

On a non-reflexive embedding with birational Gauss map for a projective variety

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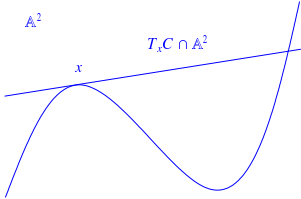
1. WHAT IS THE GAUSS MAP ?

K : alg. closed field of char. $p \geq 0$

Simple Case

$C \subset \mathbb{P}^2$; plane curve
 $x \in C_{\text{sm}}$; smooth point

$T_x C \subset \mathbb{P}^2$; the projective tangent line at x



$T_x C$ =the projective closure of the tangent line (passing through x)

Linearity Theorem

$p = 0 \Rightarrow$ general fiber of γ : linear space.

e.g. X : curve $\Rightarrow$ general fiber of γ : **1pt.**

Linearity Theorem fails in char $p > 0$!

Example.

$p > 2$.

$f: \mathbb{A}^1 \rightarrow \mathbb{P}^2; t \mapsto (1 : t : t^{2p})$

Gauss map:

$$\begin{pmatrix} 1 & x & x^{2p} \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & x^{2p} \\ 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ general fiber of γ : **2pts.** [$\gamma(x) = \gamma(-x)$]

Theorem (F [1])

$p > 0$.

$\forall Y \subset \mathbb{P}^k$ proj. variety, $\exists X \subset \mathbb{P}^N$ proj. variety s.t.

Y = general fiber of γ .

Why such phenomena occur ?

$d\gamma \equiv 0$

i.e. $\times$ The generic smoothness (Sard's Theorem) of γ

General Case

$X \subset \mathbb{P}^N$; projective variety

$x \in X_{\text{sm}}$

$T_x X \subset \mathbb{P}^N$; the projective tangent space at x

Gauss map $\gamma: X \rightarrow \mathbb{G}(n, \mathbb{P}^N) \cong \mathbb{G}(N - n - 1, \mathbb{P}^{N*})$

$$x \mapsto T_x X \mapsto (T_x X)^*$$

Calculation

$x_1, \dots, x_n; \text{local coordinate around } x \in X$

$\rho = (1 : x_1 : \dots : x_n : f_{n+1} : \dots : f_N) : X \hookrightarrow \mathbb{P}^N$

$T_x X \in \mathbb{G}(n, \mathbb{P}^N)$:

$$\begin{pmatrix} \rho \\ \partial \rho / \partial x_1 \\ \vdots \\ \partial \rho / \partial x_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & f_{n+1} & \cdots & f_N \\ 0 & 1 & 0 & \cdots & 0 & f_{n+1, x_1} & \cdots & f_{N, x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & f_{n+1, x_n} & \cdots & f_{N, x_n} \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & f_{n+1} - \sum_{j=1}^n x_j f_{n+1, x_j} & \cdots & f_N \\ 0 & 1 & 0 & \cdots & 0 & f_{n+1, x_1} & \cdots & f_{N, x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & f_{n+1, x_n} & \cdots & f_{N, x_n} \end{pmatrix}$$

3. REFLEXIVITY & KLEIMAN-PIENE'S QUESTION

$X \subset \mathbb{P}^N$ proj. variety of dim n

$CX := \{(x, H) \in X_{\text{sm}} \times \mathbb{P}^{N*} | T_x X \subset H\} \subset \mathbb{P}^N \times \mathbb{P}^{N*}$

conormal variety

$\pi: CX \rightarrow \mathbb{P}^{N*}$ conormal map

$X^* := \pi(CX) \subset \mathbb{P}^{N*}$ dual variety

$CX^* \subset \mathbb{P}^{N**} \times \mathbb{P}^{N*} = \mathbb{P}^N \times \mathbb{P}^{N*}$; conormal variety / X^*

Definition.

X is called **reflexive** if $CX = CX^*$.

Monge-Segre-Wallace criterion

X is reflexive $\Leftrightarrow \pi: CX \rightarrow X^*$ is **generically smooth**.

Proposition.

X : reflexive $\Rightarrow$ "Usual" Geometry;

(1) (Projective Duality) $X^{**} = X$.

(2) (Linearity Theorem)

general fiber of γ is linear (scheme-theoretically).

Especially, γ is generically smooth.

Relationship between π & γ

$$\begin{array}{ccc} CX & \xrightarrow{\gamma'} I_{\gamma(X)} & \xrightarrow{\pi} X^* \\ \downarrow \square & & \downarrow \\ X & \xrightarrow{\gamma} \gamma(X) \subset \mathbb{G}(N - n - 1, \mathbb{P}^{N*}) & \end{array}$$

$I_{\gamma(X)} = \{(E, H) \in \gamma(X) \times \mathbb{P}^{N*} | E \subset H\}$

$\gamma'(x, H) = (T_x X, H)$

dim $X = 1$ insep.deg of $\gamma =$ insep.deg of π .

$\Rightarrow \pi': I_{\gamma(X)} \rightarrow X^*$; generically smooth

(Kaji, Hefez, Voloch,...)

Kleiman-Piene's question ([7, pp.108–109])

Is $\pi': I_{\gamma(X)} \rightarrow X^*$ separable in higher dimension ?

KPQ: affirmative $\Rightarrow$

"gen. smoothness of $\gamma \Rightarrow$ gen. smoothness of π "
($\Leftrightarrow$ reflexivity)

dim $X = 2$

Theorem (F-Kaji [3]). Assume that $\dim X = 2$.

- γ : gen. smooth $\Rightarrow X$: reflexive
- $\exists X$ s.t. $\pi': I_{\gamma(X)} \rightarrow X^*$: inseparable.

Next Step:

What kind of variety has non-reflexive embedding with gen. smooth Gauss map ?

Main Theorem (F-Kaji [4])

$X \subset \mathbb{P}^N$; proj. variety of dim ≥ 3 , not linear.

$d\gamma \neq 0$

$\Rightarrow \exists \tau: X \hookrightarrow \mathbb{P}^M$ s.t.

$\tau(X)$: **non-reflexive** & $\gamma(\tau)$: **birational**

[$d\gamma \equiv 0$: The differential of γ is identically zero]

Example (of $d\gamma \equiv 0$)

- (The graph of a Frobenius map of) the projective space $\mathbb{P}^n$
- Fermat hypersurface of degree d with $d \equiv 1 \pmod{p}$

2. NON-REFLEXIVE EMBEDDING WITH BIRATIONAL GAUSS MAP (NREBG)

Answer to KPQ: **No !**

Counterexamples of dim ≥ 3

Example (Kaji [5, 6]).

Segre variety $X = \prod_{i=1}^r \mathbb{P}^{n_i} \xrightarrow{\text{Segre}} \mathbb{P}^M$;

($n := \dim X = \sum_{i=1}^r n_i$)

- $r \geq 2 \Rightarrow \gamma: X \rightarrow \gamma(X)$ is **isomorphic**.
- X : **non-reflexive** $\Leftrightarrow \begin{cases} 2n_i < n \text{ for } \forall i \\ n : \text{odd} \\ p = 2 \end{cases}$

Example (F [2]).

$p > 0$.

$Y \subset \mathbb{P}^n: X_0^{p+1} + \dots + X_n^{p+1} = 0$

$X = Y \times \mathbb{P}^1 \xrightarrow{\text{Segre}} \mathbb{P}^{2n+1}$

- $n \geq 2 \Rightarrow \gamma: X \rightarrow \gamma(X)$ is **isomorphic**.
- $n \geq 3 \Rightarrow X$: **non-reflexive**.

Open Question ([6, Question 2]).

γ : gen. smooth $\Rightarrow$ gen. fibers of γ : linear ?

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