

# A MIXED KNUTH CORRESPONDENCE FOR $(A, B)$ -PARTIALLY STRICT TABLEAUX

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Fix finite totally ordered sets  $\mathcal{A}, \mathcal{A}'$  throughout this article. A pair  $(U, C)$  of subsets of  $\mathcal{A}$  is called a *division* of  $\mathcal{A}$  if it satisfies

$$U \uplus C = \mathcal{A}. \quad (\text{disjoint union})$$

Henceforth, we fix a division  $(U, C)$  of  $\mathcal{A}$ , and we call elements of  $U$  *uncircled letters* and elements of  $C$  *circled letters*. Fix another division  $(A, B)$  of  $\mathcal{A}$ . Set  $k = |A|$  and  $l = |B|$  so that we have  $|\mathcal{A}| = k + l$ . We have two pairs  $(A, B)$  and  $(U, C)$  which are divisions of  $\mathcal{A}$ . We write

$$\begin{aligned} A_u &= A \cap U, & A_c &= A \cap C \\ B_u &= B \cap U, & B_c &= B \cap C \end{aligned}$$

## EXAMPLE 1.1

Set  $A = \{1, 3^\circ, 5, 7^\circ\}$ ,  $B = \{2, 4^\circ, 6, 8^\circ\}$ ,  $U = \{1, 2, 5, 6\}$  and  $C = \{3^\circ, 4^\circ, 7^\circ, 8^\circ\}$ . Then  $(A, B)$  and  $(U, C)$  are divisions of  $[8]$  and we have  $A_u = \{1, 5\}$ ,  $A_c = \{3^\circ, 7^\circ\}$ ,  $B_u = \{2, 6\}$ , and  $B_c = \{4^\circ, 8^\circ\}$ . As in this example we write elements of  $A$  in lightface and elements of  $B$  in boldface.

We take the word “ $(A, B)$ -partially strict” from [Ok], but the original definition is due to [St]. For the definition of  $(k, l)$ -semistandard tableaux see [BR] or [Re]. A *reverse plane partition*  $\pi$  is a filling of a Young diagram with letters of  $\mathcal{A}$  wherein in each row from left to right and in each column from top to bottom the letters are arranged in weakly increasing order.

## DEFINITION 1.1

Let  $\pi$  be a reverse plane partition.  $\pi$  is said to be  $(A, B)$ -partially strict if it satisfies the conditions:

- (i) For any  $m \in A$ ,  $m$  appears at most once in each column.
- (ii) For any  $m \in B$ ,  $m$  appears at most once in each row.

We call a  $(A, B)$ -partially strict reverse plane partition a  $(A, B)$ -partially strict tableau. A  $(\mathbf{P}, \emptyset)$ -partially strict skew tableau is usually called a column-strict skew tableau and a  $(\emptyset, \mathbf{P})$ -partially strict skew tableau, a row-strict skew tableau. If  $A = \{1, 2, \dots, k\}$  and  $B = \{1', 2', \dots, l'\}$ , where  $1 < 2 < \dots < k < 1' < 2' < \dots < l'$ , then a  $(A, B)$ -partially strict tableau is called a  $(k, l)$ -semistandard tableau.

## EXAMPLE 1.2

Set  $(A, B)$  to be the division given in EXAMPLE 4.1. Then

$$\pi =$$

|            |            |            |            |            |            |            |            |
|------------|------------|------------|------------|------------|------------|------------|------------|
| 1          | 1          | 1          | 2          | $^\circ 4$ | 5          | 5          | $^\circ 7$ |
| $^\circ 3$ | $^\circ 3$ | $^\circ 3$ | $^\circ 4$ | 6          | $^\circ 7$ | $^\circ 7$ |            |
| $^\circ 4$ | 5          | 5          | 6          |            |            |            |            |
| $^\circ 4$ | $^\circ 7$ | $^\circ 7$ | $^\circ 8$ |            |            |            |            |
| 5          | $^\circ 8$ |            |            |            |            |            |            |

is a  $(A, B)$ -partially strict tableau.

$$\pi' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 2' & 3' \\ \hline 4 & 4 & 1' & 2' & \\ \hline 1' & 3' & & & \\ \hline 1' & & & & \\ \hline \end{array}$$

is an exmple of  $(4, 3)$ -semistandard tableau, where  $1 < 2 < 3 < 4 < 1' < 2' < 3'$ .

**DEFINITION 1.2**

Let  $\lambda/\mu$  be a skew diagram. Let  $\mathcal{T}_{(A,B)}(\lambda/\mu)$  denote the set of all  $(A, B)$ -partially strict skew tableaux of shape  $\lambda/\mu$ . For  $\pi \in \mathcal{T}_{(A,B)}(\lambda/\mu)$  set the weight  $wt(\pi)$  of  $\pi$  to be  $\prod_{a \in A} x_a^{m_a}$  where

$$m_a = \text{the number of times } a \text{ occurs in } \pi$$

and  $x_a$ 's are indeterminates. Set

$$HS_{\lambda/\mu}^{(A,B)}(x) = \sum_{\pi \in \mathcal{T}_{(A,B)}(\lambda/\mu)} wt(\pi).$$

It is clear from the defintion that

$$HS_{\lambda/\mu}^{(A,B)}(x) = HS_{\lambda'/\mu'}^{(B,A)}(x).$$

In particular if  $A = \{1, 2, \dots, k\}$  and  $B = \{1', 2', \dots, l'\}$ , where  $1 < 2 < \dots < k < 1' < 2' < \dots < l'$ , we write  $HS_{\lambda/\mu}^{(A,B)}(x)$  as  $HS_{\lambda/\mu}^{(k,l)}(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_l)$ .

**PROPOSITION 1.1**

Set  $A = \{a_1, a_2, \dots, a_k\}$  and  $B = \{b_1, b_2, \dots, b_l\}$ . Then

$$HS_{\lambda/\mu}^{(A,B)}(x) = HS_{\lambda/\mu}^{(k,l)}(x_{a_1}, x_{a_2}, \dots, x_{a_k}, x_{b_1}, x_{b_2}, \dots, x_{b_l})$$

**Proof.**

We can easily construct a bijection between  $\mathcal{T}_{(A,B)}(\lambda/\mu)$  and  $\mathcal{T}_{(\{1,2,\dots,k\},\{1',2',\dots,l'\})}(\lambda/\mu)$  using the jeu de taquin method in [Re], Section 3, pp.266. For details see [Re]. ■

Now we define a mixed Knuth insertion.

**DEFINITION 1.3**

Let  $\pi$  be a  $(A, B)$ -partially strict tableau and let  $x \in A$ . We define  $\text{INSERT}_{(A,B;U,C)}(x)$  as follow.

If  $x \in U$ , insert  $x$  into the first row of  $\pi$ ; if  $x \in C$ , insert  $x$  into the first column of  $\pi$ . If the bumped element  $y$  is uncircled, then we insert  $y$  into the row immediately below or if the bumped element  $y$  is circled, then we insert  $y$  into the column immediately to its right by the following rules.

$y$  replace the least element which is  $> y$  if  $y \in A_u \cup B_c$ : or  $y$  replace the least element which is  $\geq y$  if  $y \in B_u \cup A_c$ .

Continue until an insertion takes place at the end of a row or column, bumping no new element. This procedure terminates in a finite number of steps. Then set  $(s, t)$  to be the cell which is added to  $\pi$ .

Similarly we define  $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$  by swapping  $U$  and  $C$  in the foregoing definition. If  $x \in U$ , insert  $x$  into the first column of  $\pi$ ; if  $x \in C$ , insert  $x$  into the first row of  $\pi$ . The uncircled letters which are bumped are inserted into the column immediately to its right and circled letters are inserted into the row immediately below by the following rule.

$y$  replace the least element which is  $> y$  if  $y \in A_c \cup B_u$ : or  $y$  replace the least element which is  $\geq y$  if  $y \in B_c \cup A_u$ .

It is easy to see that the resulting tableau is also  $(A, B)$ -partially strict. Let  $\pi \leftarrow^m x$  (resp.  $x \rightarrow^m \pi$ ) denote the tableau which is obtained after we applied  $\text{INSERT}_{(A,B;U,C)}(x)$  (resp.  $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$ ) to  $\pi$ .

### EXAMPLE 1.3

Let  $\pi$  be the  $(A, B)$ -partially strict tableau in EXAMPLE 4.2.

$$\pi \leftarrow^m 4 =$$

|    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|
| 1  | 1  | 1  | 1  | °4 | 5  | 5  | °7 |
| 2  | °3 | °3 | °3 | °4 | °7 | °7 |    |
| °4 | 5  | 5  | 6  | °8 |    |    |    |
| °4 | 6  | °7 | °7 |    |    |    |    |
| 5  | °8 |    |    |    |    |    |    |

And we have  $(s, t) = (4, 4)$ .

$$2 \rightarrow^m \pi =$$

|    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|
| 1  | 1  | 1  | 2  | °4 | 5  | 5  | °7 |
| 2  | °3 | °3 | °4 | 6  | °7 | °7 |    |
| °3 | 5  | 5  | 5  | 6  |    |    |    |
| °4 | °7 | °7 | °8 |    |    |    |    |
| °4 | °8 |    |    |    |    |    |    |

And we have  $(s, t) = (6, 1)$ .

### REMARK 1.1

In [Re] two insertion procedures are defined for  $(k, l)$ -semistandard tableaux. Set  $A = \{1, 2, \dots, k\}$  and  $B = \{1', 2', \dots, l'\}$ , where  $1 < 2 < \dots < k < 1' < 2' < \dots < l'$ . If  $U = A$  and  $C = B$ , then the insertion algorithm in Definition 4.2 is called RS1 insertion in [Re]. If  $U = \mathcal{A}$  and  $C = \emptyset$ , then the insertion algorithm is called RS2 insertion.

### DEFINITION 1.4

Let  $\pi$  be a  $(A, B)$ -partially strict tableau. Set  $m_x$  to be the number of times  $x$  occurs in  $\pi$  for each  $x \in \mathcal{A}$ . Let  $m = \sum_{x \in \mathcal{A}} m_x$ . We make a partial tableau  $pt(\pi)$  with letters in  $[m]$  from  $\pi$  as follows. If  $x \in A$ , then replace  $m_x$   $x$ 's in  $\pi$  to  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$  from left to right. If  $x \in B$ , then replace  $m_x$   $x$ 's in  $\pi$  to  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$  from top to bottom. If  $x \in U$  then  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$  are in  $U$ , and vice versa.

### EXAMPLE 1.4

If  $\pi$  is as in Example 4.2, then  $pt(\pi)$  is as in Example 2.2.

### DEFINITION 1.5

A *word with repetition* is a sequence  $w = w_1 w_2 \dots w_m$  of letters in  $\mathcal{A}$  wherein each  $a \in \mathcal{A}$  can appear more than once. Given a word with repetition  $w = w_1 w_2 \dots w_m$ , we make the insertion tableau  $\pi = \emptyset \leftarrow^m w$  for  $w$  as follows. For  $i = 1, 2, \dots, m$  we define inductively  $\pi_0 = \emptyset$  and  $\pi_i = \pi_{i-1} \leftarrow^m w_i$ . Let  $\pi = \pi_m$ .

**EXAMPLE 1.5**

$$w = {}^\circ 2 \quad {}^\circ 2 \quad 1 \quad {}^\circ 3 \quad 4 \quad {}^\circ 3 \quad {}^\circ 3 \quad 1 \quad 4 \quad {}^\circ 2 \quad 4$$

is a word with repetition and the insertion tableau for  $w$  is as follows.

$$\emptyset \leftarrow^m w = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & {}^\circ 2 & {}^\circ 3 & 4 \\ \hline {}^\circ 2 & {}^\circ 3 & {}^\circ 3 & 4 & \\ \hline {}^\circ 2 & 4 & & & \\ \hline \end{array}$$

**DEFINITION 1.6**

For a given word with repetition  $w = w_1 w_2 \dots w_m$  we make a permutation  $p(w)$  of  $[m]$  as follows. For each  $x \in \mathcal{A}$  let  $m_x$  denote the number of times  $x$  appears in  $w$ . For each  $x \in \mathcal{A}$ , if  $x \in A_u \cup B_c$  then replace all  $x$  in  $w$  by  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$  in increasing order. For each  $x \in \mathcal{A}$ , if  $x \in A_c \cup B_u$  then replace all  $x$  in  $w$  by  $\sum_{y \leq x} m_x, \sum_{y \leq x} m_x - 1, \dots, \sum_{y < x} m_x + 1$  in decreasing order. If  $x \in U$  then  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$  are in  $U$ , and vice versa.

**EXAMPLE 1.6**

Let  $w$  be as in Example 4.5.

$$p(w) = {}^\circ 3 \quad {}^\circ 4 \quad 1 \quad {}^\circ 8 \quad 9 \quad {}^\circ 7 \quad {}^\circ 6 \quad 2 \quad 10 \quad {}^\circ 5 \quad 11$$

$$\emptyset \leftarrow^m p(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & {}^\circ 8 & {}^\circ 3 & 9 \\ \hline {}^\circ 4 & {}^\circ 6 & {}^\circ 7 & 10 & \\ \hline {}^\circ 5 & 11 & & & \\ \hline \end{array}$$

The following proposition is easy to see from definitions.

**PROPOSITION 1.2**

Let  $w$  be a word with repetition. Let  $\pi$  be the insertion tableau of  $w$ . Then the following diagram commutes.

$$\begin{array}{ccc} w & \longleftrightarrow & \pi \\ \downarrow p & & \downarrow p^t \\ p(w) & \longleftrightarrow & p^t(\pi) \end{array}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively.

■

**LEMMA 1.1**

Let  $\pi$  be a  $(A, B)$ -partially strict tableau and  $x, x' \in \mathcal{A}$ . If  $\text{INSERT}_{(A, B; U, C)}(x)$ , determining  $s$  and  $t$ , is immediately followed by  $\text{INSERT}_{(A, B; U, C)}(x')$ , determining  $(s', t')$ , then

(Case 1)  $x, x' \in U$

- (a) If  $x < x'$  or  $x = x' \in A$  then we have  $s \geq s'$  and  $t < t'$ .
- (b) If  $x > x'$  or  $x = x' \in B$  then we have  $s < s'$  and  $t \geq t'$ .

(Case 2)  $x, x' \in C$

- (a) If  $x > x'$  or  $x = x' \in A$  then  $s \geq s'$  and  $t < t'$ .
- (b) If  $x < x'$  or  $x = x' \in B$  then  $s < s'$  and  $t \geq t'$ .

(Case 3)  $x \in U$  and  $x' \in C$ , we always have  $s < s'$  and  $t \geq t'$ .  
 (Case 4)  $x \in C$  and  $x' \in U$ , we always have  $s \geq s'$  and  $t < t'$ .

**Proof.**

Choose arbitrary word with repetition  $w$  such that  $\pi = \emptyset \leftarrow^m w$ . Let  $w' = wx x'$ . Then it is easy to verify the lemma by using Proposition 4.2, Corollary 2.2, and Lemma 1.1. For example, we verify Case 3. Assume that  $x \in U$  and  $x' \in C$ . Then  $x' \in C$  is changed into some negative letter  $-x'$  which is less than  $x$  so that we obtain  $s < s'$  and  $t \geq t'$  immediately by Lemma 1.1. ■

**REMARK 1.2**

In the foregoing lemma by changing  $\text{INSERT}_{(A,B;U,C)}(x)$  and  $\text{INSERT}_{(A,B;U,C)}(x')$  into  $\overline{\text{INSERT}}_{(A,B;U,C)}(x)$  and  $\overline{\text{INSERT}}_{(A,B;U,C)}(x')$ , respectively and swapping  $U$  and  $C$ , we obtain a similar result on  $\overline{\text{INSERT}}_{(A,B;U,C)}(\cdot)$ .

Fix another finite totally ordered set  $\mathcal{A}'$  and its divisions  $(A', B')$  and  $(U', C')$  such that  $|A'| = k'$  and  $|B'| = l'$ . We write

$$\begin{aligned} A'_u &= A' \cap U', & A'_c &= A' \cap C' \\ B'_u &= B' \cap U', & B'_c &= B' \cap C'. \end{aligned}$$

**DEFINITION 1.7**

Let  $a$  be a  $(k' + l') \times (k + l)$  matrix of nonnegative integers

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

whose rows are labeled by elements of  $\mathcal{A}'$  and columns are labeled by elements of  $\mathcal{A}$ .  $a$  is said to be *admissible* if it satisfies:

- (1) If  $(i, j) \in A' \times A \cup B' \times B$ ,  $a_{i,j} \in \mathbf{N}$ .
- (2) If  $(i, j) \in A' \times B \cup B' \times A$ ,  $a_{i,j} \in \{0, 1\}$ .

Let  $\mathcal{M}(A', B', A, B)$  denote the set of all admissible  $(k' + l') \times (k + l)$  matrices.

**EXAMPLE 1.7**

Let  $A' = \{2, {}^\circ 4\}$ ,  $B' = \{1, {}^\circ 3\}$ ,  $A = \{{}^\circ 3, 4\}$ , and  $B = \{1, {}^\circ 2\}$ . Then

$$a = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

is a admissible  $4 \times 4$  matrix. As in this example we write  $a_{i,j}$  such that  $(i, j) \in A' \times A \cup B' \times B$  in italic.

**DEFINITION 1.8**

Let  $a \in \mathcal{M}(A', B', A, B)$ . From  $a$  we make a two-line array

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

as follows. We arrange  $a_{u,v}$  pairs of row and column labels  $\begin{pmatrix} u \\ v \end{pmatrix}$  by the following rule.

First we assume that

$$u_1 \leq u_2 \leq \cdots \leq u_m.$$

- (1) For each  $u \in A'_u \cup B'_c$  we arrange all labels  $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$  such that  $u_i = u$  as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \dots, v_{p_r}}_{\substack{\text{elements of } C \\ \text{in decreasing order}}} \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \dots, v_{p_{r+s}}}_{\substack{\text{elements of } U \\ \text{in increasing order}}}$$

- (2) For each  $u \in A'_c \cup B'_u$  we arrange all labels  $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$  such that  $u_i = u$  as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \dots, v_{p_r}}_{\substack{\text{elements of } U \\ \text{in decreasing order}}} \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \dots, v_{p_{r+s}}}_{\substack{\text{elements of } C \\ \text{in increasing order}}}$$

It is easy to see this gives an one to one correspondence between admissible matrices and two line arrays satisfying the above conditions. We call this two line array the *matrix word* of  $a$  and denote by  $l(a)$ . The top (resp. bottom) line of  $l(a)$  is denoted by  $\hat{l}(a) = u_1, u_2, \dots, u_m$  (resp.  $\check{l}(a) = v_1, v_2, \dots, v_m$ ).

**EXAMPLE 1.8**

The two line array which correspond to the matrix  $a$  in EXAMPLE 4.5 is

$$l(a) = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & \circ 3 & \circ 3 & \circ 3 & \circ 3 & \circ 4 & \circ 4 & \circ 4 & \circ 4 \\ 1 & 1 & \circ 3 & \circ 2 & 4 & 4 & \circ 3 & \circ 2 & \circ 2 & \circ 2 & 4 & 4 & 1 & \circ 3 \end{pmatrix}.$$

**DEFINITION 1.9**

Let  $a \in \mathcal{M}(A', B', A, B)$ . From  $a$  we make a two-line array  $l(a)$  in Definition 4.8.

$$l = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

We construct a sequence of tableaux pairs:

$$(\emptyset, \emptyset) = (\pi_0, \sigma_0), (\pi_1, \sigma_1), \dots, (\pi_m, \sigma_m) = (\pi, \sigma)$$

inductively as follows. For each  $i = 1, 2, \dots, m$  form  $\pi_i$  from  $\pi_{i-1}$  by performing  $\text{INSERT}_{(A, B; U, C)}(v_i)$  on  $\pi_{i-1}$  if  $u_i$  is a uncircled letter, or performing  $\overline{\text{INSERT}}_{(A, B; U, C)}(v_i)$  on  $\pi_{i-1}$  if  $u_i$  is a circled letter. Form  $\sigma_i$  from  $\sigma_{i-1}$  by placing  $u_i$  on  $\sigma_{i-1}$  in the cell added to  $\pi_i$ . By Lemma 4.1  $\sigma$  is a  $(A', B')$ -partially strict tableau and  $\pi$  and  $\sigma$  have the same shape.

**EXAMPLE 1.9**

Let  $a$  be as in Example 4.7. Then

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \circ 2 & \circ 3 & \circ 3 & 4 & 4 \\ \hline 1 & \circ 2 & 4 & & & \\ \hline 1 & \circ 3 & & & & \\ \hline \circ 2 & 4 & & & & \\ \hline \circ 2 & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & \circ 3 & \circ 4 \\ \hline 1 & \circ 3 & \circ 4 & & & \\ \hline 2 & \circ 3 & & & & \\ \hline \circ 3 & \circ 4 & & & & \\ \hline \circ 4 & & & & & \\ \hline \end{array}$$

**DEFINITION 1.10**

Let  $a \in \mathcal{M}(A', B', A, B)$ . Let

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

be the two line array which correspond to  $a$ . We construct a biword  $w$  from  $l$  as follow. For each  $x \in \mathcal{A}$  (resp.  $x \in \mathcal{A}'$ ) let  $m_x$  (resp.  $m'_x$ ) denote the number of times  $x$  occurs in the bottom (resp. top) line of  $l$ . Replace  $m'_x$   $x$ 's in the top line of  $l$  by  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y \leq x} m_x$  from left to right. The circles are transferred unchanged in this replacement. For each  $x \in \mathcal{A}$  let  $r_x$  (resp.  $s_x$ ) be the number of pairs  $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$  such that  $v_j = x$  and  $u_i \in U$  (resp.  $u_i \in C$ ). So we have  $r_x + s_x = m_x$ . For each  $x \in \mathcal{A}$  we replace  $v_j$ 's such that  $v_j = x$  by the following rules. The circles are transferred unchanged in this replacement.

(Case 1) :  $x \in A_u \cup B_c$

Replace the  $v_j$ 's of pairs  $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$  such that  $v_j = x$  and  $u_i \in C$  by  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y < x} m_x + s_x$  from right to left. Then replace the  $v_j$ 's of pairs  $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$  such that  $v_j = x$  and  $u_i \in U$  by  $\sum_{y < x} m_x + s_x + 1, \sum_{y < x} m_x + s_x + 2, \dots, \sum_{y \leq x} m_x$  from left to right.

(Case 2) :  $x \in A_c \cup B_u$

Replace the  $v_j$ 's of pairs  $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$  such that  $v_j = x$  and  $u_i \in U$  by  $\sum_{y < x} m_x + 1, \sum_{y < x} m_x + 2, \dots, \sum_{y < x} m_x + r_x$  from right to left. Then replace the  $v_j$ 's of pairs  $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$  such that  $v_j = x$  and  $u_i \in C$  by  $\sum_{y < x} m_x + r_x + 1, \sum_{y < x} m_x + r_x + 2, \dots, \sum_{y \leq x} m_x$  from left to right.

Let  $p(a)$  denote the resulting biword.

**EXAMPLE 1.10**

Let  $a$  be as Exemple 4.7 and  $l(a)$  as Example 4.8. Then we have

$$p(a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \circ 7 & \circ 8 & \circ 9 & \circ 10 & \circ 11 & \circ 12 & \circ 13 & \circ 14 \\ 2 & 1 & \circ 8 & \circ 7 & 13 & 14 & \circ 9 & \circ 6 & \circ 5 & \circ 4 & 12 & 11 & 3 & \circ 10 \end{pmatrix}.$$

The following proposition is easy to see from definitions.

**PROPOSITION 1.3**

Let  $a \in \mathcal{M}(A', B', A, B)$ . Then the following diagram commutes.

$$\begin{array}{ccc} a & \longleftrightarrow & (\pi, \sigma) \\ \downarrow p & & \downarrow \\ p(a) & \longleftrightarrow & (pt(\pi), pt(\sigma)) \end{array}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively.

**EXAMPLE 1.11**

Let  $p(a)$  be as in Example 4.9. Then the insertion pair of  $p(a)$  is as follows.

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \circ 4 & \circ 9 & \circ 10 & 13 & 14 \\ \hline 2 & \circ 5 & 12 & & & \\ \hline 3 & \circ 8 & & & & \\ \hline \circ 6 & 11 & & & & \\ \hline \circ 7 & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & \circ 7 & \circ 14 \\ \hline 2 & \circ 8 & \circ 13 & & & \\ \hline 3 & \circ 9 & & & & \\ \hline \circ 10 & \circ 12 & & & & \\ \hline \circ 11 & & & & & \\ \hline \end{array}$$

From Proposition 4.3 we obtain the following theorem.

**THEOREM 1.1**

Fix  $A$  and its divisions  $(U, C)$  and  $(A, B)$ . Fix another  $A'$  and its divisions  $(U', C')$  and  $(A', B')$ . The map in Definition 4.9 from admissible matrices  $a \in \mathcal{M}(A', B', A, B)$  to pairs  $(\pi, \sigma)$ , where  $\pi$  is  $(A, B)$ -partially strict tableau,  $\sigma$  is  $(A', B')$ -partially strict tableau and  $\pi$  and  $\sigma$  have the same shape, is a bijection.

The following proposition is also easy to see from definitions.

**PROPOSITION 1.4**

Let  $a \in \mathcal{M}(A', B', A, B)$ . If  $p(a)$  correspond to  $a$  by the map in Definition 4.10, then the inverse biword  $p(a)^{-1}$  correspond to  $a^t$ . Here  $a^t$  denote the conjugate matrix of  $a$ .

From Proposition 4.4 we obtain the following theorem.

**THEOREM 1.2**

Fix  $A$  and its divisions  $(U, C)$  and  $(A, B)$ . Assume that  $(\pi, \sigma)$  correspond to  $a$  by the bijection in Definition 4.9, where  $a \in \mathcal{M}(A, B, A, B)$ , and  $\pi$  and  $\sigma$  are  $(A, B)$ -partially strict tableau having the same shape. Then  $(\sigma, \pi)$  correspond to  $a^t$  by the same bijection.

**EXAMPLE 1.12**

Let  $a$  be as Example 4.7. Then

$$a^t = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

and

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & \circ 2 & \circ 2 & \circ 2 & \circ 2 & \circ 3 & \circ 3 & \circ 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & \circ 4 & \circ 3 & \circ 3 & \circ 3 & 2 & 2 & \circ 3 & \circ 4 & \circ 4 & \circ 4 & 2 & 2 \end{pmatrix}.$$

It's easy to make sure that

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \circ 2 & \circ 3 & \circ 3 & 4 & 4 \\ \hline 1 & \circ 2 & 4 & & & \\ \hline 1 & \circ 3 & & & & \\ \hline \circ 2 & 4 & & & & \\ \hline \circ 2 & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & \circ 3 & \circ 4 \\ \hline 1 & \circ 3 & \circ 4 & & & \\ \hline 2 & \circ 3 & & & & \\ \hline \circ 3 & \circ 4 & & & & \\ \hline \circ 4 & & & & & \\ \hline \end{array}$$

**DEFINITION 1.11**

Fix  $A$  and its divisions  $(U, C)$  and  $(A, B)$ . Let  $a = (a_{ij})_{i,j \in A} \in \mathcal{M}(A, B, A, B)$  be an admissible symmetric matrix. We define  $\text{tr}_{(A,B)} a$  by

$$\text{tr}_{(A,B)} a = \sum_{i \in A} a_{ii} + \sum_{i \in B} \text{odd}\{a_{ii}\}$$

where  $\text{odd}\{x\} = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$ .

**COROLLARY 1.1**

Fix  $A$  and its divisions  $(U, C)$  and  $(A, B)$ . The map in Definition 4.9 gives a bijection from admissible symmetric matrices  $a \in \mathcal{M}(A, B, A, B)$  onto  $(A, B)$ -partially strict tableaux  $\pi$ . In this bijection we have

$$\text{tr}_{(A, B)} a = \text{odd}(\lambda)$$

where  $\lambda$  is the shape of  $\pi$  and  $\text{odd}(\lambda)$  stands for the number of odd length columns in  $\lambda$ .

**EXAMPLE 1.13**

Let  $A = \{1, {}^\circ 3\}$  and  $B = \{{}^\circ 2, 4\}$ . Let  $a$  be an admissible symmetric matrix given by

$$a = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ 2 & {}^\circ 2 & {}^\circ 2 & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & 4 & 4 \\ {}^\circ 3 & {}^\circ 3 & 1 & {}^\circ 3 & {}^\circ 2 & {}^\circ 2 & 4 & 1 & 1 & {}^\circ 2 & 4 & {}^\circ 3 \end{pmatrix}.$$

and

$$\pi = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & {}^\circ 2 & 4 \\ \hline {}^\circ 2 & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & \\ \hline {}^\circ 2 & 4 & & & \\ \hline {}^\circ 3 & & & & \\ \hline \end{array}$$

**COROLLARY 1.2**

Fix  $A$  and its division  $(A, B)$ .

$$\sum_{\lambda} HS_{\lambda}^{(A, B)}(x) t^{\text{odd}(\lambda)} = \prod_{\substack{(i, j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i, j) \in A \times B} (1 + x_i x_j) \prod_{i \in A} \frac{1}{1 - t x_i} \prod_{i \in B} \frac{1 + t x_i}{1 - x_i^2}$$

In particular,

$$\sum_{\lambda' \text{ even}} HS_{\lambda}^{(A, B)}(x) = \prod_{\substack{(i, j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i, j) \in A \times B} (1 + x_i x_j) \prod_{i \in B} \frac{1}{1 - x_i^2}$$

Now we investigate skew case. Let  $\text{PST}_{(A, B)}(\lambda/\mu)$  denote the set of  $(A, B)$ -partially strict skew tableaux which have skew shape  $\lambda/\mu$ .

**THEOREM 1.3**

Fix  $A$  and its divisions  $(U, C)$  and  $(A, B)$ . Fix another  $A'$  and its divisions  $(U', C')$  and  $(A', B')$ . Let  $\alpha$  and  $\beta$  be fixed partitions. Then the map

$$(a, \tau, \kappa) \mapsto (\pi, \sigma)$$

defined below is a bijection between admissible matrices  $a \in \mathcal{M}(A, B, A', B')$  with  $\tau \in \text{PST}_{(A, B)}(\alpha/\mu)$  and  $\kappa \in \text{PST}_{(A', B')}(\beta/\mu)$ , on the one hand, and  $\pi \in \text{PST}_{(A, B)}(\lambda/\beta)$  and  $\sigma \in \text{PST}_{(A', B')}(\lambda/\alpha)$ , on the other, such that  $\tau \cup \hat{l}(a) = \pi$  and  $\kappa \cup \hat{l}(a) = \sigma$ .

**Proof.**

Let the largest letter of  $\kappa \cup \widehat{l}(a)$  be  $n$ . We construct  $(\pi_r, \sigma_r)$  for  $r = 0, 1, \dots, n$  as follows. Start with  $(\pi_0, \sigma_0) = (\tau, \emptyset_\alpha)$ . Form  $\pi_r$  from  $\pi_{r-1}$  as follows.

Case 1 :  $r \in A'_u \cup B'_c$

At first we insert all the circled letters of  $\widehat{l}(a)$  paired with  $r$ 's in  $\widehat{l}(a)$ , where these circled letters are arranged in decreasing order. Next we internally insert all the letters of  $\pi_{r-1}$  corresponding to  $r$ 's in  $\sigma_{r-1}$ . If  $r \in A'_u$ , the insertion proceed left to right, and if  $r \in B'_c$ , the insertion proceed top to bottom. Finally we insert all the uncircled letters of  $\widehat{l}(a)$  paired with  $r$ 's in  $\widehat{l}(a)$ , where these uncircled letters are arranged in increasing order.

Case 2 :  $r \in A'_c \cup B'_u$

At first we insert all the uncircled letters of  $\widehat{l}(a)$  paired with  $r$ 's in  $\widehat{l}(a)$ , where these uncircled letters are arranged in decreasing order. Next we internally insert all the letters of  $\pi_{r-1}$  corresponding to  $r$ 's in  $\sigma_{r-1}$ . If  $r \in A'_c$ , the insertion proceed left to right, and if  $r \in B'_u$ , the insertion proceed top to bottom. Finally we insert all the circled letters of  $\widehat{l}(a)$  paired with  $r$ 's in  $\widehat{l}(a)$ , where these uncircled letters are arranged in increasing order.

In either case placing  $r$ 's in the appropriate cells of  $\sigma_{r-1}$  result in  $\sigma_r$ . It is not hard to see that the cells where  $r$ 's are placed are horizontal or vertical strip in  $\sigma_r$ . At last we put  $(\pi_n, \sigma_n) = (\pi, \sigma)$ . ■

**EXAMPLE 1.14**

Let  $A = \{1, \circ 2\}$ ,  $B = \{\circ 3, 4\}$ ,  $A' = \{1, \circ 3\}$ ,  $B' = \{2, \circ 4\}$ ,  $\alpha = (221)$  and  $\beta = (43)$ . Let  $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

so that the matrix word of  $a$  is  $l(a) = \begin{pmatrix} 1 & 2 & 2 & \circ 3 & \circ 3 & \circ 3 & \circ 4 \\ \circ 2 & 4 & \circ 2 & 1 & 1 & \circ 2 & \circ 3 \end{pmatrix}$ . Let

$$\tau = \begin{array}{|c|c|} \hline & 1 \\ \hline 1 & 4 \\ \hline \circ 3 & \\ \hline \end{array} \quad \kappa = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & \circ 4 \\ \hline 1 & \circ 3 & \circ 4 & \\ \hline \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & 1 & 1 & \circ 3 & \circ 4 \\ \hline & & & 1 & \circ 2 & & & \\ \hline 1 & \circ 2 & 4 & & & & & \\ \hline \circ 2 & \circ 3 & & & & & & \\ \hline \end{array} \quad \sigma = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & 1 & 1 & 2 & \circ 3 & \circ 3 & \circ 4 \\ \hline & & 2 & \circ 3 & \circ 3 & & & \\ \hline & 1 & \circ 4 & & & & & \\ \hline 2 & \circ 4 & & & & & & \\ \hline \end{array}$$

**COROLLARY 1.3**

Fix  $A$  and its division  $(A, B)$ . Fix another  $A'$  and its division  $(A', B')$ . Let  $\alpha$  and  $\beta$  be fixed partitions. Then

$$\begin{aligned} & \sum_{\lambda} HS_{\lambda/\beta}^{(A,B)}(x) HS_{\lambda/\alpha}^{(A',B')}(y) \\ &= \sum_{\mu} HS_{\alpha/\mu}^{(A,B)}(x) HS_{\beta/\mu}^{(A',B')}(y) \prod_{(i,j) \in A \times A' \cup B \times B'} \frac{1}{1 - x_i y_j} \prod_{(i,j) \in A \times B' \cup B \times A'} (1 + x_i y_j) \end{aligned}$$

**THEOREM 1.4**

Let  $A = A'$ ,  $(A, B) = (A', B')$ ,  $(U, C) = (U', C')$  and  $\alpha = \beta$  in Theorem 4.3. If  $(a, \tau, \kappa)$  correspond to  $(\pi, \sigma)$  by the bijection in Theorem 4.3 then  $(a^t, \kappa, \tau)$  correspond to  $(\sigma, \pi)$  by the same bijection.

**THEOREM 1.5**

Fix  $A$  and its divisions  $(U, C)$  and  $(A, B)$ . Let  $\alpha$  be a fixed partition. Then the mapping in Theorem 4.3 restricts to a bijection

$$(a, \tau) \leftrightarrow \pi$$

where  $a \in \mathcal{M}(A, B, A, B)$  is a symmetric matrix,  $\tau \in \text{PST}_{(A, B)}(\alpha/\mu)$ ,  $\pi \in \text{PST}_{(A, B)}(\lambda/\mu)$ , and  $\bar{l}(a) \cup \tau = \pi$ . In this bijection we always have

$$\text{tr}_{(A, B)} a + \text{odd}(\mu) = \text{odd}(\lambda)$$

**EXAMPLE 1.15**

Let  $A = \{1, {}^\circ 3\}$ ,  $B = \{{}^\circ 2, 4\}$  and  $\alpha = (221)$ . Let  $a = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$  so that the matrix word of  $a$  is

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ 2 & {}^\circ 2 & {}^\circ 2 & {}^\circ 3 & 4 \\ {}^\circ 3 & {}^\circ 2 & 4 & {}^\circ 2 & {}^\circ 2 & 1 & 1 & 1 \end{pmatrix}. \text{ Let}$$

$$\tau = \begin{array}{|c|c|} \hline & \\ \hline & 4 \\ \hline {}^\circ 3 & \\ \hline \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & {}^\circ 2 & {}^\circ 3 \\ \hline & 1 & {}^\circ 2 & 4 \\ \hline {}^\circ 2 & {}^\circ 3 & 4 & \\ \hline \end{array}$$

**COROLLARY 1.4**

Fix  $A$  and its division  $(A, B)$ . Let  $\alpha$  be a fixed partition.

$$\begin{aligned} & \sum_{\lambda} HS_{\lambda/\alpha}^{(A, B)}(x) t^{\text{odd}(\lambda)} \\ &= \sum_{\mu} HS_{\alpha/\mu}^{(A, B)}(x) t^{\text{odd}(\mu)} \prod_{\substack{(i, j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i, j) \in A \times B} (1 + x_i x_j) \prod_{i \in A} \frac{1}{1 - t x_i} \prod_{i \in B} \frac{1 + t x_i}{1 - x_i^2} \end{aligned}$$

In particular,

$$\sum_{\lambda' \text{ even}} HS_{\lambda'/\alpha}^{(A, B)}(x) = \sum_{\mu' \text{ even}} HS_{\alpha/\mu'}^{(A, B)}(x) \prod_{\substack{(i, j) \in A \times A \cup B \times B \\ i < j}} \frac{1}{1 - x_i x_j} \prod_{(i, j) \in A \times B} (1 + x_i x_j) \prod_{i \in B} \frac{1}{1 - x_i^2}$$

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