{\beta_1} \cdots \left(\frac{\partial}{i \partial x_n} \right)^{\beta_n} \end{aligned}$$

We define the pseudodifferential operator of symbol $p(x, \xi)$ by

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\xi} p(x, \xi) \hat{u}(\xi) d\xi$$

where the integration is taken in $\mathbb{R}^n$ and $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is,

$$\hat{u}(\xi) = \int e^{-ix\xi} u(x) dx$$

We denote that the set of pseudodifferential operators with symbol of class $S_{\rho,\delta}^m$ by the same notation as the symbol class.

We say that a linear operator $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is L^p bounded if there is a constant C such that

$$\|Tu\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)} \text{ for any } u \in \overline{S}$$

We denote the set of all L^p bounded operators by $\mathcal{L}(L^p(\mathbb{R}^n))$. The following theorem is known as Calderón-Vaillancourt theorem([1]).

Theorem 1. *Let $0 \leq \delta \leq \rho \leq 1, \delta < 1$. Then we have*

$$S_{\rho,\delta}^0 \subset \mathcal{L}(L^2(\mathbb{R}^n))$$

For the general L^p boundedness, we can see the following theorem([2],[4]).

Theorem 2. *Let $0 \leq \delta \leq \rho \leq 1, (\delta < 1)$ and $1 < p < \infty$. Then we have*

$$S_{\rho,\delta}^m \subset \mathcal{L}(L^p)$$

if and only if $m \leq -n(1-\rho) \left| \frac{1}{2} - \frac{1}{p} \right|$

We want to generalize these results to a class of pseudodifferential operators which is useful to the study of Schrödinger operators with magnetic potentials.

2. PRELIMINARY RESULTS

Let $a(x) = (a_1(x), \dots, a_n(x))$ be an $\mathbb{R}^n$ valued function such that $\partial_x^\alpha a_j(x)$ are bounded for any multiinteger $\alpha \neq 0$. Then we define a smooth function $\lambda(x, \xi)$ by

$$\lambda(x, \xi) = \sqrt{|\xi - a(x)|^2 + 1}$$

Then it is not difficult that the function $\lambda(x, \xi)$ satisfies

- (1) $\lambda(x, \xi) \geq 1$
- (2) $|\partial_\xi^\alpha \partial_x^\beta \lambda(x, \xi)| \leq C_{\alpha,\beta} \lambda(x, \xi)^{1-|\alpha|}$

By using the function $\lambda(x, \xi)$ we define a class $S_{\rho,\delta,\lambda}^m$ of symbols by

$$S_{\rho,\delta,\lambda}^m = \left\{ p(x, \xi) : |p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha,\beta} \lambda(x, \xi)^{m-\rho|\alpha|+\delta|\beta|} \text{ for any } \alpha; \text{ and } \beta \right\}$$

and denote

$$S_{\rho,\delta,\lambda}^\infty = \bigcup_{m \in \mathbb{R}} S_{\rho,\delta,\lambda}^m.$$

This class of symbols is useful for the study of Schrödinger operators with magnetic potentials(see for example [7]). Then it is known that if $0 \leq \delta \leq \rho \leq 1, \delta < 1$ then the class of pseudodifferential operators with symbols $S_{\rho,\delta,\lambda}^\infty$ makes an algebra. Moreover we can show the following L^2 boundedness theorem by using the method in [5].

Theorem 3. *We assume that $0 \leq \delta < \rho \leq 1$. If a symbol $p(x, \xi)$ is in $S_{\rho,\delta,\lambda}^0$, then the pseudodifferential operator $P = p(X, D_x)$ is L^2 bounded. That is, there is a constant C such that*

$$\|p(X, D_x)u\| \leq C\|u\|$$

where $\|\cdot\|$ means the usual $L^2(\mathbb{R}^n)$ norm.

3. L^p BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS

Let $a(x) = (a_1(x), \dots, a_n(x))$ be an $\mathbb{R}^n$ valued function, and let

$$(1) \quad \lambda(x, \xi) = \sqrt{|\xi - a(x)|^2 + 1}$$

In the following we don't assume that the vector function $a(x)$ is not smooth, we need only the fact that $a(x)$ is $\mathbb{R}^n$ valued and measurable.

In the following we use always C as constant independent of variables. Hence the value of C in inequalities are not the same at each occurrence. First we give simple boundedness lemmas of the pseudodifferential operators.

Lemma 1. *If the support of symbol $p(x, \xi)$ is contained in $\{(x, \xi) : |\xi - a(x)| \leq R\}$ for some positive constant R and $p(x, \xi)$ satisfies*

$$(2) \quad |p^{(\alpha)}(x, \xi)| \leq C_\alpha$$

for any α with $|\alpha| \leq n + 1$. Then the operator $p(X, D_x)$ is written as

$$p(X, D_x)u(x) = \int K(x, x - y)u(y)dy$$

where the kernel $K(x, z)$ satisfies

$$(3) \quad |k(x, z)| \leq \frac{C}{\langle z \rangle^{n+1}}$$

Proof. We can write

$$p(X, D_x)u(x) = \int K(x, x - y)u(y)dy$$

where

$$K(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \xi) d\xi$$

Then for $|\alpha| \leq n + 1$ we have

$$z^\alpha K(x, z) = (i)^{|\alpha|} \frac{1}{(2\pi)^n} \int e^{iz\xi} p^{(\alpha)}(x, \xi) d\xi$$

Hence we have

$$\begin{aligned} |z^\alpha K(x, z)| &\leq \frac{1}{(2\pi)^n} \int |p^{(\alpha)}(x, \xi)| d\xi \\ &\leq \frac{1}{(2\pi)^n} \int_{|\xi - a(x)| \leq R} C_\alpha d\xi \\ &\leq C \end{aligned}$$

where the last constant C is independent of the variable x . Thus we have the kernel estimate (3). $\square$

Because of the estimate (3), we have

$$(4) \quad \int |K(x, z)| dz \leq M$$

Therefore we have

Proposition 1. *Let $p(x, \xi)$ satisfy the same assumption as in Lemma 1, then the pseudodifferential operator $p(X, D_x)$ is L^p bounded for $1 \leq p \leq \infty$ and the bound norm is estimated by M in (4).*

For $2 \leq p \leq \infty$ we have

Lemma 2. *If the support of symbol $p(x, \xi)$ contained in $\{(x, \xi) : |\xi - a(x)| \leq R\}$ for some positive constant R and $p(x, \xi)$ satisfies the inequality (2) for $|\alpha| \leq \kappa = \left[\frac{n}{2}\right] + 1$, then the pseudodifferential operator $p(X, D_x)$ is L^p bounded for $2 \leq p \leq \infty$.*

Proof. We can write

$$p(X, D_x)u(x) = \int K(x, x-y)u(y)dy$$

where

$$K(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \xi) d\xi$$

Then by the Schwarz inequality and the Plancherel formula we have

$$\begin{aligned} \int |K(x, z)| dz &\leq C \left\{ \int \langle z \rangle^{-2\kappa} dz \right\}^{1/2} \left\{ \int \langle z \rangle^{2\kappa} |K(x, z)|^2 dz \right\}^{1/2} \\ &\leq C \sum_{|\alpha| \leq \kappa} \left\{ \int |z^\alpha K(x, z)|^2 dz \right\}^{1/2} \\ &= C \sum_{|\alpha| \leq \kappa} \left\{ \int |p^{(\alpha)}(x, \xi)|^2 d\xi \right\}^{1/2} \\ &= C \end{aligned}$$

Thus we have

$$\|p(X, D_x)u\|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}$$

In a similar way we have

$$\begin{aligned} \|p(X, D_x)u\|^2 &= \int \left| \int K(x, x-y)u(y)dy \right|^2 dx \\ &\leq \int \left\{ \int \langle x-y \rangle^{2\kappa} |K(x, x-y)|^2 \right\} \left\{ \int \langle x-y \rangle^{-2\kappa} |u(x)|^2 dy \right\} dx \\ &= C \int \left\{ \int \langle z \rangle^{2\kappa} |K(x, z)|^2 \right\} \left\{ \int \langle x-y \rangle^{-2\kappa} |u(x)|^2 dy \right\} \\ &\leq C \int \int \langle x-y \rangle^{-2\kappa} |u(x)|^2 dy dx \\ &= C \|u\|^2 \end{aligned}$$

Hence by the Riesz-Thorin interpolation we get the Lemma. $\square$

One of the main results in the present note is the following.

Theorem 4. *Let $a(x)$ be the same as in Lemma 1, and $\lambda(x, \xi)$ be defined by (1). Let $\omega(t)$ be a nonnegative and nondecreasing function on $[0, \infty)$ such that*

$$\int_0^\infty \frac{\omega(t)}{t} dt < \infty$$

We assume that a symbol $p(x, \xi)$ satisfies

$$|p^{(\alpha)}(x, \xi)| \leq C_\alpha \lambda(x, \xi)^{-|\alpha|} \omega(\lambda(x, \xi)^{-1})$$

for any α with $|\alpha| \leq n+1$. Then the pseudodifferential operator $p(X, D_x)$ is L^p bounded for $1 \leq p \leq \infty$.

Proof. By Lemma 1, we may assume that the support of the symbol $p(x, \xi)$ is contained in $\{(x, \xi) : |\xi - a(x)| \geq 2\}$. Now we take a smooth nonnegative function $f(t)$ such that the support of $f(t)$ is contained in the interval $[\frac{1}{2}, 1]$ and

$$\int_0^\infty \frac{f(t)}{t} dt = 1$$

Then since the support of the symbol $p(x, \xi)$ is contained in $\{(x, \xi) : |\xi - a(x)| \geq 2\}$, we have

$$\begin{aligned} p(X, D_x)u(x) &= \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t} dt \int \int e^{i(x-y)\xi} p(x, \xi) f(t|\xi|) u(y) dy d\xi \\ &= \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t^{n+1}} dt \int \int e^{i(\frac{x-y}{t})\xi} e^{i(x-y)a(x)} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) u(y) d\xi dy \\ &= \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t} dt \int e^{itz a(x)} K_t(x, z) u(x - tz) dz \end{aligned}$$

where

$$K_t(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi$$

If we put $\tilde{p}(x, \xi) = p(x, \frac{\xi}{t} + a(x))$, then it is easy to see that

$$|\tilde{p}^{(\alpha)}(x, \xi)| \leq C_\alpha \langle \xi \rangle^{-|\alpha|} \omega(\langle \xi \rangle^{-1})$$

for $|\alpha| \leq n+1$. Since the equality

$$z^\alpha K_t(x, z) = \frac{i^{|\alpha|}}{(2\pi)^n} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int e^{iz\xi} \tilde{p}^{(\alpha')}(x, \frac{\xi}{t}) \partial_\xi^{\alpha - \alpha'} f(|\xi|) d\xi$$

holds for $|\alpha| \leq n+1$, we have

$$\begin{aligned} |z^\alpha K_t(x, z)| &\leq \frac{1}{(2\pi)^n} \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int |\tilde{p}^{(\alpha')}(x, \frac{\xi}{t}) \partial_\xi^{\alpha - \alpha'} f(|\xi|)| d\xi \\ &\leq C \sum_{\alpha' \leq \alpha} \frac{1}{t^{|\alpha'|}} \binom{\alpha}{\alpha'} \int_{1/2 \leq |\xi| \leq 1} \left| \frac{\xi}{t} \right|^{|\alpha'|} \omega\left(\left| \frac{\xi}{t} \right|^{-1}\right) d\xi \\ &\leq C\omega(t) \end{aligned}$$

for $|\alpha| \leq n+1$. Therefore we have

$$(5) \quad |K_t(x, z)| \leq C \langle \xi \rangle^{-n-1} \omega(t)$$

By the inequality (5) and the equality

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{1}{t} dt \int e^{itz a(x)} K_t(x, z) u(x - tz) dz$$

we can see that the operator $p(X, D_x)$ is L^1 bounded and L^∞ bounded. That is inequalities

$$\begin{aligned} \|p(X, D_x)u\|_{L^1(\mathbb{R}^n)} &\leq C\|u\|_{L^1(\mathbb{R}^n)} \\ \|p(X, D_x)u\|_{L^\infty(\mathbb{R}^n)} &\leq C\|u\|_{L^\infty(\mathbb{R}^n)} \end{aligned}$$

holds. So by the Riesz-Thorin interpolation theorem we have the L^p boundedness for $1 \leq p \leq \infty$. $\square$

When $2 \leq p$, we can show a little more general result than Theorem 4, by using the Plancherel Theorem.

Theorem 5. *Let $a(x)$ and $\lambda(x, \xi)$ be the same as in Theorem 4. Let $\omega(t)$ be a nonnegative and nondecreasing function on $[0, \infty)$ such that*

$$\int_0^\infty \frac{\omega(t)}{t} dt < \infty$$

We assume that a symbol $p(x, \xi)$ satisfies

$$|p^{(\alpha)}(x, \xi)| \leq C_\alpha \lambda(x, \xi)^{-|\alpha|} \omega(\lambda(x, \xi)^{-1})$$

for any α with $|\alpha| \leq \kappa = [\frac{n}{2}] + 1$. Then the pseudodifferential operator $p(X, D_x)$ is L^p bounded for $2 \leq p \leq \infty$.

Proof. We first show the L^∞ boundedness. We write the operator $p(X, D_x)$, as in the proof of Theorem 4, by

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \int e^{itz a(x)} K_t(x, z) u(x - tz) dz$$

where

$$K_t(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi$$

Then writing $\kappa = [\frac{n}{2}] + 1$, we have

$$\begin{aligned} \int |K_t(x, z)| dz &= \int \langle z \rangle^{-\kappa} \langle z \rangle^\kappa |K_t(x, z)| dz \\ &\leq \left\{ \int \langle z \rangle^{-2\kappa} dz \right\}^{1/2} \left\{ \int \langle z \rangle^{2\kappa} |K_t(x, z)|^2 dz \right\}^{1/2} \\ &\leq C \sum_{|\alpha| \leq \kappa} \left\{ \int |z^\alpha K_t(x, z)|^2 dz \right\}^{1/2} \end{aligned}$$

Using the Plancherel equality, we have

$$\begin{aligned} \int |z^\alpha K_t(x, z)|^2 dz &= \int |\partial_\xi^\alpha \{ \tilde{p}(x, \frac{\xi}{t}) f(|\xi|) \}|^2 d\xi \\ &\leq C_\alpha \omega(t) \end{aligned}$$

Hence we have

$$|p(X, D_x)u(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}$$

In order to show L^2 boundedness of the operator $p(X, D_x)$, we use the same representation

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \int e^{itza(x)} K_t(x, z) u(x - tz) dz$$

where

$$K_t(x, z) = \frac{1}{(2\pi)^n} \int e^{iz\xi} p(x, \frac{\xi}{t} + a(x)) f(|\xi|) d\xi$$

From this representation we have

$$\|p(X, D_x)\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} \int_0^1 \frac{dt}{t} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) dz \right\|_{L^2(\mathbb{R}^n)}$$

Then we have

$$\begin{aligned} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) dz \right\|^2 &= \int \left| \int e^{itza(x)} K_t(x, z) u(x - tz) dz \right|^2 dx \\ &\leq \int \left| \int |K_t(x, z) u(x - tz)| dz \right|^2 dx \end{aligned}$$

By using the Schwarz inequality we have

$$\left| \int |K_t(x, z) u(x - tz)| dz \right|^2 \leq \int \langle z \rangle^{2\kappa} |K_t(x, z)|^2 dz \int \langle z \rangle^{-2\kappa} |u(x - tz)|^2 dz$$

As above we can see

$$\begin{aligned} \int \langle z \rangle^{2\kappa} |K_t(x, z)|^2 dz &\leq \sum_{|\alpha| \leq \kappa} \int |z^\alpha K_t(x, z)|^2 dz \\ &= \sum_{|\alpha| \leq \kappa} \int |\partial_\xi \left\{ p(x, \frac{\xi}{t} + a(x)) f(|\xi|) \right\}|^2 d\xi \\ &\leq C\omega(t)^2 \end{aligned}$$

Therefore we get

$$\begin{aligned} \left\| \int e^{itza(\cdot)} K_t(\cdot, z) u(\cdot - tz) dz \right\|^2 &\leq C \int \int \langle z \rangle^{-2\kappa} |u(x - tz)|^2 dz dx \\ &\leq C\omega(t)^2 \|u\|^2 \end{aligned}$$

Thus from the assumption of $\omega(t)$ we have the L^2 estimate

$$\|p(X, D_x)u\| \leq C\|u\|$$

Again by the Riesz-Thorin interpolation theorem we have the L^p boundedness for $2 \leq p \leq \infty$. $\square$

Remark 1. Theorems in this section we don't always assume that the vector function $a(x) = (a_1(x), \dots, a_n(x))$ satisfies the estimate

$$|\partial_x a_j(x)| \leq C$$

In several theorems we can prove the theorem under only the measurability of $a(x)$.

Remark 2. If the vector function $a(x)$ is bounded, then the symbol class $S_{\rho,\delta,\lambda}^m$ coincide with the usual Hörmander class $S_{\rho,\delta}^m$. Hence using the similar method of usual class, the L^p boundedness in Theorem 4 can be shown (see for example, [6]). Even if $a(x)$ is not bounded, we have

Proposition 2. Let $a(x)$ and $\lambda(x, \xi)$ be the same as in Theorem 4. For any smooth function φ with compact support, we have

$$\|\varphi(x)p(X, D_x)u\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}$$

4. CONJECTURE

As we see in the previous sections we can expect that the following L^p boundedness theorem.

Conjecture 1. If the vector function $a(x) = (a_1(x), \dots, a_n(x))$ satisfies

$$|\partial^\alpha a_j(x)| \leq C_\alpha$$

for any $\alpha \neq 0$. Then for $1 < p < \infty$, the operator $p(X, D)$ in $S_{1,\delta,\lambda}^0$ is L^p bounded. That is,

$$S_{1,\delta,\lambda}^0 \subset \mathcal{L}(L^p(\mathbb{R}^n))$$

holds.

As we stated in section 2 it is known that if the vector function $a(x)$ satisfies the estimates in the above conjecture, the operators in $S_{1,\delta,\lambda}^0$ with $(\delta < 1)$ are L^2 bounded. So if we can show the weak type $(1, 1)$ estimates or boundedness from $L^\infty(\mathbb{R}^n)$ to BMO , then we can get the above conjecture, that is, L^p boundedness for $1 < p < \infty$ by using the interpolation theorems (see for example [8], [3]). The fundamental conjecture is

Conjecture 2. If the vector function $a(x) = (a_1(x), \dots, a_n(x))$ satisfies

$$|\partial^\alpha a_j(x)| \leq C_\alpha$$

for any $\alpha \neq 0$. Then the operator $p(X, D)$ in $S_{1,\delta,\lambda}^0$ is bounded from $L^\infty(\mathbb{R}^n)$ to BMO , that is, there is a constant C such that

$$\|p(X, D_x)u\|_{BMO} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}$$

REFERENCES

- [1] V.P. Calderón and R. Vaillancourt, A class of bounded pseudo-differential operators Proc. Nat. Acad. Sci. U.S.A., **69** (1972) 1185–1187
- [2] C. Fefferman, L^p -bounds for pseudo-differential operators Israel J. Math., **14** (1972) 413–417
- [3] C. Fefferman and E. Stein, H^p -spaces of several variables Acta Math., **129** (1972) 137–193
- [4] L. Hörmander, Pseudo-differential operators and hypoelliptic equations, Proc., Symposium on Singular Integrals Amer. Math. Soc., **10** (1967) 138–183
- [5] H. Kumano-go, Pseudo-differential operators, MIT Press, Cambridge, Mass. and London, England, 1982
- [6] M. Nagase, On some classes of L^p -bounded pseudo-differential operators Osaka J. Math., **23** (1986) 425–440
- [7] M. Nagase and T. Umeda, On the essential selfadjointness of quantum Hamiltonians of relativistic particles in magnetic fields Sci. Rep., Col. Gen. Educ. Osaka Univ., **36** (1987)

L^p BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS

- [8] E.M.Stein, *Singular integrals and differentiability properties of functions* Princeton Univ. Press, Princeton, N.J., 1970

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560, JAPAN (大阪大学理学研究科数学教室)

E-mail address: nagase@math.wani.osaka-u.ac.jp