

Second-harmonic generation of two-dimensional elastic wave propagation in an infinite layered structure with nonlinear spring-type interfaces

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ABSTRACT

The two-dimensional elastic wave propagation in an infinite layered structure with nonlinear interlayer interfaces is analyzed theoretically to investigate the second-harmonic generation due to interfacial nonlinearity. The structure consists of identical isotropic linear elastic layers that are bonded to each other by spring-type interfaces possessing identical linear normal and shear stiffnesses but different quadratic nonlinearity parameters. Explicit analytical expressions are derived for the second-harmonic amplitudes when a single monochromatic Bloch mode propagates in the structure in arbitrary directions by applying the transfer-matrix approach and the Bloch theorem to the governing equations linearized by a perturbation method. The second-harmonic generation by a single nonlinear interface and by multiple consecutive nonlinear interfaces are shown to be profoundly influenced by the band structure of the layered structure, the fundamental Bloch wave mode, and its propagation direction. In particular, the second harmonics generated at multiple consecutive interfaces are found to grow cumulatively with the propagation distance when the phase matching occurs between the Bloch modes at the fundamental and double frequencies.

Keywords: Layered structure; Second harmonic generation; Transfer matrix method; Perturbation analysis; Spring-type interface; Bloch theorem

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1. Introduction

Nonlinear acoustic wave propagation in layered structures was analyzed theoretically by several researchers to explore the amplitude-dependent dispersion relation [1,2], the occurrence of localized solutions or solitary waves, [3–11] and the higher-harmonic generation [12–14]. Recently, these findings have been utilized in practical applications, such as the so-called acoustic diode proposed by Liang et al. [15,16], which rectifies acoustic wave propagation by combining a superlattice with a nonlinear medium. These foregoing studies [1–16] dealt with the nonlinear acoustic phenomena caused by material nonlinearity.

In the field of nondestructive testing, the nonlinear acoustics at the contact between two rough surfaces of solid media have attracted much attention as a sensitive measure to assess interfacial imperfections such as closed cracks and kissing bonds. In particular, the higher-harmonic generation at these imperfect interfaces has been investigated theoretically and experimentally [17–31]. These studies have demonstrated that the second-harmonic generation behavior at a contacting interface or thin interphase layer between two solids can reasonably be described by modeling it as a nonlinear spring-type interface [19,20,22,27–31], where the stresses are continuous while discontinuities are allowed in the displacements with the resulting displacement jumps related nonlinearly to the stresses. This modeling has also been utilized to analyze the second-harmonic generation in double interface structures by Yan et al. [32,33] and Junca and Lombard [34].

For the more general case of multiple interface problem, the authors [35–37] have analyzed the second-harmonic generation in multilayered structures with nonlinear spring-type interlayer interfaces by combining a perturbation analysis and the transfer-matrix method [38,39]. As a result, the second-harmonic generation in the presence of multiple imperfect interfaces was found to be influenced by the frequency in a more complicated manner in comparison with the single interface problem, reflecting the band structure of periodic structures. The existing studies on layered structures [32–37] are, however, limited to the simplest case of the one-dimensional

longitudinal wave propagation in a direction perpendicular to the interfaces. When the interlayer imperfections such as closed delaminations in composite laminates [40–42] are characterized using nonlinear acoustic techniques, there are some cases where the normal incidence to the interfaces is unavailable due, for example, to the surrounding environments. The second-harmonic generation at nonlinear spring-type interfaces subjected to oblique incidence of longitudinal or shear waves has been analyzed in the two-dimensional framework [20,22,43], but these foregoing works dealt with a single interface problem. Therefore, understanding of the corresponding issue for the multiple interfaces is important and expected to provide some insights into the interface evaluation of multilayered structures using oblique incidence.

The aim of this paper is to theoretically elucidate the second-harmonic generation behavior of elastic waves propagating in arbitrary directions in a multilayered structure with nonlinear spring-type interlayer interfaces. To acquire fundamental understanding of the influence of frequency and propagation direction on the second-harmonic generation, the present study focuses on the two-dimensional Bloch wave propagation in an infinitely layered structure consisting of identical isotropic linear elastic layers and nonlinear spring-type interfaces with identical linear interfacial normal and shear stiffnesses. The problem is formulated in Section 2. The current analysis is put forward by extending the procedure of the one-dimensional wave propagation case in Ref. [36]. Namely, the nonlinear governing equations are first linearized by a perturbation method, and the problems for the fundamental and its second-harmonic components are then solved using the transfer-matrix method [38,39] and the Bloch theorem to obtain explicit analytical expressions for the second-harmonic amplitudes in the layered structure in Section 3. Numerical examples are presented and discussed in Section 4 for the second-harmonic generation by a single nonlinear interface and by multiple consecutive nonlinear interfaces.

2. Formulation

The present study considers the two-dimensional plane-strain wave propagation in an infinitely layered structure consisting of identical isotropic linear elastic layers (thickness h , density ρ , longitudinal wave velocity c_L , and shear wave velocity c_T) and spring-type interlayer interfaces as shown in Fig. 1. The x_1 – x_2 Cartesian coordinate system is set in such a manner that the layering direction coincides with the x_1 axis. The displacements, denoted by $u_j(x_1, x_2, t)$ where $j = 1, 2$ and t is the time, satisfy the following two-dimensional Navier equations with neglected body forces in each layer.

$$\frac{\partial^2 u_j}{\partial t^2} = \sum_{p=1}^2 \left[c_T^2 \frac{\partial^2 u_j}{\partial x_p \partial x_p} + (c_L^2 - c_T^2) \frac{\partial^2 u_p}{\partial x_p \partial x_j} \right], \quad X_m < x_1 < X_{m+1}, \quad (1)$$

where $X_m = mh$ ($m \in \mathbb{Z}$) represent the x_1 coordinate of the interfaces. The normal and shear stresses in the layers, denoted by $\sigma_{11}(x_1, x_2, t)$ and $\sigma_{12}(x_1, x_2, t)$, are given by

$$\sigma_{11} = \rho c_L^2 \frac{\partial u_1}{\partial x_1} + \rho (c_L^2 - 2c_T^2) \frac{\partial u_2}{\partial x_2}, \quad \sigma_{12} = \rho c_T^2 \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right). \quad (2)$$

At each interface, the spring-type boundary condition is adopted (i.e., stresses are continuous while discontinuity is allowed in the displacements across the interface). Furthermore, the interfaces are assumed to possess weak quadratic nonlinearity and to exhibit an orientation-independent response in their plane (i.e., normal and shear stresses are assumed to be even and odd functions of the tangential displacement jump, respectively). In this situation, the boundary conditions at $x_1 = X_m$ are given by [22,44]

$$\sigma_{11}(X_{m+}, x_2, t) = \sigma_{11}(X_{m-}, x_2, t) = K_N \{ y_{m1}(x_2, t) - \beta_{m1} [y_{m1}(x_2, t)]^2 - \beta_{m2} [y_{m2}(x_2, t)]^2 \}, \quad (3a)$$

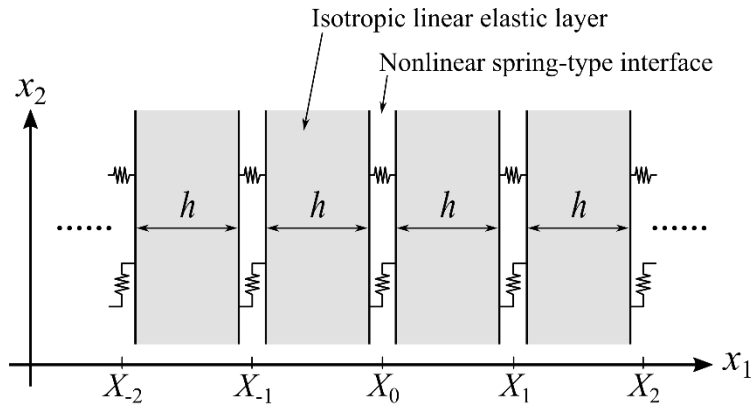
$$\sigma_{12}(X_{m+}, x_2, t) = \sigma_{12}(X_{m-}, x_2, t) = K_T [1 - \beta_{m3} y_{m1}(x_2, t)] y_{m2}(x_2, t), \quad (3b)$$

where the subscripts “ $m+$ ” and “ $m-$ ” denote the limit of a field variable when x_1 approaches X_m from $x_1 > X_m$ and $x_1 < X_m$, respectively. In Eq. (3), K_N and K_T denote the linear interfacial normal and shear stiffnesses, respectively, which are assumed to be the same for all interfaces, and β_{mk} ($k = 1, 2, 3$) are parameters representing the nonlinearity of the interface at $x_1 = X_m$. Furthermore,

$$y_{mj}(x_2, t) = u_j(X_{m+}, x_2, t) - u_j(X_{m-}, x_2, t) \quad (4)$$

denote the jumps of displacements at $x_1 = X_m$.

In this paper, the dissipative nature of the layers and the interfaces is not incorporated. In other words, the aforementioned material parameters are assumed to be $\{h, \rho, c_L, c_T, K_N, K_T\} \in \mathbb{R}_{>0}$ and $\beta_{mk} \in \mathbb{R}$, where $\mathbb{R}_{>0} = \{x \in \mathbb{R} | x > 0\}$.



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3. Second-harmonic generation due to a monochromatic Bloch wave propagation

In this section, analytical expressions for the second-harmonic amplitudes in the infinite layered structure when it is subjected to a monochromatic wave propagation in arbitrary directions are derived by extending the procedure presented for the one-dimensional wave propagation problem in Ref. [36] to the two-dimensional one.

3.1. Linearization of governing equation

Equations (1)–(4) are first linearized using a perturbation approach. It is assumed that $\max_{m, 1 \leq k \leq 3} |\beta_{mk}| a_0$, where $a_0 \in \mathbb{R}_{>0}$ is a characteristic displacement amplitude, is sufficiently small and that the solution of Eqs. (1)–(4) is given as

$$u_j = u_j^{(1)} + u_j^{(2)}, \quad \left[u_1^{(1)} \right]^2 + \left[u_2^{(1)} \right]^2 \gg \left[u_1^{(2)} \right]^2 + \left[u_2^{(2)} \right]^2, \quad (5)$$

where $u_j^{(1)}$ is the fundamental solution and $u_j^{(2)}$ is the first-order correction. Then, $u_j^{(1)}$ and $u_j^{(2)}$ satisfy Eq. (1) with u_j replaced by $u_j^{(1)}$ and $u_j^{(2)}$, respectively, and the boundary conditions for them are given by

$$\sigma_{11}^{(1)}(X_{m+}, x_2, t) = \sigma_{11}^{(1)}(X_{m-}, x_2, t) = K_N y_{m1}^{(1)}(x_2, t), \quad (6a)$$

$$\sigma_{12}^{(1)}(X_{m+}, x_2, t) = \sigma_{12}^{(1)}(X_{m-}, x_2, t) = K_T y_{m2}^{(1)}(x_2, t), \quad (6b)$$

for $u_j^{(1)}$, and

$$\sigma_{11}^{(2)}(X_{m+}, x_2, t) = \sigma_{11}^{(2)}(X_{m-}, x_2, t) = K_N \left\{ y_{m1}^{(2)}(x_2, t) - \beta_{m1} \left[y_{m1}^{(1)}(x_2, t) \right]^2 - \beta_{m2} \left[y_{m2}^{(1)}(x_2, t) \right]^2 \right\}, \quad (7a)$$

$$\sigma_{12}^{(2)}(X_{m+}, x_2, t) = \sigma_{12}^{(2)}(X_{m-}, x_2, t) = K_T \left[y_{m2}^{(2)}(x_2, t) - \beta_{m3} y_{m1}^{(1)}(x_2, t) y_{m2}^{(1)}(x_2, t) \right], \quad (7b)$$

for $u_j^{(2)}$. In Eqs. (6) and (7), $\sigma_{11}^{(r)}$, $\sigma_{12}^{(r)}$, and $y_{mr}^{(r)}$ ($r = 1, 2$) are given by Eqs. (2) and (4) with u_j replaced by $u_j^{(r)}$.

3.2. Transfer matrix of unit cell

The problem for $u_j^{(1)}$ (Navier equations and boundary conditions in Eq. (6)) governs the linear wave propagation in a perfectly periodic structure of spatial period h whose unit cell consists of one layer and one linearized spring-type interface. A solution to this problem when a monochromatic wave with angular frequency $\omega_0 \in \mathbb{R}_{>0}$ propagates in the structure is considered here by introducing the complex-value expression as

$$u_j^{(1)}(x_1, x_2, t) = \text{Re} \left\{ U_j^{(1)}(x_1) \exp[i(k_2 x_2 - \omega_0 t)] \right\}, \quad (8)$$

where i denotes the imaginary unit, and $k_2 \in \mathbb{R}$ is the x_2 component of wavevector, which is conserved throughout the layered structure due to Snell's law. Furthermore, the following displacement amplitude vector is introduced to employ the transfer-matrix formalism [38,39]:

$$\mathbf{A}_m^{(1)} \equiv \left(A_{mL-}^{(1)}, A_{mT-}^{(1)}, A_{mL+}^{(1)}, A_{mT+}^{(1)} \right)^T \quad \text{for } X_m < x_1 < X_{m+1}, \quad (9)$$

where the superscript “T” denotes the transpose. In Eq. (9), $A_{mL\pm}^{(1)}$ and $A_{mT\pm}^{(1)}$ denote the complex-valued time-harmonic displacement amplitudes of partial longitudinal and shear waves, respectively, that propagate in the corresponding layer in the $\pm x_1$ directions. Namely, $U_j^{(1)}$ are assumed to be expressed as a superposition of these partial waves as

$$\begin{aligned} \begin{pmatrix} U_1^{(1)}(x_1) \\ U_2^{(1)}(x_1) \end{pmatrix} = & A_{mL-}^{(1)} \begin{pmatrix} -p_{L1} \\ p_{L2} \end{pmatrix} \exp \left[\frac{i\omega_0 p_{L1} (X_{m+1} - x_1)}{c_L} \right] \\ & + A_{mT-}^{(1)} \begin{pmatrix} p_{T2} \\ p_{T1} \end{pmatrix} \exp \left[\frac{i\omega_0 p_{T1} (X_{m+1} - x_1)}{c_T} \right] \\ & + A_{mL+}^{(1)} \begin{pmatrix} p_{L1} \\ p_{L2} \end{pmatrix} \exp \left[\frac{i\omega_0 p_{L1} (x_1 - X_m)}{c_L} \right] \\ & + A_{mT+}^{(1)} \begin{pmatrix} p_{T2} \\ -p_{T1} \end{pmatrix} \exp \left[\frac{i\omega_0 p_{T1} (x_1 - X_m)}{c_T} \right], \end{aligned} \quad (10)$$

where p_{Lj} and p_{Tj} are the components of unit vectors representing the propagation direction of partial longitudinal and shear waves, respectively, which are given from $p_{L2}/c_L = p_{T2}/c_T =$

$$k_2/\omega_0, \quad p_{L1} = \sqrt{1 - p_{L2}^2}, \quad \text{and} \quad p_{T1} = \sqrt{1 - p_{T2}^2}.$$

The amplitude vectors in the adjacent layers are then connected as [45]

$$\mathbf{A}_m^{(1)} = \mathbf{B}(\Omega) \mathbf{A}_{m-1}^{(1)}, \quad \mathbf{B}(\Omega) = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} \quad (11)$$

where $\Omega \equiv \omega_0 h/c_T \in \mathbb{R}_{>0}$ represents the normalized angular frequency, and $\mathbf{B}(\Omega)$ is the 4×4 transfer matrix of the unit cell, whose elements can be expressed explicitly by

$$\begin{aligned} B_{11} &= [1 + i\Omega L_1(\Lambda_N, \Lambda_T)] E_L(-\Omega), & B_{12} &= i\Omega L_2(\Lambda_N, \Lambda_T) E_L(-\Omega), \\ B_{13} &= i\Omega L_1(\Lambda_N, -\Lambda_T), & B_{14} &= i\Omega L_2(-\Lambda_N, \Lambda_T) E_L(-\Omega) E_T(\Omega), \\ B_{21} &= i\Omega L_3(\Lambda_N, \Lambda_T) E_T(-\Omega), & B_{22} &= [1 + i\Omega L_4(\Lambda_N, \Lambda_T)] E_T(-\Omega), \\ B_{23} &= i\Omega L_3(\Lambda_N, -\Lambda_T) E_L(\Omega) E_T(-\Omega), & B_{24} &= i\Omega L_4(-\Lambda_N, \Lambda_T), \end{aligned} \quad (12)$$

$$\begin{aligned}
B_{31} &= i\Omega L_1(-\Lambda_N, \Lambda_T), \quad B_{32} = i\Omega L_2(-\Lambda_N, \Lambda_T), \\
B_{33} &= [1 - i\Omega L_1(\Lambda_N, \Lambda_T)]E_L(\Omega), \quad B_{34} = i\Omega L_2(\Lambda_N, \Lambda_T)E_T(\Omega), \\
B_{41} &= i\Omega L_3(\Lambda_N, -\Lambda_T), \quad B_{42} = i\Omega L_4(\Lambda_N, -\Lambda_T), \\
B_{43} &= i\Omega L_3(\Lambda_N, \Lambda_T)E_L(\Omega), \quad B_{44} = [1 - i\Omega L_4(\Lambda_N, \Lambda_T)]E_T(\Omega),
\end{aligned}$$

where

$$L_1(\Lambda_N, \Lambda_T) = -\frac{\zeta}{p_{L1}} \left[(1 - 2p_{T2}^2)^2 \Lambda_N + (2p_{L1}p_{T2})^2 \Lambda_T \right], \quad (13a)$$

$$L_2(\Lambda_N, \Lambda_T) = \frac{2\zeta p_{T2}}{p_{L1}} (1 - 2p_{T2}^2) (\zeta p_{T1} \Lambda_N - p_{L1} \Lambda_T), \quad (13b)$$

$$L_3(\Lambda_N, \Lambda_T) = L_2(\Lambda_N, \Lambda_T) \frac{p_{L1}}{\zeta p_{T1}}, \quad (13c)$$

$$L_4(\Lambda_N, \Lambda_T) = -\frac{1}{p_{T1}} \left[(2\zeta p_{T1}p_{T2})^2 \Lambda_N + (1 - 2p_{T2}^2)^2 \Lambda_T \right], \quad (13d)$$

$$E_L(\Omega) = \exp(i\Omega \zeta p_{L1}), \quad E_T(\Omega) = \exp(i\Omega p_{T1}). \quad (13e)$$

In Eqs. (12) and (13), the non-dimensional parameter $\zeta \equiv c_T/c_L \in \mathbb{R}_{>0}$ represents the ratio between the shear and longitudinal wave velocities, and

$$\Lambda_N \equiv \frac{\rho c_L^2}{2K_N h} \in \mathbb{R}_{>0}, \quad \Lambda_T \equiv \frac{\rho c_T^2}{2K_T h} \in \mathbb{R}_{>0}, \quad (14)$$

represent the relative linear normal and shear compliances of the interfaces.

3.3. Application of Bloch theorem

Application of the Bloch theorem to the present periodic structure gives

$$\mathbf{A}_m^{(1)} = \exp[iK(\Omega)h] \mathbf{A}_{m-1}^{(1)}, \quad (15)$$

where $K(\Omega) \in \mathbb{C}$ denotes the x_1 component of the Bloch wavevector (referred to as the Bloch wavenumber hereafter). Note that the x_2 component of the Bloch wavevector is assumed to coincide with k_2 mentioned above in Eq. (8).

A comparison between Eqs. (11) and (15) indicates that $\lambda(\Omega) \equiv \exp[iK(\Omega)h]$ is an eigenvalue of the transfer matrix $\mathbf{B}(\Omega)$. Using the property that the reciprocal of the eigenvalue

of $\mathbf{B}(\Omega)$ is also an eigenvalue of $\mathbf{B}(\Omega)$ [46], the characteristic equation of $\mathbf{B}(\Omega)$ becomes $\lambda^4 - I_1(\Omega)\lambda^3 + I_2(\Omega)\lambda^2 - I_1(\Omega)\lambda + 1 = 0$, which then reduces to a simpler form as

$$\cos[2K(\Omega)h] - I_1(\Omega)\cos[K(\Omega)h] + \frac{I_2(\Omega)}{2} = 0, \quad (16)$$

where $I_1(\Omega)$ and $I_2(\Omega)$ denote the first and second invariants of $\mathbf{B}(\Omega)$, respectively. Equation (16) has four solutions for $K(\Omega)$ in the range of $-\pi/h < \text{Re}[K(\Omega)] \leq \pi/h$, corresponding to two pairs of Bloch waves propagating or decaying in the opposite x_1 directions. As Eq. (16) reduces to a quadratic equation for $\eta(\Omega) \equiv \cos[K(\Omega)h]$ with the aid of the cosine double-angle identity, $K(\Omega)$ can be obtained as

$$K(\Omega) = \frac{2q\pi + \arg[\lambda(\Omega)]}{h} - i\frac{\ln|\lambda(\Omega)|}{h}, \quad q \in \mathbb{Z}, \quad (17)$$

where

$$\lambda(\Omega) = \eta(\Omega) \pm i\sqrt{1 - \eta(\Omega)^2}, \quad (18)$$

and

$$\eta(\Omega) = \frac{I_1(\Omega)}{4} \pm \sqrt{\left[\frac{I_1(\Omega)}{4}\right]^2 - \frac{I_2(\Omega) - 2}{4}}. \quad (19)$$

In Eq. (19), the invariants of the transfer matrix are given explicitly from Eqs. (12) and (13) as

$$\frac{I_1(\Omega)}{4} = \frac{1}{2} \left\{ \cos(\Omega\zeta p_{L1}) - \Omega^2\zeta^2 \left[(1 - 2p_{T2}^2)^2 \Lambda_N + (2p_{L1}p_{T2})^2 \Lambda_T \right] \text{sinc}(\Omega\zeta p_{L1}) \right. \quad (20a)$$

$$\left. + \cos(\Omega p_{T1}) - \Omega^2 \left[(2\zeta p_{T1}p_{T2})^2 \Lambda_N + (1 - 2p_{T2}^2)^2 \Lambda_T \right] \text{sinc}(\Omega p_{T1}) \right\},$$

$$\begin{aligned} \frac{I_2(\Omega) - 2}{4} &= 8\Omega^2\zeta^2 p_{T2}^2 (1 - 2p_{T2}^2)^2 \Lambda_N \Lambda_T \\ &+ \left[1 - 8\Omega^2\zeta^2 p_{T2}^2 (1 - 2p_{T2}^2)^2 \Lambda_N \Lambda_T \right] \cos(\Omega\zeta p_{L1}) \cos(\Omega p_{T1}) \\ &+ \Omega^4\zeta^2 \Lambda_N \Lambda_T \left[(1 - 2p_{T2}^2)^4 + (4\zeta p_{L1}p_{T1}p_{T2}^2)^2 \right] \text{sinc}(\Omega\zeta p_{L1}) \text{sinc}(\Omega p_{T1}) \end{aligned} \quad (20b)$$

$$- \Omega^2\zeta^2 \left[(1 - 2p_{T2}^2)^2 \Lambda_N + (2p_{L1}p_{T2})^2 \Lambda_T \right] \text{sinc}(\Omega\zeta p_{L1}) \cos(\Omega p_{T1})$$

$$- \Omega^2 \left[(2\zeta p_{T1}p_{T2})^2 \Lambda_N + (1 - 2p_{T2}^2)^2 \Lambda_T \right] \cos(\Omega\zeta p_{L1}) \text{sinc}(\Omega p_{T1}).$$

From Eq. (17), $K(\Omega)$ becomes real when $|\lambda(\Omega)| = 1$ (i.e., when $|\text{Re}[\eta(\Omega)]| \leq 1$ and $\text{Im}[\eta(\Omega)] = 0$ from Eq. (18)), which corresponds to the propagating Bloch mode. In any other cases (i.e., either when $|\text{Re}[\eta(\Omega)]| > 1$ and $\text{Im}[\eta(\Omega)] = 0$ or when $\text{Im}[\eta(\Omega)] \neq 0$), $K(\Omega)$ possesses non-zero imaginary parts [46], indicating that the corresponding Bloch wave decays in the x_1 direction as it propagates in the periodic structure. In particular, it is found from Eq. (20) that $I_1(\Omega)$ and $I_2(\Omega)$ are always real because the non-dimensional parameters $\Omega, \zeta, \Lambda_N, \Lambda_T, p_{L2}$, and p_{T2} are real while p_{L1} and p_{T1} are either real or pure imaginary. Thus, $\text{Im}[\eta(\Omega)] \neq 0$ is equivalent to $I_1(\Omega)^2 - 4I_2(\Omega) + 8 < 0$ from Eq. (19), resulting that all four Bloch waves become decaying modes.

In what follows, the wavenumbers of Bloch waves propagating (or decaying) in the positive and negative x_1 directions are denoted by $K^{b+}(\Omega)$ and $K^{b-}(\Omega)$ ($b = 1, 2$), and the corresponding eigenvectors by $\mathbf{v}^{b+}(\Omega)$ and $\mathbf{v}^{b-}(\Omega)$, respectively. They, hence, satisfy the following relation.

$$\mathbf{B}(\Omega)\mathbf{v}^{b\pm}(\Omega) = \exp[\pm iK^{b\pm}(\Omega)h]\mathbf{v}^{b\pm}(\Omega), \quad (21)$$

where the double sign applies in the same order, and $\mathbf{v}^{b\pm}$ are assumed to be normalized in such a manner that $\|\mathbf{v}^{b\pm}\| \equiv \sqrt{(\mathbf{v}^{b\pm})^\dagger \mathbf{v}^{b\pm}} = 1$, where the superscript “ $\dagger$ ” denotes the Hermitian transpose.

Note that in the case of the one-dimensional Bloch wave propagation in the layering direction, the Bloch modes propagating in the opposite directions can explicitly be distinguished from their dispersion relation [36]. In the present case of arbitrary propagation directions, however, such distinction is not straightforward only from Eqs. (17)–(20). Hence, $K^{b+}(\Omega)$ and $K^{b-}(\Omega)$ are determined in this paper as follows. When the Bloch waves are decaying modes, those decaying in the positive and negative x_1 directions are simply identified as $\text{Im}[K^{b+}(\Omega)] > 0$ and $\text{Im}[K^{b-}(\Omega)] < 0$, respectively. On the other hand, when the Bloch waves are propagating modes (i.e., when $\text{Im}[K^{b\pm}(\Omega)] = 0$), their propagation directions are determined by the sign of the time-

averaged energy flux density in the x_1 direction [47] given by $-\text{Re} \left[\sum_{j=1}^2 \overline{V_j^{(1)}}(x_1) \Sigma_{1j}^{(1)}(x_1) \right] / 2$, where “ $\bar{\cdot}$ ” denotes the complex conjugate and $V_j^{(1)}$ and $\Sigma_{1j}^{(1)}$ are the components of time-harmonic particle velocity and stress of the corresponding Bloch wave, respectively. Specifically, the Bloch modes propagating in the positive and negative x_1 directions are identified as $\text{Re} \left[(\mathbf{v}^{b+})^\dagger \mathbf{W} \mathbf{v}^{b+} \right] > 0$ and $\text{Re} \left[(\mathbf{v}^{b-})^\dagger \mathbf{W} \mathbf{v}^{b-} \right] < 0$, respectively, where $\mathbf{W}$ is a 4×4 matrix given by [48]

$$\mathbf{W}(\Omega) = \begin{bmatrix} \overline{\mathbf{H}(\Omega)}(\mathbf{P}^-)^\dagger \mathbf{J}^- \mathbf{H}(\Omega) & \overline{\mathbf{H}(\Omega)}(\mathbf{P}^-)^\dagger \mathbf{J}^+ \\ (\mathbf{P}^+)^\dagger \mathbf{J}^- \mathbf{H}(\Omega) & (\mathbf{P}^+)^\dagger \mathbf{J}^+ \end{bmatrix}. \quad (22)$$

In Eq. (22), $\mathbf{P}^\pm$, $\mathbf{J}^\pm$, and $\mathbf{H}$ are the 2×2 matrices given by

$$\mathbf{P}^\pm = \begin{bmatrix} \pm p_{L1} & p_{T2} \\ p_{L2} & \mp p_{T1} \end{bmatrix}, \mathbf{J}^\pm = \begin{bmatrix} 1 - 2p_{T2}^2 & \pm 2\zeta p_{T1} p_{T2} \\ \pm 2\zeta p_{L1} p_{T2} & -1 + 2p_{T2}^2 \end{bmatrix}, \mathbf{H}(\Omega) = \begin{bmatrix} E_L(\Omega) & 0 \\ 0 & E_T(\Omega) \end{bmatrix}, \quad (23)$$

where the double sign applies in the same order.

3.4. Second-harmonic amplitudes in the layered structure

Now, the solution to the problem for $u_j^{(1)}$ is assumed to be a single Bloch mode propagating (or decaying) in the positive x_1 direction. Furthermore, without loss of generality, the magnitude of displacement amplitude vector in the layer occupying $X_0 < x_1 < X_1$ is assumed to be $\|\mathbf{A}_0^{(1)}\| = a_0$, where a_0 is the characteristic displacement amplitude used in Section 3.1, i.e., $\mathbf{A}_0^{(1)} = a_0 \mathbf{v}^{b+}(\Omega)$.

The problem for $u_j^{(2)}$ (Navier equations and boundary conditions in Eq. (7)) governs the linear wave propagation in the same periodic structure as in the case for $u_j^{(1)}$, except for the presence of driving terms that consist of the direct current and double frequency components on the right-hand side of Eq. (7). Note here that since the present interest is in analyzing the second-harmonic generation, the direct current components of driving terms are omitted below. A solution to this problem is sought in the frequency domain again by introducing the complex-value

expression as

$$u_j^{(2)}(x_1, x_2, t) = \text{Re} \left\{ U_j^{(2)}(x_1) \exp[2i(k_2 x_2 - \omega_0 t)] \right\}. \quad (24)$$

Let us first consider an auxiliary problem that the interface at $x_1 = X_n$ ($n \in \mathbb{Z}$) solely possesses the nonlinearity. In this situation, the displacement amplitudes of second harmonics in the layers on both sides of the nonlinear interface can be connected as

$$\mathbf{A}_n^{(2)} = \mathbf{B}(2\Omega) \mathbf{A}_{n-1}^{(2)} - \beta_{n1} a_0^2 \exp[2inK^{b+}(\Omega)h] \mathbf{b}_n^{b+}(\Omega), \quad (25)$$

where $\mathbf{A}_n^{(2)}$ is given by Eq. (9) with the subscript “ m ” and superscript “(1)” replaced by “ n ” and “(2)”, respectively, with $A_{nL\pm}^{(2)}$ and $A_{nT\pm}^{(2)}$ representing the complex-valued second-harmonic amplitudes of the partial longitudinal and shear waves propagating in the $\pm x_1$ directions, respectively. In Eq. (25), $\mathbf{b}_n^{b+}(\Omega)$ is given by

$$\mathbf{b}_n^{b+}(\Omega) = \begin{pmatrix} \left\{ -\frac{1-2p_{T2}^2}{p_{L1}} \{ [g_1^{b+}(\Omega)]^2 + \gamma_n [g_2^{b+}(\Omega)]^2 \} + 2\zeta p_{T2} \chi_n g_1^{b+}(\Omega) g_2^{b+}(\Omega) \right\} E_L(-2\Omega) \\ \left\{ 2p_{T2} \{ [g_1^{b+}(\Omega)]^2 + \gamma_n [g_2^{b+}(\Omega)]^2 \} + \frac{1-2p_{T2}^2}{p_{T1}} \chi_n g_1^{b+}(\Omega) g_2^{b+}(\Omega) \right\} E_T(-2\Omega) \\ \frac{1-2p_{T2}^2}{p_{L1}} \{ [g_1^{b+}(\Omega)]^2 + \gamma_n [g_2^{b+}(\Omega)]^2 \} + 2\zeta p_{T2} \chi_n g_1^{b+}(\Omega) g_2^{b+}(\Omega) \\ 2p_{T2} \{ [g_1^{b+}(\Omega)]^2 + \gamma_n [g_2^{b+}(\Omega)]^2 \} - \frac{1-2p_{T2}^2}{p_{T1}} \chi_n g_1^{b+}(\Omega) g_2^{b+}(\Omega) \end{pmatrix}, \quad (26)$$

where $\gamma_n \equiv \beta_{n2}/\beta_{n1}$, $\chi_n \equiv \beta_{n3}/\beta_{n1}$ ($\beta_{n1} \neq 0$ is assumed), and

$$\begin{pmatrix} g_1^{b+}(\Omega) \\ g_2^{b+}(\Omega) \end{pmatrix} = \Omega \times \begin{bmatrix} (1-2p_{T2}^2) \Lambda_N \zeta E_L(\Omega) & -2\zeta^2 p_{T1} p_{T2} \Lambda_N E_T(\Omega) & (1-2p_{T2}^2) \Lambda_N \zeta & 2\zeta^2 p_{T1} p_{T2} \Lambda_N \\ -2p_{L1} p_{T2} \Lambda_T E_L(\Omega) & -(1-2p_{T2}^2) \Lambda_T E_T(\Omega) & 2p_{L1} p_{T2} \Lambda_T & -(1-2p_{T2}^2) \Lambda_T \end{bmatrix} \mathbf{v}^{b+}(\Omega). \quad (27)$$

From a physical reasoning that the second harmonics generated at the nonlinear interface should propagate away (or decay with the distance) from it, the second-harmonic wave fields for $x_1 > X_n$ and $x_1 < X_n$ are assumed to be a superposition of Bloch waves propagating (or decaying)

in the positive and negative x_1 directions, respectively. Therefore, $\mathbf{A}_n^{(2)}$ and $\mathbf{A}_{n-1}^{(2)}$ are expressed as

$$\mathbf{A}_n^{(2)} = -\beta_{n1} a_0^2 \exp[2inK^{b+}(\Omega)h] [F_{n,b+}^{1+} \mathbf{v}^{1+}(2\Omega) + F_{n,b+}^{2+} \mathbf{v}^{2+}(2\Omega)], \quad (28a)$$

$$\mathbf{A}_{n-1}^{(2)} = -\beta_{n1} a_0^2 \exp[2inK^{b+}(\Omega)h] [F_{n,b+}^{1-} \mathbf{v}^{1-}(2\Omega) + F_{n,b+}^{2-} \mathbf{v}^{2-}(2\Omega)], \quad (28b)$$

where $F_{n,b+}^{1\pm}$ and $F_{n,b+}^{2\pm}$ are complex constants to be determined by solving the system of equations given by

$$[\mathbf{v}^{1+}(2\Omega) \quad \mathbf{v}^{2+}(2\Omega) \quad \mathbf{v}^{1-}(2\Omega) \quad \mathbf{v}^{2-}(2\Omega)] \begin{pmatrix} F_{n,b+}^{1+} \\ F_{n,b+}^{2+} \\ -\exp[-iK^{1+}(2\Omega)h] F_{n,b+}^{1-} \\ -\exp[-iK^{2+}(2\Omega)h] F_{n,b+}^{2-} \end{pmatrix} = \mathbf{b}_n^{b+}(\Omega). \quad (29)$$

Since $a_0 \exp[inK^{b+}(\Omega)h]$ represents the amplitude of fundamental Bloch wave impinging on the nonlinear interface and $\|\mathbf{v}^{1\pm}(2\Omega)\| = \|\mathbf{v}^{2\pm}(2\Omega)\| = 1$ as mentioned above in Eq. (21), the factors $|F_{n,b+}^{1\pm}|$ and $|F_{n,b+}^{2\pm}|$ in Eq. (28) represent the modal amplitudes of the corresponding second-harmonic Bloch waves generated at the nonlinear interface per unit nonlinearity parameter and unit squared incident amplitude. These factors are thus referred to as generation efficiency of the second-harmonic Bloch waves hereafter.

Once $\mathbf{A}_n^{(2)}$ and $\mathbf{A}_{n-1}^{(2)}$ are determined, the second-harmonic amplitudes in the layer occupying $X_m < x_1 < X_{m+1}$ due to the fundamental Bloch wave propagation with wavenumber $K^{b+}(\Omega)$ are given from Eqs. (11) and (21) with Ω replaced by 2Ω and Eq. (28) as

$$\begin{aligned} \mathbf{A}_m^{(2)} &= \beta_{n1} a_0^2 \exp[2inK^{b+}(\Omega)h] \\ &\times \begin{cases} -\sum_{s=1}^2 \exp[i(m-n)K^{s+}(2\Omega)h] \mathbf{E}_{n,b+}^{s+}(2\Omega), & m \geq n, \\ \sum_{s=1}^2 \exp[-i(m-n)K^{s+}(2\Omega)h] \mathbf{E}_{n,b+}^{s-}(2\Omega), & m \leq n-1, \end{cases} \end{aligned} \quad (30)$$

where

$$\mathbf{E}_{n,b+}^{s+}(2\Omega) = F_{n,b+}^{s+} \mathbf{v}^{s+}(2\Omega), \quad (31a)$$

$$\mathbf{E}_{n,b+}^{s-}(2\Omega) = -\exp[-iK^{s+}(2\Omega)h] F_{n,b+}^{s-} \mathbf{v}^{s-}(2\Omega). \quad (31b)$$

Since the problem for $u_j^{(2)}$ is linear, the solution to the original problem involving multiple nonlinear interfaces at $x_1 = X_m$ ($m \in \mathbb{Z}$) is formally written by superposing the above auxiliary solution in Eq. (30) for different n as

$$\begin{aligned} \mathbf{A}_m^{(2)} = & -a_0^2 \sum_{s=1}^2 \left\{ \exp[imK^{s+}(2\Omega)h] \sum_{n=-\infty}^m \beta_{n1} \exp[in\Delta K_{b+}^{s+}(\Omega)h] \mathbf{E}_{n,b+}^{s+}(2\Omega) \right. \\ & \left. - \exp[-imK^{s+}(2\Omega)h] \sum_{n=m+1}^{\infty} \beta_{n1} \exp[4in\tilde{K}_{b+}^{s+}(\Omega)h] \mathbf{E}_{n,b+}^{s-}(2\Omega) \right\}, \end{aligned} \quad (32)$$

where

$$\Delta K_{b+}^{s+}(\Omega) \equiv 2K^{b+}(\Omega) - K^{s+}(2\Omega), \quad \tilde{K}_{b+}^{s+}(\Omega) \equiv \frac{1}{4}[2K^{b+}(\Omega) + K^{s+}(2\Omega)]. \quad (33)$$

4. Numerical examples and discussions

4.1. Dispersion relation of Bloch waves

Equation (32) indicates that the second-harmonic amplitudes in the infinite layered structure with nonlinear interfaces are influenced by the propagation characteristics of Bloch waves. These are first examined here to ease the interpretation of the second-harmonic generation behavior to be discussed in Sections 4.2 and 4.3.

The x_1 component of the Bloch wavevector can be calculated from Eqs. (17)–(20) with given x_2 component k_2 . Note that k_2 is assumed to be real as stated in Section 3.2, indicating that the x_2 component of the unit vector representing the propagation direction of partial shear waves, given by $p_{T2} = k_2 h / \Omega$ from Snell's law, is also real.

In terms of p_{T2} , the domain of k_2 can be divided into three cases according to the propagation characteristics of partial waves in the layers: first, when $|p_{T2}| \leq \zeta$, both partial longitudinal and shear waves are the propagating modes with real p_{L1} and p_{T1} . Their

propagation angles relative to the x_1 axis are given by $\theta_L \equiv \sin^{-1} p_{L2} = \sin^{-1}(p_{T2}/\zeta)$ and $\theta_T \equiv \sin^{-1} p_{T2}$ for longitudinal and shear waves, respectively. Second, when $\zeta < |p_{T2}| \leq 1$, the partial shear waves are the only propagating mode since the longitudinal waves become evanescent with $p_{L1} = i\sqrt{(p_{T2}/\zeta)^2 - 1}$. Third, when $|p_{T2}| > 1$, not only p_{L1} but also $p_{T1} = i\sqrt{p_{T2}^2 - 1}$ becomes pure imaginary, resulting that all partial waves become evanescent.

The Bloch wave is generally given as a linear combination of partial waves propagating or decaying in the constituting layers. For the periodic structure considered in the present study, however, each Bloch wave is dominated either by the partial longitudinal or shear wave [46], and the Bloch wave propagation characteristics can reasonably be classified into the aforementioned three cases: both Bloch modes, only the shear-dominated Bloch mode, and no Bloch modes can be the propagating modes in the first, second, and third cases, respectively, except in the vicinity of their boundaries $|p_{T2}| = \zeta$ and 1. The boundaries where the number of possible propagating Bloch modes changes deviate slightly from $|p_{T2}| = \zeta$ and 1 depending mainly on the interfacial compliances (they strictly coincide with $|p_{T2}| = \zeta$ and 1 when the interfaces are perfectly bonded with $\mathcal{A}_N = \mathcal{A}_T = 0$).

In the following numerical examples, the ratio of shear to longitudinal wave velocities is fixed as $\zeta = 0.5$, for which in terms of θ_T the aforementioned first and second cases correspond to $|\theta_T| \leq 30^\circ$ and $30^\circ < |\theta_T| \leq 90^\circ$, respectively. Note that the third case is not considered below, because the present interest is in examining the frequency dependence of the second-harmonic generation when the fundamental Bloch wave propagates in the infinite layered structure.

The dispersion relation of two Bloch modes propagating or decaying in the positive x_1 direction when $\mathcal{A}_N = 0.1$ and $\mathcal{A}_T = 0.1$ is shown in Figs. 2(a) and 2(b) for the special case of $\theta_T = 0^\circ$, corresponding to the Bloch waves propagating perpendicularly to the interfaces. Likewise, the results for $\theta_T = 20^\circ$ and 40° as representative values for $|\theta_T| \leq 30^\circ$ and $30^\circ < |\theta_T| \leq 90^\circ$ are shown in Figs. 3 and 4, respectively. In Figs. 2–4, the longitudinal- and shear-dominated Bloch

wavenumbers are denoted by K^{1+} and K^{2+} , respectively. The frequency ranges in which $\text{Im}[K^{s+}(\Omega)] = 0$ ($s = 1, 2$) are the pass bands, and those in which $\text{Im}[K^{s+}(\Omega)] > 0$ with finite bandwidth are the stop bands.

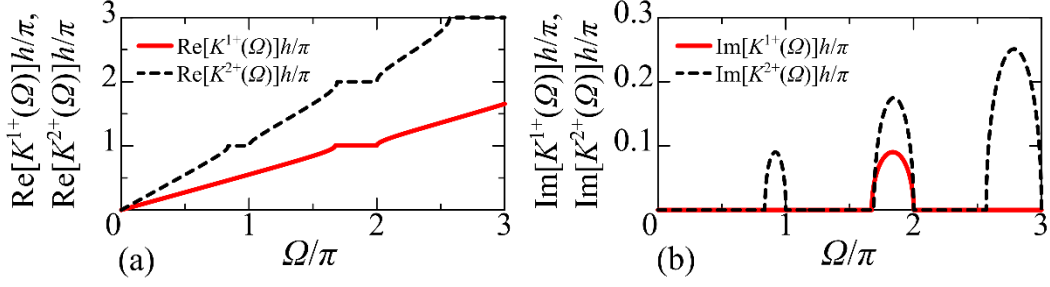


Fig. 2 Dispersion relation of Bloch waves when $A_N = 0.1$, $A_T = 0.1$, and $\theta_T = 0^\circ$. (a) Real part and (b) imaginary part of Bloch wavenumbers.

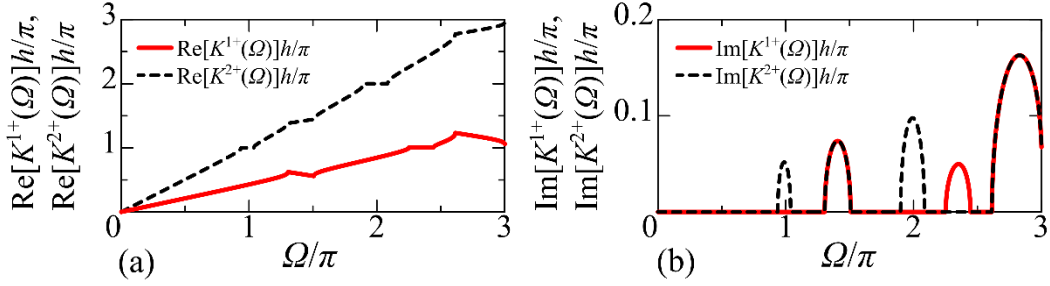


Fig. 3 Dispersion relation of Bloch waves when $A_N = 0.1$, $A_T = 0.1$, and $\theta_T = 20^\circ$. (a) Real part and (b) imaginary part of Bloch wavenumbers.

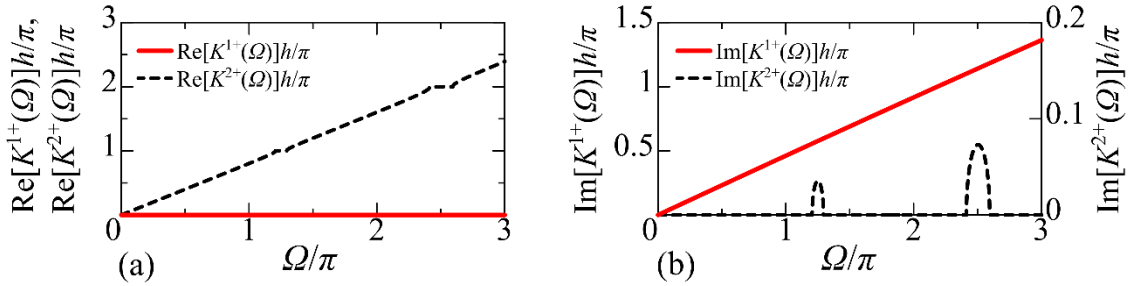


Fig. 4 Dispersion relation of Bloch waves when $A_N = 0.1$, $A_T = 0.1$, and $\theta_T = 40^\circ$. (a) Real part and (b) imaginary part of Bloch wavenumbers.

When $\theta_T = 0^\circ$ in Fig. 2, the propagation angle of longitudinal wave is also $\theta_L = 0^\circ$. Namely, all partial waves propagate perpendicularly to the interfaces. In this case, two solutions to Eq. (19), denoted by $\eta_1(\Omega)$ and $\eta_2(\Omega)$, are given explicitly by

$$\eta_1(\Omega) = \cos(\Omega\zeta) - \Omega\zeta\Lambda_N \sin(\Omega\zeta) \quad \text{for } K^{1+}(\Omega), \quad (34a)$$

$$\eta_2(\Omega) = \cos\Omega - \Omega\Lambda_T \sin\Omega \quad \text{for } K^{2+}(\Omega). \quad (34b)$$

The Bloch modes governed by Eqs. (34a) and (34b) consist only of the longitudinal and shear waves, respectively, due to the absence of mode conversion at the interfaces. Equation (34a) coincides with the one derived in Ref. [36] for the one-dimensional longitudinal wave propagation in the layering direction of infinite layered structures. As mentioned earlier in Section 3.3, the periodic structure considered in this paper generally has two types of stop bands: one associated with $\text{Im}[\eta(\Omega)] \neq 0$ and the other with $|\text{Re}[\eta(\Omega)]| > 1$ and $\text{Im}[\eta(\Omega)] = 0$. In the present case of $\theta_T = 0^\circ$, however, $\eta_1(\Omega)$ and $\eta_2(\Omega)$ in Eq. (34) are both real, resulting that only the stop bands associated with $|\text{Re}[\eta(\Omega)]| > 1$ and $\text{Im}[\eta(\Omega)] = 0$ appear at around $\Omega/\pi = 0.9, 1.8$, and 2.7 in Fig. 2(b). In these frequency bands, the real parts of the Bloch wavenumbers in Fig. 2(a) become $\text{Re}[K^{b+}(\Omega)]h/\pi = l$ ($l \in \mathbb{N}^+$) [46].

When $\theta_T = 20^\circ$ in Fig. 3, the propagation angle of partial longitudinal wave is $\theta_L = 43.2^\circ$, and the Bloch modes now consist of both longitudinal and shear waves due to the occurrence of mode conversion at the interfaces. In Figs. 3(a) and 3(b), not only the stop bands associated with $|\text{Re}[\eta(\Omega)]| > 1$ and $\text{Im}[\eta(\Omega)] = 0$ centered at around $\Omega/\pi = 1, 2$, and 2.4 but also the ones with $\text{Im}[\eta(\Omega)] \neq 0$ are formed at around $\Omega/\pi = 1.4$ and 2.8 , wherein $\text{Re}[K^{1+}(\Omega) + K^{2+}(\Omega)]h/\pi = l$ and $\text{Im}[K^{1+}(\Omega)] = \text{Im}[K^{2+}(\Omega)]$ [46].

The stop bands shown above in Figs. 2 and 3 are caused by the constructive interference of the partial waves scattered from the interfaces [46]. On the other hand, when $\theta_T = 40^\circ$ in Fig. 4, the partial longitudinal waves become evanescent. Accordingly, the longitudinal-dominated Bloch wave becomes a decaying mode ($\text{Im}[K^{1+}(\Omega)] > 0$) without wave interference in the entire frequency range. The other Bloch mode is dominated by the propagating shear waves, and only

the stop bands associated with $|\operatorname{Re}[\eta(\Omega)]| > 1$ and $\operatorname{Im}[\eta(\Omega)] = 0$ appear at around $\Omega/\pi = 1.2$ and 2.5 in Fig. 4(b).

4.2. Second-harmonic generation by a single nonlinear interface

On the basis of the Bloch wave propagation characteristics shown in Section 4.1, the frequency dependence of the second-harmonic generation due to a single nonlinear interface is examined here.

As a numerical example, the second-harmonic amplitudes in the layer occupying $X_{10} < x_1 < X_{11}$ ($m = 10$) are calculated from Eq. (30) when $\mathcal{A}_N = 0.1$, $\mathcal{A}_T = 0.1$, and the interface at $x_1 = X_7$ ($n = 7$) solely possesses the nonlinearity of $\beta_{n1} = \beta \in \mathbb{R}_{>0}$, $\gamma_n = 1$, and $\chi_n = 1$. All other interfaces are assumed to be linear, i.e., $\beta_{k1} = \gamma_k = \chi_k = 0$ for $k \neq n$. The results for $\theta_T = 0^\circ$, 20° , and 40° when the fundamental Bloch wavenumber is $K^{b+}(\Omega)$ ($b = 1, 2$) in Figs. 2–4 are plotted as functions of the normalized frequency in Figs. 5–7, respectively. In these figures, the factors governing the second-harmonic amplitudes as well as the resulting second-harmonic displacements at $x_1 = X_{10+}$ that are calculated from

$$\mathbf{U}^{(2)}(X_{m+}) = [\mathbf{P}^- \mathbf{H}(2\Omega) \quad \mathbf{P}^+] \mathbf{A}_m^{(2)}, \quad \mathbf{U}^{(2)}(x_1) \equiv \left(U_1^{(2)}(x_1), U_2^{(2)}(x_1) \right)^T, \quad (35)$$

are also shown. It is noted here that the second-harmonic generation when $\theta_T = 0^\circ$ with the fundamental Bloch wavenumber $K^{1+}(\Omega)$ has been analyzed by Biwa and Ishii [36]. In this paper, these results are also shown to examine the influence of θ_T on the second-harmonic generation behavior.

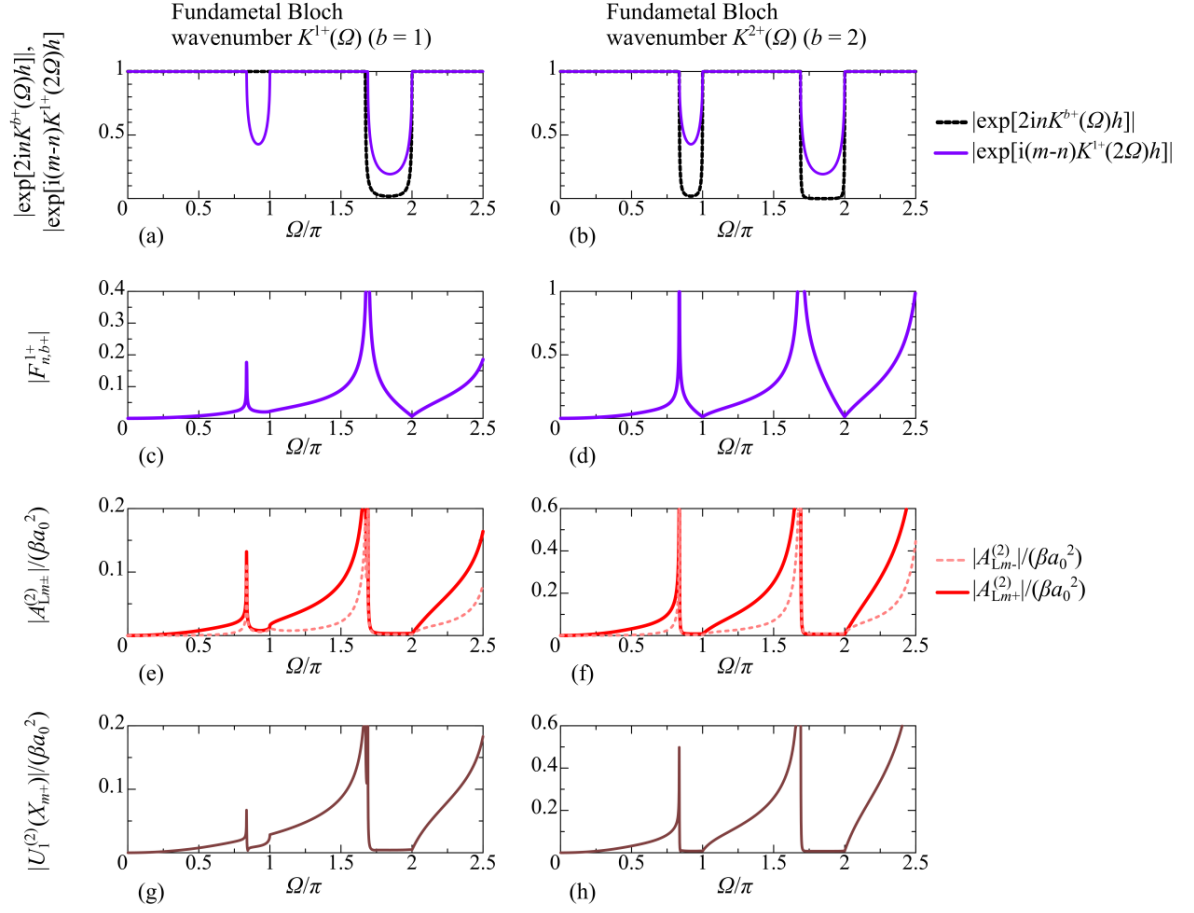


Fig. 5 Frequency dependence of the second-harmonic generation by a single nonlinear interface when $n = 7$, $m = 10$, $A_N = 0.1$, $A_T = 0.1$, and $\theta_T = 0^\circ$. (a) and (b) Propagation characteristics of the fundamental and the second-harmonic Bloch waves. (c) and (d) Generation efficiency of the second-harmonic Bloch wave at the nonlinear interface. (e) and (f) Second-harmonic amplitudes in the layer occupying $X_m < x_1 < X_{m+1}$. (g) and (h) Second-harmonic displacement at $x_1 = X_{m+}$. The left and right columns correspond to the cases when the fundamental Bloch wavenumber is $K^{1+}(\Omega)$ and $K^{2+}(\Omega)$, respectively.

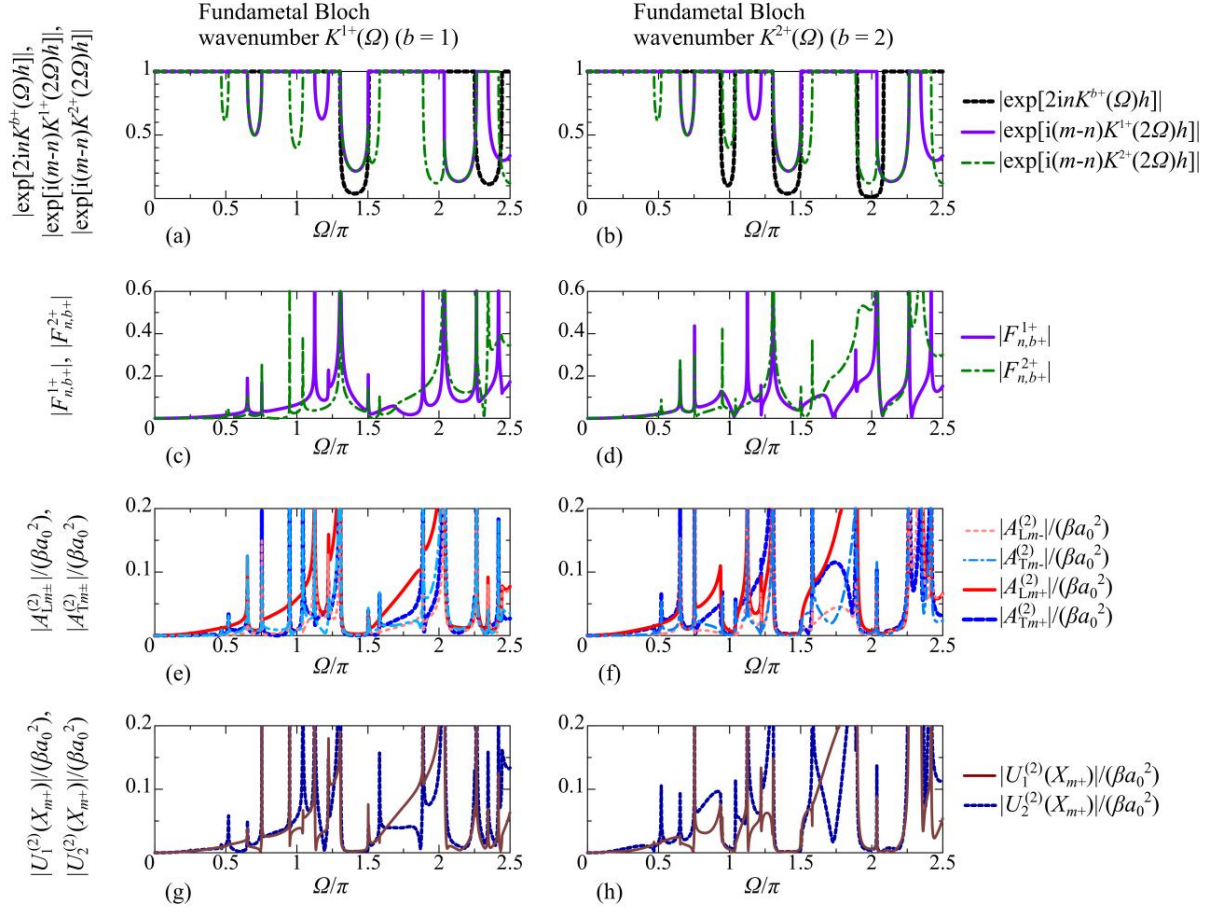


Fig. 6 Frequency dependence of the second-harmonic generation by a single nonlinear interface when $n = 7$, $m = 10$, $A_N = 0.1$, $A_T = 0.1$, and $\theta_T = 20^\circ$. (a) and (b) Propagation characteristics of the fundamental and the second-harmonic Bloch waves. (c) and (d) Generation efficiency of the second-harmonic Bloch waves at the nonlinear interface. (e) and (f) Second-harmonic amplitudes in the layer occupying $X_m < x_1 < X_{m+1}$. (g) and (h) Second-harmonic displacements at $x_1 = X_{m+1}$. The left and right columns correspond to the cases when the fundamental Bloch wavenumber is $K^{1+}(\Omega)$ and $K^{2+}(\Omega)$, respectively.

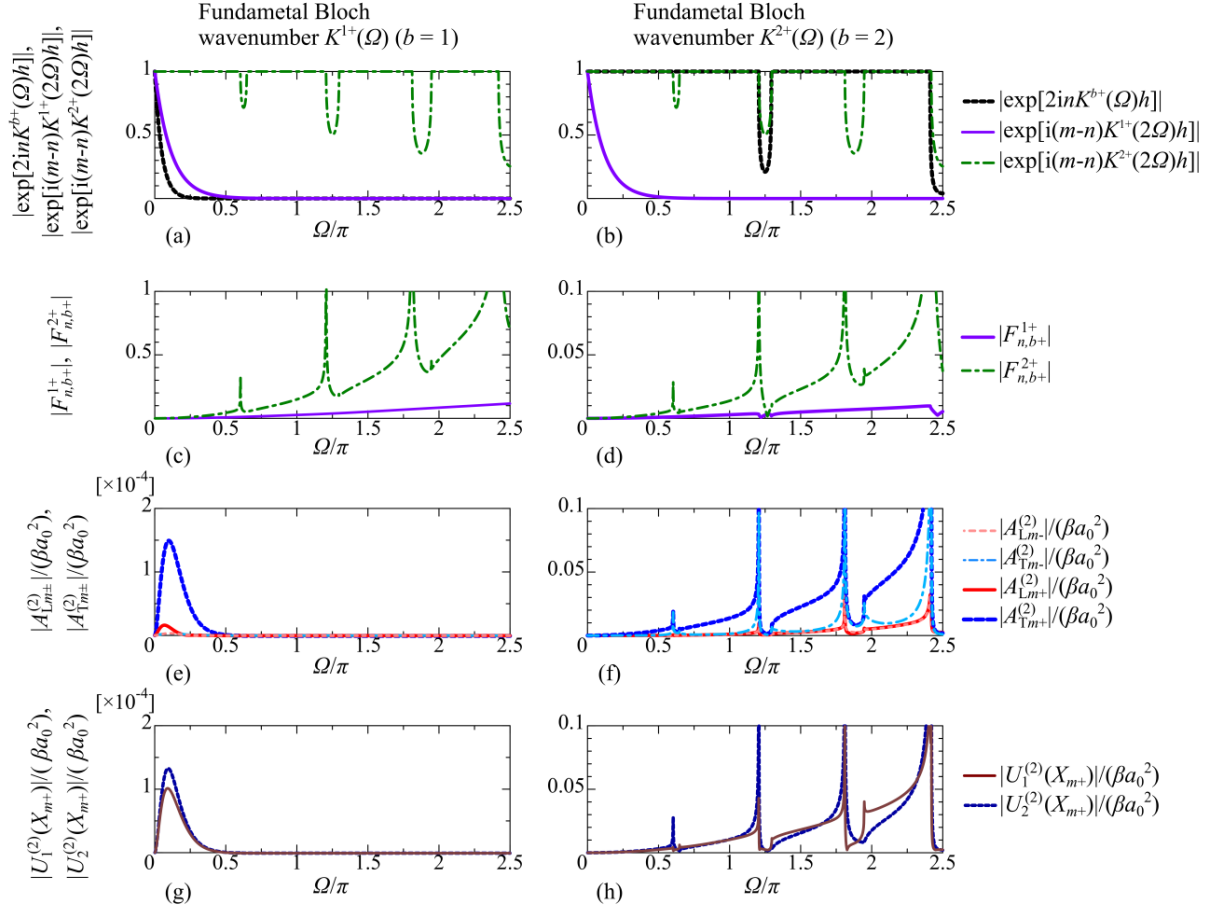


Fig. 7 Frequency dependence of the second-harmonic generation by a single nonlinear interface when $n = 7$, $m = 10$, $A_N = 0.1$, $A_T = 0.1$, and $\theta_T = 40^\circ$. (a) and (b) Propagation characteristics of the fundamental and the second-harmonic Bloch waves. (c) and (d) Generation efficiency of the second-harmonic Bloch waves at the nonlinear interface. (e) and (f) Second-harmonic amplitudes in the layer occupying $X_m < x_1 < X_{m+1}$. (g) and (h) Second-harmonic displacements at $x_1 = X_{m+}$. The left and right columns correspond to the cases when the fundamental Bloch wavenumber is $K^{1+}(\Omega)$ and $K^{2+}(\Omega)$, respectively.

When $\theta_T = 0^\circ$ and the fundamental Bloch wave has the wavenumber $K^{1+}(\Omega)$ that is governed by the longitudinal waves, the second-harmonic generation at the nonlinear interface depends only on the nonlinearity parameter β_{n1} and is influenced neither by γ_n nor χ_n since $g_2^{1+}(\Omega) = 0$ in Eq. (26). Likewise, when $\theta_T = 0^\circ$ and the fundamental Bloch wave with wavenumber $K^{2+}(\Omega)$ consists only of the shear waves, the second-harmonic generation depends only on $\beta_{n1}\gamma_n (= \beta_{n2})$ since $g_1^{2+}(\Omega) = 0$ in Eq. (26). In both cases, the factors $F_{n,b+}^{2\pm}$ vanish from Eqs. (26) and (29). Hence, the second harmonics in Figs. 5(e)–5(h) consist only of the longitudinal

waves with the associated displacement components only in the x_1 direction, and reflect the band structure of the corresponding fundamental and second-harmonic Bloch waves as discussed in Ref. [36]. It should be noted here that the coincidence of stop band locations between the fundamental shear and the second-harmonic longitudinal Bloch waves in Fig. 5(b) is due to a particular choice of $\zeta = 0.5$ and $\Lambda_N = \Lambda_T$ (see Eqs. (34a) and (34b)).

By contrast, when $\theta_T = 20^\circ$ (Fig. 6), not only $F_{n,b+}^{1+}$ but also $F_{n,b+}^{2+}$ generally become nonzero as shown in Figs. 6(c) and 6(d), indicating that the second-harmonic wave fields consist of both longitudinal- and shear-dominated Bloch modes with their propagation characteristics represented by $|\exp[i(m-n)K^{1+}(2\Omega)h]|$ and $|\exp[i(m-n)K^{2+}(2\Omega)h]|$, respectively, in Figs. 6(a) and 6(b). This leads to the appearance of more stop bands in Figs. 6(a) and 6(b) compared to Figs. 5 (a) and 5(b) for $\theta_T = 0^\circ$. The resulting second harmonics in Figs. 6(e)–6(h), hence, exhibit more complicated frequency dependence than those in Figs. 5(e)–5(h), whereas their generation mechanism can be explained in the same manner as in the case of $\theta_T = 0^\circ$.

When $\theta_T = 40^\circ$ (Fig. 7) and the fundamental Bloch wave has the wavenumber $K^{1+}(\Omega)$, which is dominated by the evanescent longitudinal waves, the factor $|\exp[2inK^{1+}(\Omega)h]|$ in Fig. 7(a) rapidly decreases to a vanishingly low level as the frequency increases from $\Omega = 0$. Hence, the second-harmonic generation at the nonlinear interface is allowed only in the low frequency range of about $\Omega/\pi < 0.5$. In this range, however, the generation efficiency of second harmonics at the nonlinear interface is very small in Fig. 7(c). As a result, the observed second harmonics become very small as can be seen in the vertical scales in Figs. 7(e) and 7(g). On the other hand, when the fundamental Bloch wave is the other shear-dominated mode with $K^{2+}(\Omega)$, either or both of generation and propagation efficiency of the second-harmonic longitudinal-dominated Bloch mode with $K^{1+}(2\Omega)$ (i.e., $|F_{n,1+}^{1+}|$ in Fig. 7(d) and $|\exp[i(m-n)K^{1+}(2\Omega)h]|$ in Fig. 7(b)) are vanishingly small in the pass bands of fundamental wave. Therefore, almost only the shear-dominated Bloch mode with $K^{2+}(2\Omega)$ is involved in the second harmonics in Figs. 7(f) and 7(h), resulting that their frequency dependence becomes as simple as that for $\theta_T = 0^\circ$ in Fig. 5 again.

It should be noted that the generation efficiency of the second harmonics at the nonlinear interface in Figs. 5(c), 5(d), 6(c), 6(d), 7(c), and 7(d) and the resulting second-harmonic components in Figs. 5(e)–5(h), 6(e)–6(h), and 7(e)–7(h) have sharp peaks when the double frequency 2Ω is at the boundaries between pass and stop bands. This trend was also presented by Biwa and Ishii [36] regarding the longitudinal Bloch wave propagation in the layering direction. This can be attributed to the fact that at such frequencies, the transfer matrix $\mathbf{B}(2\Omega)$ has repeated eigenvalues and the 4×4 matrix consisting of its eigenvectors on the left-hand side of Eq. (29) becomes rank deficient. The present perturbation analysis, hence, loses its validity at these frequencies where the second-harmonic wave fields exhibit diverging behavior.

4.3. Second-harmonic generation by multiple consecutive nonlinear interfaces

Consider next the second-harmonic generation due to multiple consecutive interfaces having identical nonlinearity. The interfaces are assumed to possess the following nonlinearity parameters.

$$\beta_{k1} = \beta, \quad \gamma_k = 1, \quad \chi_k = 1 \quad \text{for} \quad 1 \leq k \leq N, \quad (36a)$$

$$\beta_{k1} = 0, \quad \gamma_k = 0, \quad \chi_k = 0 \quad \text{for} \quad k \leq 0, k \geq N + 1, \quad (36b)$$

where $N \in \mathbb{N}^+$ denotes the number of nonlinear interfaces. In this situation, Eq. (32) reduces to

$$\mathbf{A}_m^{(2)} = \beta a_0^2 \sum_{s=1}^2 \Psi(4\tilde{K}_{b+}^{s+}(\Omega)h, N) \exp[-imK^{s+}(2\Omega)h] \mathbf{E}_{1,b+}^{s-}(2\Omega) \quad \text{for } m \leq 0, \quad (37a)$$

$$\begin{aligned} \mathbf{A}_m^{(2)} = \beta a_0^2 \sum_{s=1}^2 \{ & \Psi(4\tilde{K}_{b+}^{s+}(\Omega)h, N - m) \exp[-imK^{s+}(2\Omega)h] \mathbf{E}_{1,b+}^{s-}(2\Omega) \\ & - \Psi(\Delta K_{b+}^{s+}(\Omega)h, m) \exp[imK^{s+}(2\Omega)h] \mathbf{E}_{1,b+}^{s+}(2\Omega) \} \quad \text{for } 1 \leq m \leq N - 1, \end{aligned} \quad (37b)$$

$$\mathbf{A}_m^{(2)} = -\beta a_0^2 \sum_{s=1}^2 \Psi(\Delta K_{b+}^{s+}(\Omega)h, N) \exp[imK^{s+}(2\Omega)h] \mathbf{E}_{1,b+}^{s+}(2\Omega) \quad \text{for } m \geq N, \quad (37c)$$

where Ψ is the function defined by [36]

$$\Psi(\xi, M) \equiv \begin{cases} M, & \xi = 2q\pi \ (q \in \mathbb{Z}), \\ \frac{1 - \exp(iM\xi)}{\exp(-i\xi) - 1}, & \text{otherwise.} \end{cases} \quad (38)$$

It is seen from Eq. (37b) that the second-harmonic wave field for $1 \leq m \leq N-1$ consists of the Bloch waves propagating in the positive and negative x_1 directions. As represented by the factors $\Psi(\Delta K_{b+}^{s+}(\Omega)h, m)$ and $\Psi(4\tilde{K}_{b+}^{s+}(\Omega)h, N-m)$, their amplitudes generally oscillate against m with the wavenumber governed by $\Delta K_{b+}^{s+}(\Omega)$ and $\tilde{K}_{b+}^{s+}(\Omega)$, respectively. On the other hand, when $\Delta K_{b+}^{s+}(\Omega)h = 2q\pi$ and $\text{Im}[K^{s+}(2\Omega)h] = 0$, the corresponding second-harmonic Bloch wave propagating in the positive x_1 direction grows linearly as it propagates in the region $1 \leq m \leq N-1$. Likewise, the one propagating in the negative x_1 direction grows when $\tilde{K}_{b+}^{s+}(\Omega)h = q\pi/2$ and $\text{Im}[K^{s+}(2\Omega)h] = 0$. These conditions indicate the phase matching between the fundamental and the second-harmonic Bloch waves when they are both the propagating modes.

In Fig. 8, the frequency dependence of $\Delta K_{b+}^{s+}(\Omega)$ and $\tilde{K}_{b+}^{s+}(\Omega)$ when $\theta_T = 0^\circ, 20^\circ$, and 40° is shown for fixed $\mathcal{A}_N = 0.1$ and $\mathcal{A}_T = 0.1$. Note that in Figs. 8(a) and 8(b) for $\theta_T = 0^\circ$, $\Delta K_{b+}^{1+}(\Omega)$ and $\tilde{K}_{b+}^{1+}(\Omega)$ are depicted only because the corresponding second harmonics consist only of the longitudinal Bloch mode with wavenumber $K^{1+}(2\Omega)$ as mentioned in Section 4.2.

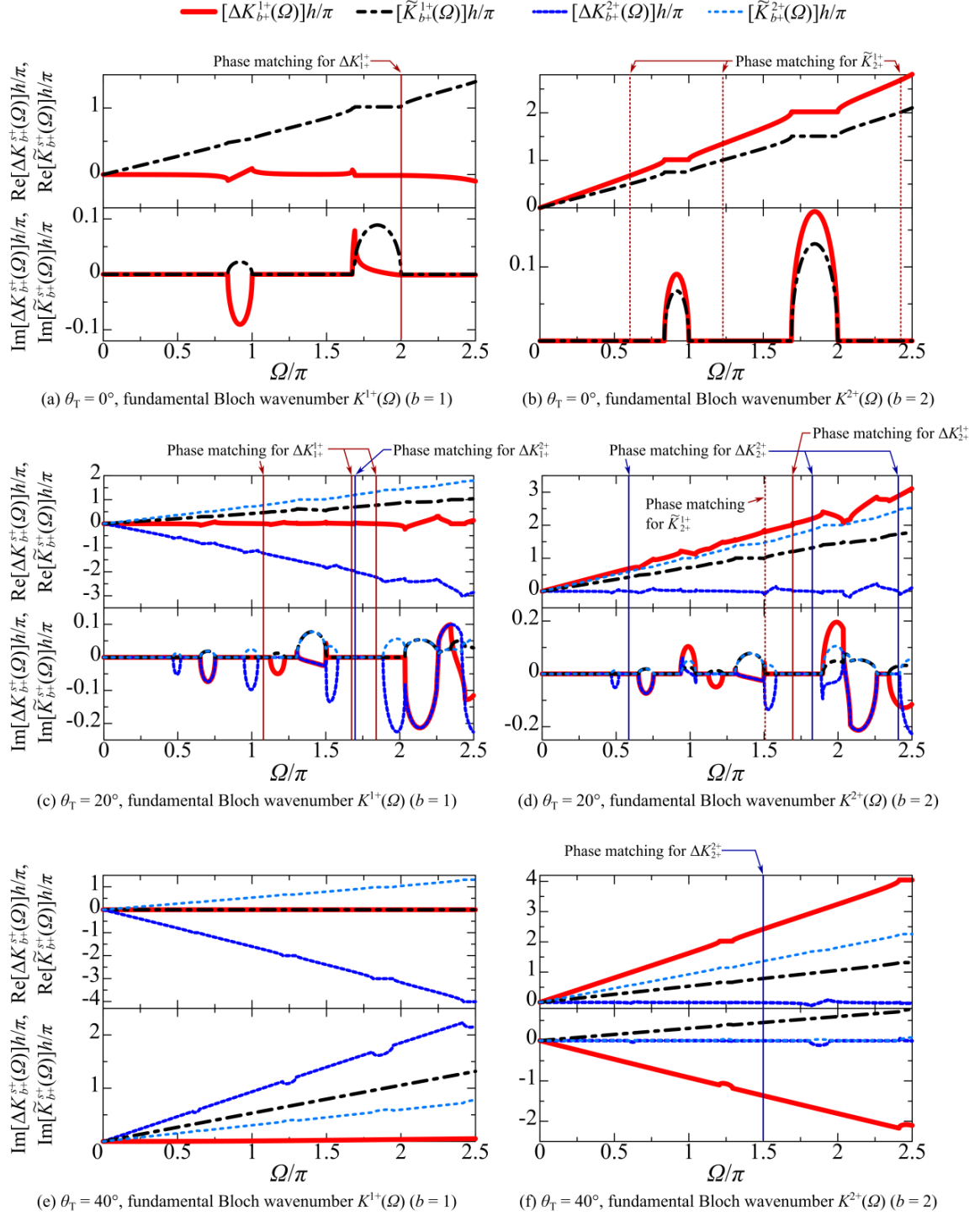


Fig. 8 Frequency dependence of $\Delta K_{b+}^{s+}(\Omega)$ and $\tilde{K}_{b+}^{s+}(\Omega)$ ($s, b = 1, 2$) when $A_N = 0.1$ and $A_T = 0.1$. (a) and (b) $\theta_T = 0^\circ$. (c) and (d) $\theta_T = 20^\circ$. (e) and (f) $\theta_T = 40^\circ$. The left and right columns correspond to the cases when the fundamental Bloch wavenumber is $K^{1+}(\Omega)$ and $K^{2+}(\Omega)$, respectively. The frequencies satisfying the phase matching condition are indicated by vertical

lines.

In the case of the one-dimensional longitudinal wave propagation analyzed in the previous study [36], the phase matching occurs only at $\Omega/\pi = 2q$ as shown in Fig. 8(a). At these frequencies, however, no second harmonics are generated because of their null generation efficiency at the nonlinear interfaces (see Fig. 5(c)). By contrast, when the fundamental Bloch wave is the shear mode in Fig. 8(b), the frequencies satisfying the phase matching condition for $\tilde{K}_{2+}^{1+}(\Omega)$ are numerically found at $\Omega/\pi = 0.6$ ($\tilde{K}_{2+}^{1+}h = \pi/2$), $\Omega/\pi = 1.23$ ($\tilde{K}_{2+}^{1+}h = \pi$), and $\Omega/\pi = 2.42$ ($\tilde{K}_{2+}^{1+}h = 2\pi$), where $F_{n,2+}^{1+}$ becomes nonzero in Fig. 5(d). Such frequencies are also found when $\theta_T = 20^\circ$ as indicated in Figs. 8(c) and 8(d). Compared to the cases when $\theta_T = 0^\circ$ (Figs. 8(a) and 8(b)), there exist more phase-matched frequencies when $\theta_T = 20^\circ$ (Figs. 8(c) and 8(d)) due mainly to the presence of two relevant second-harmonic Bloch modes. When $\theta_T = 40^\circ$ in Fig. 8(e), no phase matching occurs because the fundamental Bloch wave dominated by the evanescent longitudinal waves becomes the decaying mode in the entire frequency range. As a result, the phase matching can be achieved only between the shear-dominated fundamental and second-harmonic Bloch modes in Fig. 8(f).

To demonstrate the above observation, the second-harmonic displacements are calculated for each layer by

$$\mathbf{U}^{(2)}(x_1) = [\mathbf{P}^-\mathbf{L}(1 - \hat{x}_1) \quad \mathbf{P}^+\mathbf{L}(\hat{x}_1)]\mathbf{A}_m^{(2)} \quad \text{for } X_m < x_1 < X_{m+1}, \quad (39)$$

where $\hat{x}_1 \equiv (x_1 - X_m)/h$ and

$$\mathbf{L}(\phi) = \begin{bmatrix} \exp(2i\Omega\zeta p_{L1}\phi) & 0 \\ 0 & \exp(2i\Omega p_{T1}\phi) \end{bmatrix}. \quad (40)$$

The spatial distributions of the second-harmonic displacements obtained from Eqs. (39) and (40) when the number of nonlinear interfaces is $N = 30$ are shown in Fig. 9 for several different frequencies in Fig. 8(d).

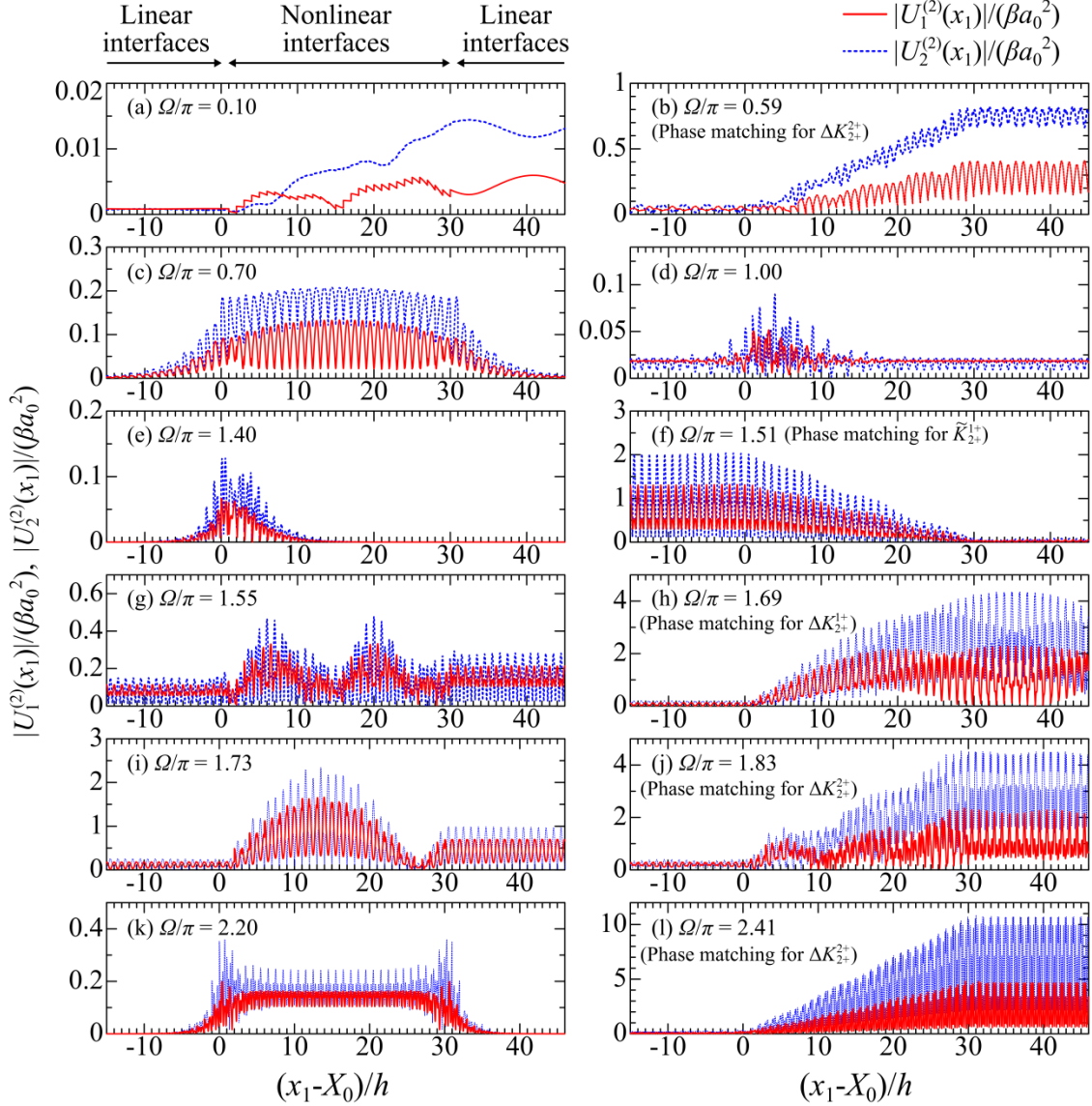


Fig. 9 Spatial distributions of the second-harmonic displacements in the x_1 direction for different normalized frequencies when $N = 30$, $A_N = 0.1$, $A_T = 0.1$, $\theta_T = 20^\circ$, and the fundamental Bloch wavenumber is $K^{2+}(\Omega)$. (a) $\Omega/\pi = 0.10$. (b) $\Omega/\pi = 0.59$. (c) $\Omega/\pi = 0.70$. (d) $\Omega/\pi = 1.00$. (e) $\Omega/\pi = 1.40$. (f) $\Omega/\pi = 1.51$. (g) $\Omega/\pi = 1.55$. (h) $\Omega/\pi = 1.69$. (i) $\Omega/\pi = 1.73$. (j) $\Omega/\pi = 1.83$. (k) $\Omega/\pi = 2.20$. (l) $\Omega/\pi = 2.41$.

In Fig. 9, the spatial distribution of the second harmonics depends markedly on the fundamental frequency due to the band structure of the fundamental and second-harmonic Bloch waves [36]. Among them, when $\Omega/\pi = 0.59$ in Fig. 9(b), $\Omega/\pi = 1.83$ in Fig. 9(j), and $\Omega/\pi = 2.41$ in Fig. 9(l) where the phase matching condition is satisfied for $\Delta K_{2+}^{2+}(\Omega)$ (i.e., $\Delta K_{2+}^{2+}(\Omega)h = 0$

from Fig. 8(d) and $\text{Im}[K^{2+}(2\Omega)h] = 0$ from Fig. 6(d)), the second harmonics in $1 < (x_1 - X_0)/h < 30$ grow cumulatively with the distance in the positive x_1 direction. This trend can also be seen in Fig. 9(h) for $\Omega/\pi = 1.69$, where the phase matching occurs for $\Delta K_{2+}^{1+}(\Omega)$ (i.e., $\Delta K_{2+}^{1+}(\Omega)h = 2\pi$ and $\text{Im}[K^{1+}(2\Omega)h] = 0$). Furthermore, when $\Omega/\pi = 1.51$, $\tilde{K}_{2+}^{1+}(\Omega)h = \pi$ ($2K^{2+}(\Omega)h = -2K^{2-}(\Omega)h = 2.94\pi$ and $K^{1+}(2\Omega)h = 1.06\pi$) meets the phase matching condition in Fig. 8(d). Hence, the second harmonics in Fig. 9(f) exhibit the cumulative growth in the negative x_1 direction in $1 < (x_1 - X_0)/h < 30$. It is noted here that $|U_1^{(2)}(x_1)|$ in Figs. 9(h) and 9(j) appear not to increase monotonically with the distance in $1 < (x_1 - X_0)/h < 30$. This is due mainly to the fact that the phase matching occurs between the fundamental Bloch mode and only one of the second-harmonic Bloch modes propagating in the positive and negative x_1 directions, and that the other second-harmonic Bloch modes oscillate in $1 < (x_1 - X_0)/h < 30$.

From Eq. (37c), the second-harmonic Bloch waves for $m \geq N$ increase in proportion to the number of consecutive nonlinear interfaces N when they satisfy the aforementioned phase matching condition $\Delta K_{b+}^{s+}(\Omega)h = 2q\pi$ and $\text{Im}[K^{s+}(2\Omega)h] = 0$. In Fig. 10, the second-harmonic displacements $\|\mathbf{U}^{(2)}(x_1)\|$ at $x_1 = X_{80+}$ when $N = 10, 20, 30$, and 40 are shown as functions of the normalized fundamental frequency for $\theta_T = 0^\circ, 20^\circ$, and 40° with fixed $A_N = 0.1$ and $A_T = 0.1$.

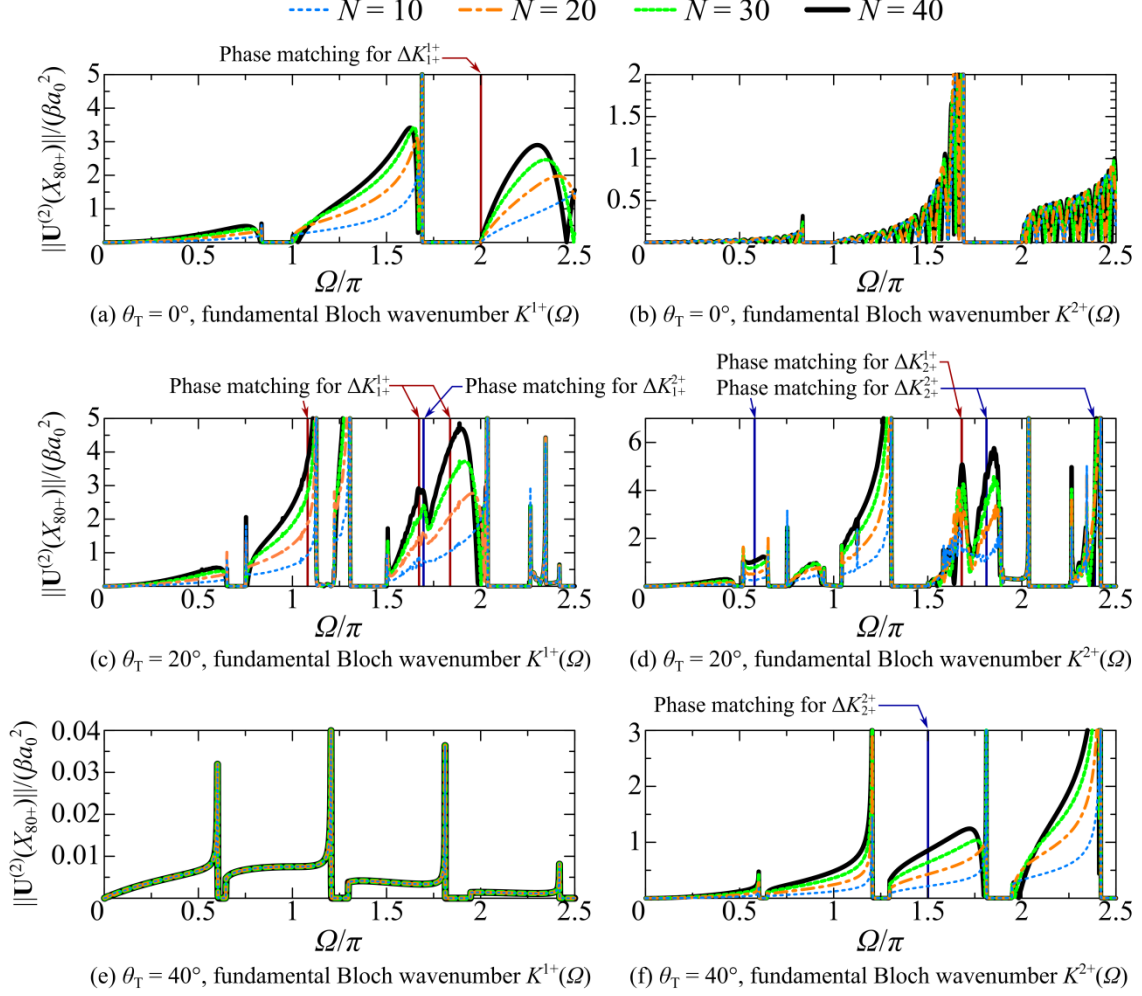


Fig. 10 Frequency dependence of the second-harmonic displacements at $x_1 = X_{80+}$ for different numbers of consecutive nonlinear interfaces $N = 10, 20, 30$, and 40 when $A_N = 0.1$ and $A_T = 0.1$. (a) and (b) $\theta_T = 0^\circ$. (c) and (d) $\theta_T = 20^\circ$. (e) and (f) $\theta_T = 40^\circ$. The left and right columns correspond to the cases when the fundamental Bloch wavenumber is $K^{1+}(\Omega)$ and $K^{2+}(\Omega)$, respectively. The frequencies satisfying the phase matching condition for ΔK_{b+}^{s+} ($b, s = 1, 2$) are indicated by vertical lines.

In Fig. 10, the second-harmonic displacements exhibit sharp peaks at the boundaries between the pass and stop bands of the double frequency due to the diverging nature of the generation efficiency at the nonlinear interfaces $F_{n,b+}^{1+}$ and $F_{n,b+}^{2+}$ ($b = 1, 2$) as discussed in Section 4.2.

When $\theta_T = 0^\circ$ and the fundamental Bloch wave has the wavenumber $K^{2+}(\Omega)$, the frequencies satisfying the phase matching condition for $\Delta K_{2+}^{1+}(\Omega)$ are not numerically found in the present frequency range as shown in Fig. 8(b). As a result, the magnitude of the second-harmonic

displacements in Fig. 10(b) is insensitive to N and exhibits fine oscillatory nature against the frequency with the wavenumber governed by $N\Delta K_{2+}^{1+}(\Omega)$. The influence of N is also insignificant in Fig. 10(e) for $\theta_T = 40^\circ$ with the fundamental Bloch wavenumber $K^{1+}(\Omega)$, because the second-harmonic generation is allowed only for the nonlinear interfaces near $x_1 = X_0$ due to the decaying nature of the fundamental Bloch wave. On the other hand, in Fig. 10(a), the second harmonics tend to grow with N when both fundamental and double frequencies lie in the pass bands except near the boundaries between the pass and stop bands, because $|\Delta K_{1+}^{1+}(\Omega)|$ in Fig. 8(a) is so small that the approximate phase matching occurs in these frequency ranges. Likewise, $\Delta K_{1+}^{1+}(\Omega)$ in Fig. 8(c), $\Delta K_{2+}^{2+}(\Omega)$ in Fig. 8(d), and $\Delta K_{2+}^{2+}(\Omega)$ in Fig. 8(f) nearly meet the phase matching condition except in the vicinity of the stop bands, wherein the second harmonics in Figs. 10(c), 10(d), and 10(f) increase monotonically with N . In particular, unlike in the case of $\theta_T = 0^\circ$ (at $\Omega/\pi = 2$ in Fig. 10(a)), the second harmonics exhibit almost linear growth with N when the exact phase matching occurs for $\theta_T = 20^\circ$ and 40° (at $\Omega/\pi = 1.08, 1.67, 1.70$, and 1.84 in Fig. 10(c), at $\Omega/\pi = 0.60, 1.69, 1.83$, and 2.40 in Fig. 10(d), and at $\Omega/\pi = 1.50$ in Fig. 10(f)).

Finally, the frequency dependence of the second-harmonic displacements when $\theta_T = 20^\circ$ and the fundamental Bloch wavenumber is $K^{1+}(\Omega)$ is depicted in Fig. 11 for different non-dimensional linear interfacial compliances, $\mathcal{A}_N$ and $\mathcal{A}_T$. As discussed in Ref. [36], the monotonic growth of the second harmonics with N is seen in each pass band for the smaller compliances in Fig. 11(a), whereas it is limited to the low frequency range of about $\Omega/\pi < 0.5$ and the immediate vicinity of the phase-matched frequencies for the larger compliances in Figs. 11(b) and 11(c) due to their more remarkable dispersive nature. Note that in contrast to the one-dimensional longitudinal wave propagation case [36] where the phase-matched frequencies are fixed at $\Omega/\pi = 2q$, the ones for the two-dimensional case are generally influenced by the linear interfacial compliances (and also by the ratio of shear to longitudinal wave velocities in the constituting layers) as shown in Fig. 11.

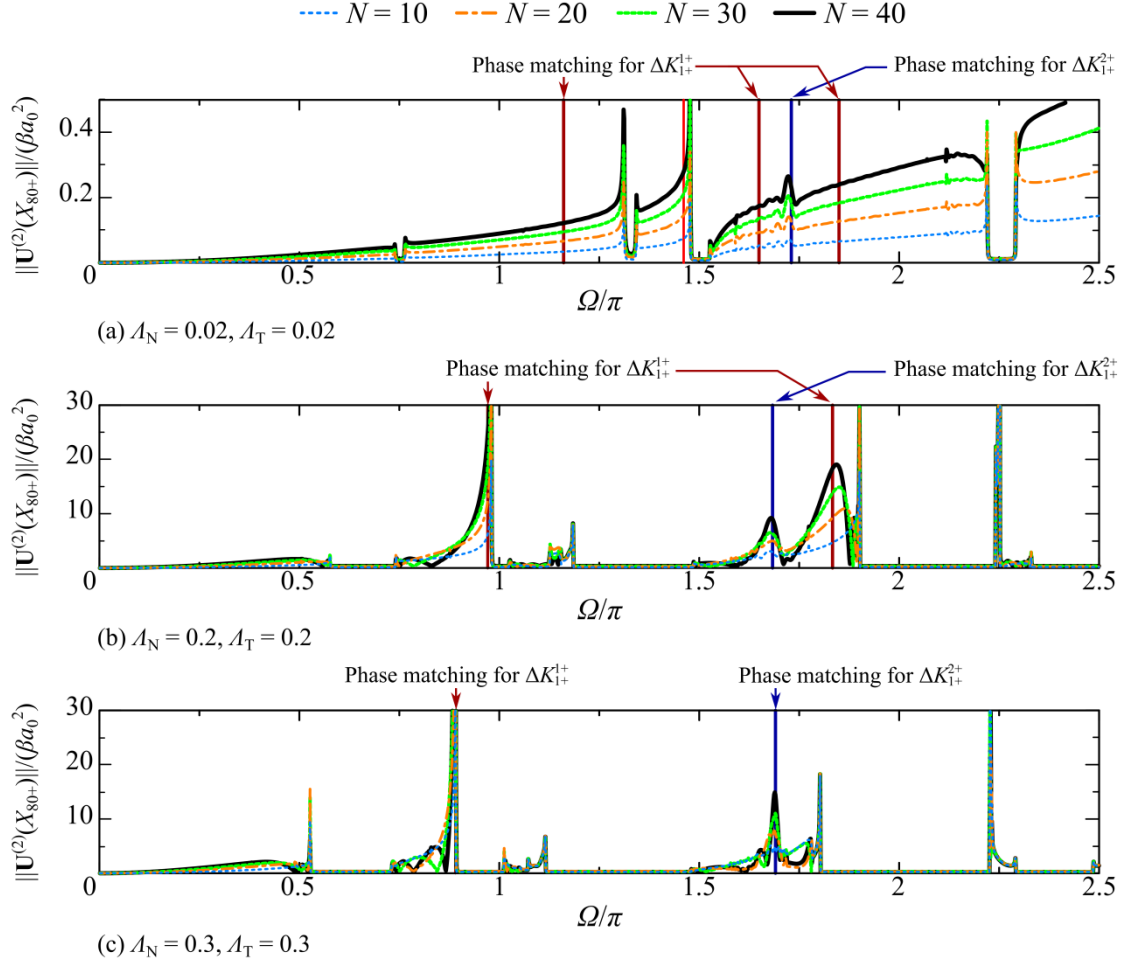


Fig. 11 Frequency dependence of the second-harmonic displacements at $x_1 = X_{80+}$ for different numbers of consecutive nonlinear interfaces $N = 10, 20, 30$, and 40 when $\theta_T = 20^\circ$ and the fundamental Bloch wavenumber is $K_1^+(\Omega)$. (a) $A_N = 0.02$ and $A_T = 0.02$. (b) $A_N = 0.2$ and $A_T = 0.2$. (c) $A_N = 0.3$ and $A_T = 0.3$. The frequencies satisfying the phase matching condition for ΔK_{1+}^{s+} ($s = 1, 2$) are indicated by vertical lines.

5. Conclusions

The second-harmonic generation of the two-dimensional elastic wave propagation in an infinite layered structure consisting of isotropic linear elastic layers and nonlinear spring-type interfaces has been analyzed theoretically. Using a perturbation approach and the transfer-matrix method, explicit analytical expressions have been derived for the second-harmonic amplitudes when a single monochromatic Bloch mode propagates in the structure in arbitrary directions. The influence of Bloch wave propagation direction on the frequency dependence of the second

harmonics generated by a single nonlinear interface and by multiple consecutive nonlinear interfaces has been explained based on the evanescent nature of partial waves in the layers. In particular, the second harmonics generated at multiple consecutive interfaces have been shown to grow monotonically with the propagation distance when the phase matching occurs between the fundamental and second-harmonic Bloch modes.

The present study investigated the second-harmonic generation in infinitely layered structures to gain fundamental understanding. The analysis presented here can be extended to the case of finitely layered structures, which is an important issue to be explored from a practical point of view for the nondestructive diagnosis of actual layered structures using nonlinear acoustic method. The previous study [37] has shown that the one-dimensional second-harmonic generation in the layering direction of finitely layered structures is closely related to that of infinitely extended layered structures [36], and that the cumulative growth of second harmonics can occur even in the case of a finite number of layers. In light of these results, the findings of the present study are expected to be helpful when analyzing the second-harmonic generation in arbitrary directions of finite layered structures, which is left for the future study.

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