

## Bifurcation Equations of 3-Dimensional Piecewise-Linear Vector Fields

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### ABSTRACT

Global bifurcation equations of 3-dimensional 3-region continuous piecewise-linear vector fields with odd symmetry are given.

The bifurcation equations cover all sort of bifurcation sets, i.e., homoclinicity, heteroclinicity, saddle-node bifurcation, period-doubling bifurcation and Hopf bifurcation.

#### 1. Normal forms for 3-dimensional 3-region systems.

Consider a 3-dimensional 3-region continuous piecewise linear vector field  $f: \mathbb{R}^3 \dashrightarrow \mathbb{R}^3$  with odd symmetry  $f(-x) = -f(x)$ . Assume  $f$  has no proper linear subspace which is invariant under linear vector field in the middle region, and which is parallel to the boundary. If  $f$  has eigenvalues  $\mu_1, \mu_2, \mu_3$  at the middle region, and  $\nu_1, \nu_2, \nu_3$  at the outer regions, the normal form theorem for piecewise linear vector fields guarantees that such a vector field is uniquely determined up to the linearly conjugacy as follows;

$$f(x) = Ax + \frac{1}{2} p \{ |\langle \alpha, x \rangle - 1| + (\langle \alpha, x \rangle - 1) \}$$

$$\begin{aligned}
& - \frac{1}{2} p (|\langle \alpha, x \rangle + 1| - (\langle \alpha, x \rangle + 1)) \\
& = \begin{cases} B(x-P) & (x \in R_+) \\ Ax & (x \in R_0) \\ B(x+P) & (x \in R_-) \end{cases} \quad (1.1)
\end{aligned}$$

where

$$R_{\pm} = \{x \in \mathbb{R}^3 : \pm \langle \alpha, x \rangle - 1 > 0\},$$

$$R_0 = \{x \in \mathbb{R}^3 : |\langle \alpha, x \rangle| \leq 1\},$$

$$\alpha = {}^T(1, 0, 0), \quad p = {}^T(c_1, c_2, c_3),$$

$$P = {}^T(1 - a_3/b_3, c_1 a_3/b_3, c_2 a_3/b_3),$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ c_3 + a_3 & a_2 & a_1 \end{pmatrix} = A + p {}^T \alpha$$

$$a_1 = \mu_1 + \mu_2 + \mu_3, \quad a_2 = -(\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1), \quad a_3 = \mu_1 \mu_2 \mu_3,$$

$$b_1 = v_1 + v_2 + v_3, \quad b_2 = -(v_1 v_2 + v_2 v_3 + v_3 v_1), \quad b_3 = v_1 v_2 v_3,$$

$$c_1 = b_1 - a_1, \quad c_2 = b_2 - a_2 + c_1 a_1, \quad c_3 = b_3 - a_3 + a_2 c_1 + a_1 c_2.$$

Define the boundaries  $U_1$  and  $U_{-1}$  by

$$U_{\pm 1} = \{x \in \mathbb{R}^3 : \langle \alpha, x \rangle = \pm 1\}.$$

We separate  $U_1$  and  $U_{-1}$  into two parts respectively;

$$U_1^+ = \{x \in U_1 : \langle \alpha, Ax \rangle > 0\}, \quad U_1^- = \{x \in U_1 : \langle \alpha, Ax \rangle < 0\},$$

$$U_{-1}^+ = \{x \in U_{-1} : \langle -\alpha, Ax \rangle > 0\}, \quad U_{-1}^- = \{x \in U_{-1} : \langle -\alpha, Ax \rangle < 0\}.$$

Define points  $C_i \in U_1$  ( $i=1,2,3$ ) by

$$C_i = {}^T(1, \mu_i, \mu_i^2), \quad (1.2)$$

then the vector  $\overrightarrow{OC_i}$  is a eigenvector of A with respect to  $\mu_i$  ( $i=1,2,3$ ). Define points  $D_i \in U_1$  ( $i=1,2,3$ ) by

$$D_i = {}^T(1, v_i a_3/b_3, v_i(v_i - c_1)a_3/b_3), \quad (1.3)$$

then the vector  $\overrightarrow{PD_i}$  is a eigenvector of B with respect to  $v_i$  ( $i=1,2,3$ ).

## 2. Return time coordinate.

Assume the matrices A and B are regular.

(1) For  $x \in U_1^+$ , assume that there are  $y, z \in U_1^-$  such that

$$y = e^{-At}x \quad \text{where } t = \inf\{t' > 0 : |\langle e_1, e^{-At'}x \rangle| = 1\}$$

$$z = e^{Bs}(x-P)+P \quad \text{where } s = \inf\{s' > 0 : \langle e_1, e^{Bs'}(x-P)+P \rangle = 1\}.$$

Since  $Aw = B(w-P)$  for all  $w \in U_1$  by continuity of the vector field we have

$$\begin{aligned} z &= A^{-1}A(e^{Bs}(x-P)+P) = A^{-1}Be^{Bs}(x-P) = A^{-1}e^{Bs}B(x-P) \\ &= A^{-1}e^{Bs}Ax = e^{Cs}x \end{aligned}$$

where  $C = A^{-1}BA$ . Since  $x, y$  and  $z$  belong to  $U_1$ ,

$${}^T\alpha e^{-At}x = 1, \quad {}^T\alpha x = 1, \quad {}^T\alpha e^{Cs}x = 1,$$

or equivalently

$$[e_1 {}^T\alpha e^{-At} + e_2 {}^T\alpha + e_3 {}^T\alpha e^{Cs}]x = h_1$$

where  $e_1 = {}^T(1,0,0)$ ,  $e_2 = {}^T(0,1,0)$ ,  $e_3 = {}^T(0,0,1)$ ,  $h_1 = {}^T(1,1,1)$ .

If the matrix [...] is regular, we denote the inverse matrix by

$$K(t,s) = [e_1 {}^T\alpha e^{-At} + e_2 {}^T\alpha + e_3 {}^T\alpha e^{Cs}]^{-1}. \quad (2.1)$$

Then we have

$$x = K(t,s)h_1. \quad (2.2)$$

The  $(t,s)$  is called the *return time coordinate* of  $x$ .

(2) For  $x \in U_1^+$ , assume that there are  $y \in U_1^-$  and  $z \in U_1^-$  such that

$$y = e^{-At}x \quad \text{where } t = \inf\{t' > 0 : |\langle e_1, e^{-At'}x \rangle| = 1\}$$

$$z = e^{Bs}(x-P)+P \quad \text{where } s = \inf\{s' > 0 : \langle e_1, e^{Bs'}(x-P)+P \rangle = 1\}.$$

Since  $Aw = B(w-P)$  for all  $w \in U_1$  by continuity of the vector field, we have

$$\begin{aligned} z &= A^{-1}A(e^{Bs}(x-P)+P) = A^{-1}Be^{Bs}(x-P) = A^{-1}e^{Bs}B(x-P) \\ &= A^{-1}e^{Bs}Ax = e^{Cs}x \end{aligned}$$

where  $C = A^{-1}BA$ . Since  $x, z \in U_1$  and  $y \in U_1^-$ ,

$$T_{\alpha}e^{-At}x = -1, \quad T_{\alpha}x = 1, \quad T_{\alpha}e^{Cs}x = 1,$$

or equivalently

$$[e_1^T \alpha e^{-At} + e_2^T \alpha + e_3^T \alpha e^{Cs}]x = h_2 \quad (2.3)$$

where  $e_1 = {}^T(1, 0, 0)$ ,  $e_2 = {}^T(0, 1, 0)$ ,  $e_3 = {}^T(0, 0, 1)$ ,  $h_2 = {}^T(-1, 1, 1)$ .

If the matrix [...] is regular, we denote the inverse matrix by

$$K(t, s) = [e_1^T \alpha e^{-At} + e_2^T \alpha + e_3^T \alpha e^{Cs}]^{-1}.$$

Then we have

$$x = K(t, s)h_2. \quad (2.4)$$

The  $(t, s)$  is also called the *return time coordinate* of  $x$ .

### 3. Periodic orbit.

(1) The type of  $U_1 \rightarrow U_1$ .

Assume  $x \in U_1^+$  has a return time coordinate  $(t_1, s_1)$  as follows;

$$x = K(t_1, s_1)h_1 \quad \text{where}$$

$$t_1 = \inf\{t > 0 : |\langle e_1, e^{-At}x \rangle| = 1\} \quad \text{and} \quad (3.1a)$$

$$s_1 = \inf\{s > 0 : \langle e_1, e^{Bs}(x-P)+P \rangle = 1\}. \quad (3.1b)$$

If an equation between 6 eigenvalues  $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$  and  $(t_1, s_1)$  defined by

$$e^{-At_1} K(t_1, s_1) = e^{Cs_1} K(t_1, s_1) \quad (3.2)$$

holds under the open condition (3.1), the point  $x = K(t_1, s_1)$  is a periodic point for a vector field determined by  $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$ . This periodic point is called of the type  $U_1 \rightarrow U_1$ . For the other type of periodic points, the equations are derived by similar way.

(2) The type of  $U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1$ .

$$e^{Cs_1} K(t_1, s_1) h_1 = e^{-At_2} K(t_2, s_2) h_1, \quad (3.3a)$$

$$e^{Cs_2} K(t_2, s_2) h_1 = e^{-At_1} K(t_1, s_1) h_1. \quad (3.3b)$$

The open conditions for the first return times;

$$\begin{aligned} |T_{\alpha e^{-At}} K(t_i, s_i) h_1| &\neq 1 \text{ for all } t \text{ with } 0 < t < t_i, \\ T_{\alpha e^{Cs}} K(t_i, s_i) h_1 &\neq 1 \text{ for all } s \text{ with } 0 < s < s_i \quad (i = 1, 2). \end{aligned}$$

(3) The type of  $U_1 \rightarrow U_1 \rightarrow U_{-1} \rightarrow U_{-1} \rightarrow U_1$ .

$$e^{Cs_1} K(t_1, s_1) h_2 = -e^{-At_2} K(t_2, s_2) h_2, \quad (3.4a)$$

$$-e^{Cs_2} K(t_2, s_2) h_2 = e^{-At_1} K(t_1, s_1) h_2. \quad (3.4b)$$

The open conditions for the first return times;

$$\begin{aligned} |T_{\alpha e^{-At}} K(t_i, s_i) h_2| &\neq 1 \text{ for all } t \text{ with } 0 < t < t_i, \\ T_{\alpha e^{Cs}} K(t_i, s_i) h_2 &\neq 1 \text{ for all } s \text{ with } 0 < s < s_i \quad (i = 1, 2). \end{aligned}$$

(4) The type of  $U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_{-1} \rightarrow U_{-1} \rightarrow U_1$ .

$$e^{Cs_1} K(t_1, s_1) h_2 = e^{-At_2} K(t_2, s_2) h_1, \quad (3.5a)$$

$$e^{Cs_2} K(t_2, s_2) h_1 = -e^{-At_3} K(t_3, s_3) h_2, \quad (3.5b)$$

$$-e^{Cs_3} K(t_3, s_3) h_2 = e^{-At_1} K(t_1, s_1) h_2. \quad (3.5c)$$

The open conditions for first return times;

$$|T_{\alpha} e^{-At} K(t_i, s_i) h_j| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_i,$$

$$T_{\alpha} e^{Cs} K(t_i, s_i) h_j \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_i \quad (i = 1, 2),$$

where  $j = 1$  ( $i = 2$ );  $j = 2$  ( $i = 1, 3$ ).

#### 4. Bifurcation conditions of periodic orbits.

Let  $x = K(t_1, s_1) h_1$  be a periodic point with a period  $t_1 + s_1 + \dots + t_n + s_n$  as stated in § 3. Then bifurcation conditions for this periodic point are given as follows. Set

$$M = e^{Bs_n} e^{At_n} e^{Bs_{n-1}} e^{At_{n-1}} \dots e^{Bs_1} e^{At_1},$$

$$T = \text{Trace}(M) \quad \text{and} \quad D = \det(M).$$

(1) *Saddle-node bifurcation;*

$$2 - T + D = 0. \quad (4.1)$$

(2) *Period doubling bifurcation;*

$$T + D = 0. \quad (4.2)$$

(3) *Hopf bifurcation;*

$$D - 1 = 0 \quad \text{and} \quad (1-T)^2 - 4 < 0. \quad (4.3)$$

#### 5. Homoclinic orbits at 0.

Suppose  $\mu_1$  is positive real, and  $\mu_2$  and  $\mu_3$  are negative reals or complex conjugate pair with negative real part. Since an eigen vector for  $\mu_1$  is given by (1.2), a 2-dimensional stable manifold  $W^s(0)$  and a 1-dimensional unstable manifold  $W^u(0)$  for  $0 = T(0, 0, 0)$  are given as follows;

$$W^u(0) = \{ x \in \mathbb{R}^3 : x = r \overrightarrow{OC}_1, T_{\alpha} x - 1 \leq 0, r \in \mathbb{R} \},$$

$$W^s(0) = \{ x \in \mathbb{R}^3 : x = r\vec{OC}_2 + r'\vec{OC}_3, T_{\alpha x-1} \leq 0, r, r' \in \mathbb{R} \}.$$

Set

$$u = T(0, (\mu_2 + \mu_3)/(\mu_2 \mu_3), -1/(\mu_2 \mu_3)),$$

then the intersection  $W^s(0) \cap U_1$  is given by

$$W^s(0) \cap U_1 = \{x=(x,y,z) \in \mathbb{R}^3 : T_{ux-1} = 0, x=1\}.$$

Then a condition under which a homoclinic orbit at 0 exists is given as follows.

(1) The type of  $0 \rightarrow U_1 \rightarrow U_1 \rightarrow 0$ .

$$T_{\alpha e}^{Cs_0} C_1 = 1, \quad (5.1a)$$

$$T_{ue}^{Cs_0} C_1 = 1. \quad (5.1b)$$

The open conditions for the first return times;

$$T_{\alpha e}^{Cs} C_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$T_{\alpha e}^{At} e^{Cs_0} C_1 \neq 1 \text{ for all } t > 0.$$

(2) The type of  $0 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow 0$ .

$$T_{\alpha e}^{Cs_0} C_1 = 1, \quad (5.2a)$$

$$e^{Cs_0} C_1 = e^{-At_1} K(t_1, s_1) h_1, \quad (5.2b)$$

$$T_{ue}^{Cs_1} K(t_1, s_1) h_1 = 1. \quad (5.2c)$$

The open conditions for the first return times;

$$T_{\alpha e}^{Cs} C_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$|T_{\alpha e}^{-At} K(t_1, s_1) h_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1,$$

$$T_{\alpha e}^{Cs} K(t_1, s_1) h_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_1.$$

$$|T_{\alpha e}^{At} e^{Cs_1} K(t_1, s_1) h_1| \neq 1 \quad \text{for all } t > 0.$$

(3) The type of  $0 \rightarrow U_1 \rightarrow U_1 \rightarrow U_{-1} \rightarrow U_{-1} \rightarrow 0$ .

$$T_{\alpha e}^{Cs_0} C_1 = 1, \quad (5.3a)$$

$$e^{Cs_0} C_1 = -e^{-At_1} K(t_1, s_1) h_2, \quad (5.3b)$$

$$T_{ue}^{Cs_1} K(t_1, s_1) h_2 = 1. \quad (5.3c)$$

The open conditions for the first return times;

$$T_{\alpha e}^{Cs} C_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$|T_{\alpha e}^{-At} K(t_1, s_1) h_2| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1,$$

$$T_{\alpha e}^{Cs} K(t_1, s_1) h_2 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_1.$$

$$|T_{\alpha e}^{At} e^{Cs_1} K(t_1, s_1) h_2| \neq 1 \quad \text{for all } t > 0.$$

## 6. Homoclinic orbits at $P^+$ .

For eigenvalues of  $B$ , assume  $v_1$  is negative real, and  $v_2$  and  $v_3$  are positive reals or complex conjugate pair with positive real part. Since an eigen vector for  $v_1$  is given by (1.3), a 1-dimensional stable manifold  $W^s(P)$  and a 2-dimensional unstable manifold  $W^u(P)$  for  $P$  are given as follows;

$$W^s(P) = \{ x \in \mathbb{R}^3 : x = r \overline{PD}_1 + P, T_{\alpha x-1} \geq 0, r \in \mathbb{R} \},$$

$$W^u(P) = \{ x \in \mathbb{R}^3 : x = r \overline{PD}_2 + r' \overline{PD}_3 + P, T_{\alpha x-1} \geq 0, r, r' \in \mathbb{R} \}.$$

Set

$$v = T(0, (v_2 + v_3 - c_1) b_3 / (v_2 v_3 a_3), -b_3 / (v_2 v_3 a_3)),$$

then the intersection  $W^u(P) \cap U_1$  is given by

$$W^u(P) \cap U_1 = \{ x = (x, y, z) \in \mathbb{R}^3 : T_{vx-1} = 0, x=1 \}.$$

Set

$$P^+ = P \quad \text{and} \quad P^- = -P.$$

Then a condition under which a homoclinic orbit at  $P^+$  exists is given as follows.

(1) The type of  $P^+ \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow P^+$ .

$$T_{\alpha} e^{-At_1} K(t_1, s_1) h_1 = 1, \quad (6.1a)$$

$$e^{Cs_1} K(t_1, s_1) h_1 = e^{-At_2} D_1. \quad (6.1b)$$

The open conditions for the first return times;

$$T_{\alpha} e^{Cs} e^{-At_1} K(t_1, s_1) h_1 \neq 1 \text{ for all } s > 0,$$

$$|T_{\alpha} e^{-At} K(t_1, s_1) h_1| \neq 1 \text{ for all } t \text{ with } 0 < t < t_1,$$

$$T_{\alpha} e^{Cs} K(t_1, s_1) h_1 \neq 1 \text{ for all } s \text{ with } 0 < s < s_1,$$

$$|T_{\alpha} e^{-At} D_1| \neq 1 \text{ for all } t \text{ with } 0 < t < t_2.$$

## 7. Heteroclinic orbits.

For eigenvalues of  $A$ , assume  $\mu_1$  is positive real, and  $\mu_2$  and  $\mu_3$  are negative reals or complex conjugate pair with negative real part. For eigenvalues of  $B$ , assume  $\nu_1$  is negative real, and  $\nu_2$  and  $\nu_3$  are positive reals or complex conjugate pair with positive real part. Then a condition under which a heteroclinic orbit from  $O$  or  $P^-$  to  $P^+$  exists is given as follows.

(1) The type of  $O \rightarrow U_1 \rightarrow P^+$ .

$$C_1 = D_1. \quad (7.1)$$

(2) The type of  $O \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow P^+$ .

$$T_{\alpha} e^{Cs_0} C_1 = 1, \quad (7.2a)$$

$$e^{Cs_0} C_1 = e^{-At_1} D_1, \quad (7.2b)$$

$$T_{\alpha e}^{-At_1} D_1 = 1. \quad (7.2c)$$

The open conditions for the first return times;

$$T_{\alpha e}^{Cs} C_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$|T_{\alpha e}^{-At} D_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1.$$

(3) The type of  $0 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow U_1 \rightarrow P^+$ .

$$T_{\alpha e}^{Cs_0} C_1 = 1, \quad (7.3a)$$

$$e^{Cs_0} C_1 = e^{-At_1} K(t_1, s_1) h_1, \quad (7.3b)$$

$$e^{Cs_1} K(t_1, s_1) h_1 = e^{-At_2} D_1, \quad (7.3c)$$

$$T_{\alpha e}^{-At_2} D_1 = 1. \quad (7.3d)$$

The open conditions for the first return times;

$$T_{\alpha e}^{Cs} C_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$|T_{\alpha e}^{-At} K(t_1, s_1) h_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1,$$

$$T_{\alpha e}^{Cs} K(t_1, s_1) h_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_1.$$

$$|T_{\alpha e}^{-At} D_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_2.$$

(4) The type of  $P^- \rightarrow U_{-1} \rightarrow U_1 \rightarrow P^+$ .

$$T_{\alpha e}^{-At_0} C_1 = 1, \quad (7.4a)$$

$$T_{ue}^{-At_0} C_1 = -1. \quad (7.4b)$$

The open conditions for the first return times;

$$|T_{\alpha e}^{-At} C_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_0.$$

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