

Applications of algebraic geometry

to

combinatorics

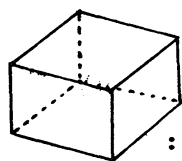
By

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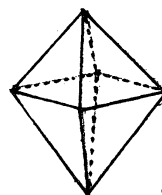
(Notes by Y. Matsuura)

In this lecture I would like to explain Stanley's idea in the application of algebraic geometry to the problem of characterizing the f -vectors of simplicial convex polytopes.

Let P be a compact convex polytope in the d -dimensional Euclidean space $\mathbb{R}^d$. P is called simplicial if every face of P except P itself is a simplex.



: not simplicial ,



: simplicial .

Let $(f_0, f_1, \dots, f_{d-1})$ be a sequence of non-negative integers. If there exists a simplicial convex d -polytope P with f_i faces of dimension i for $1 \leq i \leq d-1$, then we call $(f_0, \dots, f_{d-1})$ the f -vector of P .

Problem. Characterize the possible f -vectors.

More precisely, we want to find necessary and sufficient conditions on a sequence of integers for the existence of a simplicial convex polytope with those number of faces.

Before describing an answer to this problem, we give some definitions. For an f -vector $(f_0, f_1, \dots, f_{d-1})$ of a simplicial

convex d -polytope P , we define

$$h_i = \sum_{j=0}^i \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}$$

where we set $f_{-1} = 1$. The sequence of integers $(h_0, h_1, \dots, h_d)$ is called the h -vector of P .

If k and i are positive integers, then k can be written uniquely in the form

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \dots + \binom{n_j}{j}$$

where $n_i > n_{i-1} > \dots > n_j \geq j \geq 1$. We define

$$k^{<i>} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \dots + \binom{n_j+1}{j+1}$$

with $0^{<i>} = 0$.

A sequence $(k_0, k_1, \dots, k_d)$ of non-negative integers is called an M -vector if $k_0 = 1$ and $0 \leq k_{i+1} \leq k_i^{<i>}$ for $1 \leq i \leq d-1$.

The conjecture of McMullen, now a theorem, is stated as follows :

Theorem. A sequence $(h_0, h_1, \dots, h_d)$ of integers is the h -vector of a simplicial convex d -polytope if and only if the following conditions are satisfied :

- 1) $h_i = h_{d-i}$ for $0 \leq i \leq d$ (the Dehn - Sommerville equations),
- 2) $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_{[d/2]} - h_{[d/2]-1})$ is an M -vector.

The sufficiency of McMullen's conditions was proved by Billera and Lee [6]. Soon after, Stanley proved the necessity in 1980.

Remark. 1) The notions of f -vector and h -vector make sense for

any triangulation of the sphere S^{d-1} .

2) The Dehn - Sommerville equation $h_0 = h_d$ for $i = 0$ is just the famous Euler - Poincaré formula.

3) It is known that the Dehn - Sommerville equations continue to hold for any triangulation of the sphere S^{d-1} .

Stanley's Proof of the necessity

We recall the following result essentially due to Macaulay:

(Macaulay). A sequence $(k_0, k_1, \dots, k_d)$ of integers is an M -vector if and only if there exists a finitely generated commutative graded algebra R over a field K , generated by R_1 , such that $\dim_K R_i = k_i$.

By this fact, it is sufficient to find a homogeneous algebra R such that $\dim_K R_i = h_i - h_{i-1}$ for $0 \leq i \leq [d/2]$.

We are now in a position to explain Stanley's argument. He realized this algebra R as the cohomology ring of a certain projective variety X obtained from a given simplicial convex polytope.

Let P be a simplicial convex d -polytope. We may 1) assume P is embedded in $\mathbb{R}^d$, 2) assume the origin is in the interior and 3) change the vertices a little so that they have rational coordinates. --- (This does not affect the combinatorial structure of P .) ----- For each face of P , we consider the rational convex polyhedral cone, obtained as the union of all rays from the origin through points of that face. These cones form a complete (the union is $\mathbb{R}^d$) simplicial fan. Then, by the theory of toric varieties, we obtain a projective variety X of

dimension d :

$$X = \text{dfn. } \bigcup_C \text{Spec}(R_C)$$

where C runs over all cones of the above fan

$$\text{and } R_C = \mathbb{C}[x_1^{t_1} \cdot x_2^{t_2} \cdot \dots \cdot x_d^{t_d} \mid c = (c_1, c_2, \dots, c_d) \in C, \\ \sum_i c_i t_i \geq 0].$$

X is locally a quotient of $\mathbb{C}^d$ by a finite group action. Hence, in general, X is singular. Fortunately, it is known that there are two nice facts on the structure of the cohomology groups of X :

The first is :

(Danilov [4]). 1) $H^{2i+1}(X, \mathbb{C}) = 0$ for $0 < 2i+1 < 2d$,

2) $\dim_{\mathbb{C}} H^{2i}(X, \mathbb{C}) = h_i$ for $0 \leq 2i \leq 2d$, where $(h_0, h_1, \dots, h_d)$ is the h -vector of P .

From Poincare duality for the complex cohomology of X it follows that the Dehn - Sommerville equations hold for the h -vector of P .

By setting $A_i = H^{2i}(X, \mathbb{C})$ we have a commutative graded algebra $A = \bigoplus_i A_i$ over $\mathbb{C}$, generated by A_1 , such that $\dim_{\mathbb{C}} A_i = h_i$.

The second is :

(Steenbrink [5]). The hard Lefschetz theorem holds for X .

This theorem means that there is an element $w \in H^2(X, \mathbb{C}) = A_1$

(the cohomology class of an ample divisor on X) such that

$w^{d-2i}: H^{2i}(X, \mathbb{C}) \longrightarrow H^{2d-2i}(X, \mathbb{C})$ given by the cup product with w^{d-2i} is an isomorphism for $0 \leq i \leq [d/2]$. In particular,

$w : A_i \longrightarrow A_{i+1}$ is injective for $0 \leq i \leq [d/2]$. Now let $R = A/wA$. Then $\dim_{\mathbb{C}} R_i = h_i - h_{i-1}$ for $i \leq [d/2]$. This completes Stanley's proof.

The upper bound conjecture

There is a theorem, conjectured by Motzkin, giving an upper bound for the number of i -dimensional faces f_i , fixing f_0 , of a finite simplicial complex.

Theorem (The upper bound conjecture). Let Δ be a triangulation of the sphere S^{d-1} with n vertices and $(h_0, h_1, \dots, h_d)$ be the h -vector of Δ . Then we have

$$h_i \leq \binom{n-d+i-1}{i} \quad \text{for } 0 \leq i \leq d.$$

(McMullen proved a somewhat more general statement in 1970.)

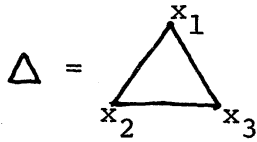
Stanley gave a different proof for this result by using the "Reisner - Stanley ring" theory in 1975. The sketch of his argument is as follows:

Let Δ be an abstract simplicial complex with vertices $\{x_1, x_2, \dots, x_n\}$ and K be a field. We associate with Δ a graded K -algebra $K[\Delta]$, called the Reisner - Stanley ring, in the following manner: Let $K[x_1, x_2, \dots, x_n]$ be the polynomial ring over K on the vertices of Δ . Let I_Δ be the ideal generated by all monomials $x_{i_1} x_{i_2} \dots x_{i_s}$ with $i_1 < i_2 < \dots < i_s$ and $\{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$ not a face of Δ . We define

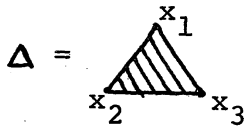
$$K[\Delta] = K[x_1, x_2, \dots, x_n] / I_\Delta$$

with $\deg x_i = 1$ for all i .

Examples.



$$K[\Delta] = K[x_1, x_2, x_3] / (x_1 x_2 x_3)$$



$$K[\Delta] = K[x_1, x_2, x_3]$$

$\text{Spec } K[\Delta]$ is the union of various coordinate t -planes with various t 's. planes.

To describe Reisner's theorem, we recall that if $\sigma \in \Delta$, then the link of σ is the subcomplex

$$\text{link } \sigma =_{\text{dfn.}} \{ \tau \in \Delta \mid \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset \}$$

In particular, $\text{link } \emptyset = \Delta$.

(Reisner). The following are equivalent:

- 1) $K[\Delta]$ is a Cohen-Macaulay ring,
- 2) $\tilde{H}^i(\text{link}(\sigma), K) = 0$ for $0 \leq i \leq \dim(\text{link}(\sigma)) - 1$ and for all $\sigma \in \Delta$,

where $\tilde{H}^i(*, K)$ is the reduced cohomology group with coefficients in K .

As an immediate consequence of this, we have the following result:

Corollary. If the geometric realization $|\Delta|$ of Δ is a sphere, then $K[\Delta]$ is Cohen-Macaulay.

Let $H_{K[\Delta]}(m)$ ($m \in \mathbb{N}$) denote the Hilbert function of $K[\Delta]$. It is known that

$$H_{K[\Delta]}(m) = \begin{cases} 1 & m = 0 \\ \sum_{i=0}^{d-1} \binom{m-1}{i} f_i & m > 0 \end{cases},$$

where $(f_0, f_1, \dots, f_{d-1})$ is the f -vector of Δ . Hence, it is easy to see that

$$(1-t)^d \sum_{m=0}^{\infty} H_{K[\Delta]}(m) t^m = h_0 + h_1 t + h_2 t^2 + \dots + h_d t^d.$$

By another result essentially due to Macaulay, it follows that if $K[\Delta]$ is Cohen-Macaulay then $(h_0, h_1, \dots, h_d)$ is an M -vector with $h_1 = n-d$. Finally Stanley showed that if $K[\Delta]$ is Cohen-Macaulay then the upper bound conjecture holds for Δ by using the notion of "an order ideal of monomials". In particular, if Δ is a triangulation of the sphere S^{d-1} , then the upper bound conjecture holds for Δ .

Open problems. 1) Find a simple proof of the necessity of McMullen's conditions.

2) Is $(h_0, h_1 - h_0, \dots, h_{[d/2]} - h_{[d/2]-1})$ always an M -vector for any triangulation of the sphere S^{d-1} ? (Note that we used the Hard Lefschetz theorem for the projective toric variety X . A triangulation of S^{d-1} not coming from a simplicial polytope also gives rise to a toric variety X in a similar manner. But X need not be projective in this case; hence Stanley's proof does not go through in general.)

References

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