

ON THE PARTIAL REGULARITY OF BOUNDED WEAK SOLUTIONS TO NONLINEAR DEGENERATE PARABOLIC SYSTEMS OF P-HARMONIC TYPE

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ABSTRACT. We establish partial regularity for bounded weak solutions of nonlinear parabolic systems of p-harmonic type. It's necessary to consider L^q -estimate for the spatial gradient of solutions by carefully using so-called Gehring inequality.

1. Introduction.

In this paper we establish Hölder estimates for bounded weak solutions to nonlinear degenerate parabolic systems of the form

$$\frac{\partial u^i}{\partial t} - \operatorname{div}(|Du|^{p-2} Du^i) = f^i(t, x, u, Du), \quad 1 \leq i \leq n \quad (1.1)$$

in an open set $Q = (0, T) \times \Omega \subset R^{m+1}$, $m \geq 2$. Here Ω is an open set in R^m , $x \in \Omega \subset R^m$, $t > 0$, T is a given positive number, $u = (u^1, u^2, \dots, u^n)$ is a mapping: $Q \rightarrow R^n$ and $Du = (D_1 u, D_2 u, \dots, D_m u)$, $D_\alpha u = \partial u / \partial x^\alpha$ ($1 \leq \alpha \leq m$) is the spatial gradient of u , p is any positive number satisfying

$$2 < p < \infty$$

and $f(t, x, u, p)$ is a Carathéodory function: $(0, T) \times \Omega \times R^n \times R^{nm} \rightarrow R^n$, satisfying the growth condition with some positive constant a

$$|f(t, x, u, p)| \leq a|p|^p \quad (1.2)$$

Let us introduce the parabolic metric with some positive constant θ

$$\delta_\theta(z_1, z_2) = \max(|x_1 - x_2|, |t_1 - t_2|^{1/\theta}), \quad z_i = (t_i, x_i), \quad i = 1, 2. \quad (1.3)$$

and denote by $H^k(\cdot, \delta_\theta)$ the k -dimensional Hausdorff measure with respect to δ_θ . Here we recall some function spaces: Hölder space $C^{0,\mu}(Q, \delta_\theta)$, denoted the spaces of Hölder continuous functions in Q (with respect to the metric δ_θ) with an exponent μ , the usual Lebesgue space $L^p(\Omega) = L^p(\Omega, R^n)$ and Sobolev spaces: $W_p^k(\Omega) = W_p^k(\Omega, R^n)$, $\dot{W}_p^k(\Omega) = \dot{W}_p^k(\Omega)(\Omega, R^n)$, $V_{2,p}(Q) = L^\infty((0, T); L^2(\Omega)) \cap L^p((0, T); W_p^1(\Omega))$, $W_{2,p}^{1,1}(Q) = W_2^1((0, T); L^2(\Omega)) \cap L^p((0, T); \dot{W}_p^1(\Omega))$. By a weak solution u of (1.1) in Q we mean a vector-valued function $u = (u^1, u^2, \dots, u^n) \in V_{2,p}(Q) \cap L^\infty(Q)$ satisfying (1.1) in the weak sense:

$$\iint_Q [-u^i \partial_t \varphi^i + |Du|^{p-2} Du^i D\varphi^i] dt dx = \iint_Q f^i \varphi^i dt dx \quad \text{for any } \varphi \in W_{2,p}^{1,1}(Q) \cup L^\infty(Q). \quad (1.4)$$

In (1.4) and in what follows, the summation notation over repeated indices is adopted.

Then our main theorem is the following:

Theorem. *Let u be a bounded weak solution of (1.1), set $M = \sup_Q |u|$ and assume that*

$$1 > 2aM. \quad (1.5)$$

Then there exist positive constants $\varepsilon, \beta, 0 < \beta < 1$, and an open set $Q_0 \subset Q$ such that $u \in C_{\text{loc}}^{0,\beta}(Q_0, \delta_2)$ and $H^{m-\varepsilon}(Q - Q_0, \delta_2) = 0$.

The proof of Theorem relies on a perturbation argument (see [8],[9],[13]) and an L^q -estimate for $|Du|$ which is of some interest in itself (refer to [9]). We prove such L^q -estimate by exploiting so-called Gehring-inequality in Sect.3 (see [8],[9]).

Remark. In a scalar case everywhere regularity for bounded weak solutions is established without assuming (1.5) (see [4],[14]). In a case of $p = 2$ the analogue result is obtained in [9], [10].

Some standard notations: For $z_0 = (t_0, x_0) \in Q$ and $r, \tau > 0$

$$B_r(x_0) = \{x \in R^n : |x - x_0| < r\}, \quad Q_{r,\tau}(z_0) = (t_0 - \tau, t_0) \times B_r(x_0).$$

For $\theta > 0$ and $z_0 \in Q$, $r > 0$ put the cylinders

$$Q_r^\theta(z_0) = (t_0 - r^\theta, t_0) \times B_r(x_0). \quad (1.6)$$

When $\theta = p$ we let $Q_r(z_0) = Q_r^p(z_0)$. In the above notations the center points x_0, z_0 are omitted when no confusion may arise. For an integrable function $f : Q \rightarrow \mathbb{R}^n$ and a measurable set $A \subset Q$

$$\bar{f}_A = \frac{1}{|A|} \int_A f dz$$

where $|A|$ denote Lebesgue measure of A . For any positive number l we mean by $[l]$ the greatest positive integer not greater than l .

2. Some preminialies.

In this section we collect a few results which we shall use in the following.

We now introduce another function space. Assume that Ω is 'of type A' (see [9],[11]), namely there exists a constant $A > 0$ such that for any $R > 0$ and all $x_0 \in \Omega$

$$|\Omega \cap B_R(x_0)| \geq AR^n$$

and denote by $L^{p,\mu}(Q)$, $p \geq 1, \mu > 0$, the space of all functions u in $L^p(Q)$ satisfying

$$([u]_{p,\mu,Q})^p = \sup_{z_0 \in Q, R > 0} R^{-\mu} \iint_{Q_R^e(z_0)} |u - \bar{u}_{Q_R^e(z_0)}|^p dz < \infty. \quad (2.1)$$

$L^{p,\mu}(Q)$ is a Banach space with the norm

$$\{|u|_{L^p(Q)}^p + ([u]_{p,\mu,Q})^p\}^{1/p}.$$

These spaces have been introduced in [8] for the Euclidean metric and in [3] for a general class of metrics including the parabolic one δ_θ . We have the following result ([3], Theorem 3.1).

Proposition 2.1. *The spaces $L^{p,m+\theta+\theta\mu}(Q)$ and $C^{0,\mu}(Q, \delta_\theta)$, $0 < \mu < 1$ are topological and algebraically isomorphic.*

We actually exploit Proposition 2.1 on a local cylinder.

Let us now recall the estimate for solutions of nonlinear degenerate parabolic systems (refer to [5],[13]).

Proposition 2.2. Let v be a weak solution of (1.1) with $f \equiv 0$ in some cylinder $Q_R^\theta \subset Q$ where $\theta = 2 + \alpha(p - 2)$ with $\alpha > 0$. Then, for $0 < \alpha < 1$, there exist positive constants γ , $q > 1$ depending only on m, p and α such that

$$\iint_{Q_r^\theta} |Dv|^p dt dx \leq \gamma \left(\frac{r}{R} \right)^{m+\theta-\alpha p} \left\{ \iint_{Q_R^\theta} |Dv|^p dt dx + 1 \right\}. \quad (2.2)$$

holds for all $0 < r < R/2$.

Finally we need the following result that can be proved similarly as [8], Prop. 5.1 (also refer to [9]) only by changing Euclidean cubes with parabolic ones:

Proposition 2.3. Let g be a nonnegative L^q -integrable function defined in some cylinder Q_R with some $q > 1$. Let us suppose that g satisfies with some positive constants: $b > 1, \delta > 0$

$$\begin{aligned} \frac{1}{|Q_r^\theta|} \iint_{Q_r^\theta(z_0)} g^q dt dx &\leq b \left(\frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} g dt dx \right)^q + \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} f^q dt dx \\ &+ \delta \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} g^q dt dx \end{aligned} \quad (2.3)$$

for all $z_0 \in Q_R$ and any $0 < r < (1/4)\text{dist}(z_0, \partial Q_R)$. Then there exist positive constants γ, ε , depending on b, q, δ and m , and δ_0 , depending only on q and m , such that, if $\delta < \delta_0$, $g \in L^{\tilde{q}}(Q_{R/4})$ for $\tilde{q} \in [q, q + \varepsilon)$ and

$$\left(\frac{1}{|Q_{R/4}|} \iint_{Q_{R/4}} g^{\tilde{q}} dt dx \right)^{1/\tilde{q}} \leq \gamma \left(\frac{1}{|Q_R|} \iint_{Q_R} g^q dt dx \right)^{\frac{1}{q}} + \left(\frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta(z_0)} f^{\tilde{q}} dt dx \right)^{\frac{1}{\tilde{q}}} \quad (2.4)$$

Now we state a fundamental inequality for solutions to (1.1). In the following Q_R is a arbitrarily fixed cylinder such that $Q_R \subset Q$, $0 < R \leq 1$. We also take a positive number θ as $0 < \theta \leq p$ and $\chi = \chi(x)$ as a function in $C_0^\infty(B_2)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on B_1 and $|D\chi| \leq 2$. We denote by $\chi_{x_0, 2r}$ the function $\chi_{x_0, 2r}(x) = \chi((x - x_0)/r)$ for any $x_0 \in Q$ and replace the notation $\chi_{x_0, 2r}$ by χ when no confusion may arise. We also use the weighted means of u in $B_{2r}(x_0)$ as

$$\begin{aligned} \bar{u}_{B_{2r}(x_0)}^\chi(t) &= \int_{B_{2r}(x_0)} u(t, x) \chi_{x_0, 2r}^p(x) dx / \int_{B_{2r}(x_0)} \chi_{x_0, 2r}^p(x) dx, \quad x_0 \in \Omega, \\ \bar{u}_{Q_{2r}^\theta(z_0)}^\chi &= \iint_{Q_{2r}^\theta(z_0)} u(t, x) \chi_{x_0, 2r}^p(x) dt dx / \iint_{Q_{2r}^\theta(z_0)} \chi_{x_0, 2r}^p(x) dt dx, \quad z_0 \in Q. \end{aligned} \quad (2.5)$$

Lemma 2.4. (Caccioppoli type estimate) *There exists a positive constant γ depending only on m , M and θ such that*

$$\begin{aligned} & \sup_{t_0 - r^\theta < t < t_0} \int_{B_r(x_0) \times \{t\}} |u - \bar{u}_{B_{2r}(x_0)}^\chi|^2 dx + \iint_{Q_r^\theta(t_0, x_0)} |Du|^p dt dx \\ & \leq \gamma \left(r^{-\theta} \iint_{Q_{2r}^\theta(t_0, x_0)} |u - \bar{u}_{B_{2r}(x_0)}^\chi(t)|^2 dt dx + r^{-p} \iint_{Q_{2r}^\theta(t_0, x_0)} |u - \bar{u}_{B_{2r}(x_0)}^\chi(t)|^p dt dx \right) \end{aligned} \quad (2.6)$$

holds for any $Q_{2r}^\theta(t_0, x_0) \subset Q_R$.

Proof. Let $\tau \in C^\infty(R, R)$ depend only on a time-variable t satisfying $0 \leq \tau \leq 1$, $\tau = 1$ on $[t_0 - r^\theta, t_0]$, $\tau = 0$ on $t < t_0 - (2r)^\theta$ and $|\partial_t \tau| \leq 2/(2^\theta - 1)r^{-\theta}$. Testing (1.1) by a function $\varphi = (u - \bar{u}_{B_{2r}(x_0)}^\chi(t))\chi^p \tau^p \mathbf{1}_{-\infty, t_0}$, we have

$$\begin{aligned} & \int_{B_{2r} \times \{t_0\}} |u - \bar{u}_{B_{2r}}^\chi|^2 \chi^p \tau^p dx + \iint_{Q_{2r}^\theta} [|Du|^p - f(t, x, u, Du)(u - \bar{u}_{B_{2r}}^\chi(t))] \chi^p \tau^p dt dx \\ & \leq \gamma \iint_{Q_{2r}^\theta} |u - \bar{u}_{B_{2r}}^\chi|^2 \chi^p \tau^{p-1} \partial_t \tau dt dx \\ & \quad + \gamma \iint_{Q_{2r}^\theta} |Du|^{p-2} Du D\chi(u - \bar{u}_{B_{2r}}^\chi) \chi^{p-1} \tau^p dt dx. \end{aligned} \quad (2.7)$$

Since by our choice of a test function the remaining term

$$\int_{t_0 - 2r^\theta}^{t_0} \left[\int_{B_{2r}} (u - \bar{u}_{B_{2r}}^\chi(t)) \chi^p dx \right] \partial_t \bar{u}_{B_{2r}}^\chi(t) \tau^p dt$$

is equal to zero, we obtain the lemma from applying Young's inequality and (1.5) to (2.7). Note that the time derivative $\partial_t \bar{u}_{B_{2r}}^\chi(t)$ is integrable. In fact, testing the identity by $\varphi = \chi^p \mathbf{1}_{(t, t_0)}$ one immediately sees that $\bar{u}_{B_{2r}}^\chi(t)$ is absolutely continuous.

Remark. $(u - \bar{u}_{B_{2r}(x_0)}^\chi(t))\chi^p \tau^p \mathbf{1}_{-\infty, t_0}$ is not admissible as a test function in (1.1). But, by substituting $[(u - \bar{u}_{B_{2r}(x_0)}^\chi(t))_h \chi^p \tau^p \mathbf{1}_{-\infty, t_0}^\varepsilon]_h$ where $\eta_h(t) = (1/h) \int_t^{t+h} \eta(s) ds$, $\eta_h(t) = (1/h) \int_{t-h}^t \eta(s) ds$ and $\mathbf{1}_{-\infty, t_0}^\varepsilon \in C^\infty(R)$, $\mathbf{1}_{-\infty, t_0}^\varepsilon = 1$ on $t < t_0 - \varepsilon$, $\mathbf{1}_{-\infty, t_0}^\varepsilon = 0$ on $t > t_0$, (which is admissible as a test function in (1.1)) into (1.1), and calculating similarly as (2.7) and letting $h, \varepsilon \downarrow 0$ in the resulting inequality, we have (2.7).

Lemma 2.5. *There exists a positive constant γ depending only on m , M and θ such that*

$$\begin{aligned} & \sup_{t_0 - r^\theta < t < t_0} \int_{B_r(x_0) \times \{t\}} |u - \bar{u}_{B_r(x_0)}^\chi(t)|^2 dx \\ & \leq \gamma \left(r^{2-\theta} \iint_{Q_{2r}^\theta(t_0, x_0)} |Du|^2 dt dx + \iint_{Q_{2r}^\theta(t_0, x_0)} |Du|^p dt dx \right) \end{aligned} \quad (2.8)$$

holds for any $Q_{2r}^\theta(t_0, x_0) \subset Q_R$.

Proof. As in the proof of Lemma 2.4, testing (1.1) with

$$(u - \bar{u}_{B_{2r}(x_0)}^\chi(t)) \chi^p \tau^p \mathbf{1}_{-\infty, t_0},$$

we obtain, from applying a simple variation of Poincaré inequality for the resulting inequality,

$$\begin{aligned} & \sup_{t_0 - 2r^\theta < t < t_0} \int_{B_r(x_0) \times \{t\}} |u - \bar{u}_{B_{2r}(x_0)}^\chi(t)|^2 dx \\ & \leq \gamma \left(r^{2-\theta} \iint_{Q_{2r}^\theta(t_0, x_0)} |Du|^2 dt dx + \iint_{Q_{2r}^\theta(t_0, x_0)} |Du|^p dt dx \right). \end{aligned}$$

Since, for any $t \in (t_0 - r^\theta, t_0)$

$$\begin{aligned} & \int_{B_r(x_0) \times \{t\}} |u - \bar{u}_{B_r(x_0)}^\chi(t)|^2 dx \\ & \leq \int_{B_{2r}(x_0) \times \{t\}} |u - \bar{u}_{B_{2r}(x_0)}^\chi(t)|^2 dx + 2|B_r| |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{B_r}^\chi(t)|^2 \\ & \leq \gamma \int_{B_r(x_0) \times \{t\}} |u - \bar{u}_{B_{2r}(x_0)}^\chi(t)|^2 dx. \end{aligned} \quad (2.9)$$

the result follows.

Lemma 2.6. *There exists a positive constant γ depending only on m and M such that*

$$\sup_{t_0 - r^\theta < t < t_0} \int_{B_r \times \{t\}} |u(t, x) - \bar{u}_{B_r}^\chi(t)|^p dx \leq \gamma r^{p(\theta-p)/(p-1)} \iint_{Q_{2r}^\theta} |Du|^p dt dx \quad (2.10)$$

holds for any $Q_{2r}^\theta \subset Q_R$.

Proof. Let τ be the same function as in Lemma 2.4. Testing (1.1) with

$$\varphi = (u - \bar{u}_{Q_{2r}^\theta}^\chi) |u - \bar{u}_{Q_{2r}^\theta}^\chi|^{p-2} \chi^p \tau^p$$

(note Remark after Lemma 2.4) and using Young's inequality, we have

$$\begin{aligned}
& (1/p) \int_{B_{2r}} |u - \bar{u}_{Q_{2r}^\theta}^\chi|^p \chi^p \tau^{p-1} dx - (1/p) \iint_{Q_{2r}^\theta} |u - \bar{u}_{Q_{2r}^\theta}^\chi|^p \chi^p \partial_t \tau \tau^{p-1} dt dx \\
& + (1 - p\varepsilon) \iint_{Q_{2r}^\theta} |Du|^p |u - \bar{u}_{Q_{2r}^\theta}^\chi|^{p-2} \chi^p \tau^p dt dx \\
& + (p-2)/4 \iint_{Q_{2r}^\theta} |Du|^{p-2} |D|u - \bar{u}_{Q_{2r}^\theta}^\chi|^2 |u - \bar{u}_{Q_{2r}^\theta}^\chi|^{p-4} \chi^p \tau^p dt dx \\
& - p\gamma(p, \varepsilon) \iint_{Q_{2r}^\theta} |u - \bar{u}_{Q_{2r}^\theta}^\chi|^{2(p-1)} |D\chi|^p \tau^p dt dx \leq a \iint_{Q_{2r}^\theta} |Du|^p |u - \bar{u}_{Q_{2r}^\theta}^\chi|^{p-1} \chi^p \tau^p dt dx.
\end{aligned}$$

Putting ε so small in the above and noticing $p > 2$, we obtain from the boundedness of u

$$\begin{aligned}
& \sup_{t_0 - r^\theta < t < t_0} \int_{B_r} |u(t, x) - \bar{u}_{Q_{2r}^\theta}^\chi|^p dx \leq \gamma \iint_{Q_{2r}^\theta} |u(t, x) - \bar{u}_{Q_{2r}^\theta}^\chi|^p \partial_t \tau dt dx \\
& + \gamma \iint_{Q_{2r}^\theta} |u(t, x) - \bar{u}_{Q_{2r}^\theta}^\chi|^{2(p-1)} |D\chi|^p dt dx + a(2M)^{p-1} \iint_{Q_{2r}^\theta} |Du|^p dt dx.
\end{aligned} \tag{2.11}$$

Note the following estimate: For $t_0 - (2r)^\theta < s < t < t_0$

$$\begin{aligned}
& \int_{B_r \times \{t\}} |u - \bar{u}_{B_{2r}}^\chi(t)|^p dx \\
& \leq 2^{p-1} \int_{B_r \times \{t\}} |u - \bar{u}_{Q_{2r}^\theta}^\chi|^p dx + 2^{p-1} |B_r| |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{Q_{2r}^\theta}^\chi|^p, \\
& \iint_{Q_{2r}^\theta} |u - \bar{u}_{Q_{2r}^\theta}^\chi|^p dt dx \\
& \leq 2^{p-1} \iint_{Q_{2r}^\theta} |u - \bar{u}_{B_{2r}}^\chi(t)|^p dt dx + 2^{p-1} |B_{2r}| \int_{t_0 - (2r)^\theta}^{t_0} |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{Q_{2r}^\theta}^\chi|^p dt.
\end{aligned} \tag{2.12}$$

Now we estimate $|\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{Q_{2r}^\theta}^\chi|^p$ for $t_0 - (2r)^\theta < t < t_0$. Testing the identity (1.1) by

$$\chi^p \mathbf{1}_{s,t}(\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{B_{2r}}^\chi(s)) |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{B_{2r}}^\chi(s)|^{p-2}, \quad t, s \in (t_0 - 2r^\theta, t_0)$$

and noting the boundedness of u , we have, for any $t_0 - (2r)^\theta < s < t < t_0$

$$|B_{2r}| |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{B_{2r}}^\chi(s)|^p \leq \gamma(M)(r^{(\theta-p)/(p-1)} + 1) \iint_{Q_{2r}^\theta} |Du|^p dt dx. \tag{2.13}$$

Noticing that $\bar{u}_{Q_{2r}^\theta}^\chi = \int_{t_0 - (2r)^\theta}^{t_0} \bar{u}_{B_{2r}}^\chi(s) ds / (2r)^\theta$, we find that, for any $t_0 - (2r)^\theta < t < t_0$

$$|\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{Q_{2r}^\theta}^\chi|^p \leq \sup_{t_0 - (2r)^\theta < s < t < t_0} |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{B_{2r}}^\chi(s)|^p, \tag{2.14}$$

so that, substituting (2.13) into (2.14) gives that

$$\sup_{t_0 - (2r)^\theta < s < t < t_0} |\bar{u}_{B_{2r}}^\chi(t) - \bar{u}_{B_{2r}}^\chi(s)|^p \leq \gamma |B_{2r}|^{-1} r^{(\theta-p)/(p-1)} \iint_{Q_{2r}^\theta} |Du|^p dt dx. \quad (2.15)$$

Combining (2.12) and (2.15) with (2.11), we obtain from the boundedness of u and a simple variation of Poincaré inequality

$$\sup_{t_0 - (2r)^\theta < s < t < 0} \int_{B_r \times \{t\}} |u(t, x) - \bar{u}_{B_{2r}}^\chi(t)|^p \leq \gamma(M) r^{\theta-p+(\theta-p)/(p-1)} \iint_{Q_{2r}^\theta} |Du|^p dt dx,$$

where we note $0 < \theta \leq p$ and $0 < r < 1$. Noting (2.9) in the proof of Lemma 2.5, the result immediately follows.

3. L^q - estimates.

Take a cylinder $Q_R \subset Q$, $0 < R \leq 1$, arbitrarily and fix it. Now we prove

Lemma 3.1. (Reverse Hölder inequality) *There exist positive constants γ and ε such that $|Du| \in L_{\text{loc}}^{p+\varepsilon}(Q_{R/4})$. Moreover there exist exponents $0 < \tilde{p} < p$ and $1 < \bar{p}$ such that*

$$\left(\frac{1}{|Q_{R/4}|} \iint_{Q_{R/4}} |Du|^{p+\varepsilon} dt dx \right)^{1/(p+\varepsilon)} \leq \gamma \left\{ \left(\frac{1}{|Q_R|} \iint_{Q_R} |Du|^p dt dx \right)^{1/p} + \left(\iint_{Q_R} |Du|^{\bar{p}} dt dx \right)^{\bar{p}} \right\}. \quad (3.1)$$

Proof. In the following θ is a positive constant satisfying $\theta \leq p$, which is chosen exactly later. Taking a exponent γ_1, α_2 as follows

$$\begin{aligned} \gamma_1 &= \frac{p}{m} \left(2 + \frac{1}{m+2} \right), \\ \max \left\{ \frac{2}{p+2}, \frac{2}{m+2}, \frac{2\gamma_1}{m+2} / \left(\frac{2\gamma_1}{m+2} + \frac{m}{m+2} \right) \right\} &< \alpha_2 < 1. \end{aligned} \quad (3.2)$$

Moreover we set

$$\alpha_1 = \frac{\gamma_1}{\alpha_2 + \gamma_1}, \quad \beta_1 = \frac{m}{m - (1 - \alpha_2)(m+2)}, \quad \beta_2 = \frac{m}{(1 - \alpha_2)(m+2)}. \quad (3.3)$$

Noting that

$$0 < \alpha_1, \alpha_2 < 1,$$

$$\beta_1, \beta_2 > 1 \text{ and } 1/\beta_1 + 1/\beta_2 = 1$$

and using Hölder inequality, Lemma 2.6 and a simple variation of Sobolev inequality, we have, for any $Q_{4r}^\theta \subset Q_R$

$$\begin{aligned}
& \iint_{Q_{2r}^\theta} |u - \bar{u}_{B_{2r}}^\chi(t)|^p dt dx \leq \sup_{t_0 - 2r^\theta < t < t_0} \left(\int_{B_{2r}(x_0) \times \{t\}} |u - \bar{u}_{B_{2r}}^\chi|^p \chi^p \tau^p dx \right)^{1-\alpha_1} \\
& \quad \times \int_{t_0 - (2r)^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}_{B_{2r}}^\chi|^p dx \right)^{\alpha_1} dt \\
& \leq \left(r^{p(\theta-p)/(p-1)} \iint_{Q_{4r}^\theta} |Du|^p dt dx \right)^{1-\alpha_1} \int_{t_0 - (2r)^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}_{B_{2r}}^\chi|^{\alpha_2 \beta_1 p} dx \right)^{\alpha_1/\beta_1} \\
& \quad \times \left(\int_{B_{2r}} |u - \bar{u}_{B_{2r}}^\chi|^{p(1-\alpha_2)\beta_2} dx \right)^{\alpha_1/\beta_2} dt \\
& \leq \gamma r^{p(1-\alpha_2)\alpha_1} r^{p(\theta-p)(1-\alpha_1)/(p-1)} \left(\iint_{Q_{4r}^\theta} |Du|^p dt dx \right)^{1-\alpha_1} \left(\int_{B_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dx \right)^{\alpha_1/\beta_2} dt \\
& \quad \times \int_{t_0 - (2r)^\theta}^{t_0} \left(\int_{B_{2r}} |Du|^{\alpha_2 \beta_1 m p / (m + \alpha_2 \beta_1 p)} dx \right)^{\alpha_1(m + \alpha_2 \beta_1 p) / \beta_1 m} \\
& \leq \gamma r^{p(1-\alpha_2)\alpha_1} r^{p(\theta-p)(1-\alpha_1)/(p-1)} |B_{2r}|^{\alpha_1(m + \alpha_2 \beta_1 p) / \beta_1 m - \alpha_1 \alpha_2} \\
& \quad \times \left(\iint_{Q_{4r}^\theta} |Du|^p dt dx \right)^{1-\alpha_1} \int_{t_0 - (2r)^\theta}^{t_0} \left(\int_{B_{2r}} |Du|^p dx \right)^{\alpha_1 \alpha_2} \left(\int_{B_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dx \right)^{\frac{\alpha_1}{\beta_2}} dt \\
& \leq \gamma r^{p(1-\alpha_2)\alpha_1} r^{\frac{p(\theta-p)(1-\alpha_1)}{p-1}} |B_{2r}|^{\frac{\alpha_1(m + \alpha_2 \beta_1 p)}{\beta_1 m} - \alpha_1 \alpha_2} \left(\iint_{Q_{4r}^\theta} |Du|^p dt dx \right)^{1-\alpha_1 + \alpha_1 \alpha_2} \\
& \quad \times \left[\int_{t_0 - (2r)^\theta}^{t_0} \left(\int_{B_{2r}} |Du|^{p(1-\alpha_2)\beta_2} dx \right)^{\frac{\alpha_1}{\beta_2(1-\alpha_1\alpha_2)}} dt \right]^{1-\alpha_1\alpha_2} \\
& \leq \gamma r^{p(1-\alpha_2)\alpha_1} r^{p(\theta-p)(1-\alpha_1)/(p-1)} |B_{2r}|^{\alpha_1(m + \alpha_2 \beta_1 p) / \beta_1 m - \alpha_1 \alpha_2} r^{\theta(1-\alpha_1\alpha_2 - \alpha_1/\beta_2)(1-\alpha_1\alpha_2)} \\
& \quad \times \left(\iint_{Q_{4r}^\theta} |Du|^p dt dx \right)^{1-\alpha_1 + \alpha_1 \alpha_2} \left(\iint_{Q_{2r}^\theta} |Du|^{p(1-\alpha_2)\beta_2} dt dx \right)^{\alpha_1/\beta_2}
\end{aligned} \tag{3.4}$$

By applying Young's inequality for (3.4), the latter is

$$\leq \delta r^p \iint_{Q_{4r}^\theta} |Du|^p dt dx + \gamma(\delta) r^p |Q_r^\theta| \left(\frac{1}{|Q_{2r}^\theta|} \iint_{Q_{2r}^\theta} |Du|^{mp/(m+2)} dt dx \right)^{(m+2)/m} \tag{3.5}$$

We estimate $\iint_{Q_{2r}^\theta} |u - \bar{u}_{B_{2r}}^\chi(t)|^2 dt dx$ for any $Q_{4r}^\theta \subset Q_R$. By Hölder inequality and Lemma 2.5, we have

$$\begin{aligned}
& \iint_{Q_{2r}^\theta} |u - \bar{u}_{B_{2r}}^\chi(t)|^2 dt dx \\
& \leq \left(\sup_{t_0-2r^\theta < t < t_0} \int_{B_{2r} \times \{t\}} |u - \bar{u}_{B_{2r}}^\chi(t)|^2 dx \right)^{1-\alpha_1} \int_{t_0-2r^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}_{B_{2r}}^\chi(t)|^2 dx \right)^{\alpha_1} dt \\
& \leq \gamma \left(r^{-\theta} \iint_{Q_{4r}^\theta} |u - \bar{u}_{B_{4r}}^\chi(t)|^2 dt dx \right)^{1-\alpha_1} \int_{t_0-2r^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}_{B_{2r}}^\chi(t)|^2 dx \right)^{\alpha_1} dt \\
& + \gamma \left(r^{-p} \iint_{Q_{4r}^\theta} |u - \bar{u}_{B_{4r}}^\chi(t)|^p dt dx \right)^{1-\alpha_1} \int_{t_0-2r^\theta}^{t_0} \left(\int_{B_{2r}} |u - \bar{u}_{B_{2r}}^\chi(t)|^2 dx \right)^{\alpha_1} dt \\
& = I_1 + I_2.
\end{aligned} \tag{3.6}$$

First we consider I_1 . Set $\alpha_1, \alpha_2, \beta_1$ and β_2 as follows:

$$\begin{aligned}
0 < \alpha_1 < \min\{1/2, 2/m\}, \quad 0 < \alpha_2 < 1 \\
\frac{p}{p-2+2\alpha_2} \leq \beta_1 < \frac{m}{\alpha_2(m-2)} \quad \beta_2 = \frac{\beta_1}{\beta_1-1}.
\end{aligned} \tag{3.7}$$

We also set θ as

$$\theta = \left(2 - \frac{m\alpha_1}{\beta_2} \right) / \left(1 + \frac{\alpha_1}{\beta_2} \right). \tag{3.8}$$

Note that

$$2(1-\alpha_2)\beta_2 \leq p, \quad \beta_1, \beta_2 > 1,$$

so that, calculating similarly as in (3.4) gives that

$$I_1 \leq \gamma r^\theta |Q_r^\theta| \left(\frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta} |Du|^2 dt dx \right)^{1-\alpha_1+\alpha_1\alpha_2} \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{\alpha_1/\beta_2}$$

Noting that

$$\frac{p}{2(1-\alpha_1+\alpha_1\alpha_2)} > 1$$

and using Young's and Hölder inequalities, we obtain

$$\begin{aligned}
I_1 & \leq \delta r^\theta |Q_r^\theta| \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta} |Du|^p dt dx \\
& + \gamma(\delta) r^\theta |Q_r^\theta| \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\alpha_2)\beta_1} dt dx \right)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))}.
\end{aligned} \tag{3.9}$$

Next, to estimate I_2 we put the exponents as follows:

θ and α_1 are the same as in (3.7) and (3.8),

$$1 - \frac{p(2-\theta)}{2(m+\theta)} < \tilde{\alpha}_2 < 1, \quad (3.10)$$

$$\frac{p}{p-2+2\tilde{\alpha}_2} < \tilde{\beta}_1 < \min\left\{\frac{m}{\tilde{\alpha}_2(m-2)}, \frac{m+\theta}{2-\theta} / \left(\frac{m+\theta}{2-\theta} - 1\right)\right\}, \quad \tilde{\beta}_2 = \frac{\tilde{\beta}_1}{\tilde{\beta}_1 - 1}.$$

Noting that

$$\tilde{\beta}_1, \tilde{\beta}_2 > 1, \quad 2(1-\tilde{\alpha}_2)\tilde{\beta}_2 \leq p$$

and estimating similarly as (3.4), we have

$$\begin{aligned} I_2 &\leq \gamma r^{(2-\theta)\alpha_1 - (m+\theta)\alpha_1/\tilde{\beta}_1} r^\theta |Q_r^\theta| \\ &\times \left(\frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta} |Du|^p dt dx \right)^{1-\alpha_1} \left(\frac{1}{|Q_{2r}^\theta|} \iint_{Q_{2r}^\theta} |Du|^2 dt dx \right)^{\alpha_1 \tilde{\alpha}_2} \\ &\times \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{\alpha_1/\tilde{\beta}_2} \end{aligned}$$

Note that

$$(2-\theta)\alpha_1 - (m+\theta)\alpha_1/\tilde{\beta}_1 \geq 0.$$

Since

$$\frac{1}{1-\alpha_1+2\alpha_1\tilde{\alpha}_2/p} > 1,$$

from Young's and Hölder inequality it follows that

$$I_2 \leq \delta r^\theta |Q_r^\theta| \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta} |Du|^p dt dx + \gamma(p, \delta) r^\theta |Q_r^\theta| \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)} \quad (3.11)$$

Combining (3.9) and (3.11) with (3.6), we have

$$\begin{aligned} &\iint_{Q_{2r}^\theta} |u - \bar{u}_{B_{2r}}^X(t)|^2 dt dx \\ &\leq \delta r^\theta |Q_r^\theta| \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta} |Du|^p dt dx + \gamma(p, \delta) r^\theta |Q_r^\theta| \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)} \\ &+ \gamma(p, \delta) r^\theta |Q_r^\theta| \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))} \end{aligned} \quad (3.12)$$

Thus, substituting (3.5) and (3.12) into (2.6) in Lemma 2.4 we obtain, for any $Q_{4r}^\theta \subset Q_R$

$$\begin{aligned} & \frac{1}{|Q_r^\theta|} \iint_{Q_r^\theta} |Du|^p dt dx \\ & \leq \delta \frac{1}{|Q_{4r}^\theta|} \iint_{Q_{4r}^\theta} |Du|^p dt dx + \gamma(p, \delta) \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))} \\ & + \gamma(p, \delta) \left(\frac{1}{|Q_{2r}^\theta|} \iint_{Q_{2r}^\theta} |Du|^{\frac{mp}{m+2}} dt dx \right)^{\frac{m+2}{m}} + \gamma(p, \delta) \left(\iint_{Q_{2r}^\theta} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{\frac{p}{\tilde{\beta}_2(p-2\tilde{\alpha}_2)}} \end{aligned} \quad (3.13)$$

The desired estimate follows from Prop.2.3 with setting $g = |Du|^{mp/(m+2)}$, $q = (m+2)/m$ and

$$\begin{aligned} f = & \gamma \left\{ \left(\iint_{Q_R} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))} \right. \\ & \left. + \left(\iint_{Q_R} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)} \right\}^{1/q} \end{aligned}$$

4. Proof of Theorem.

In the following we take $Q_{R_0}^2(\bar{t}, \bar{x}) \subset Q$, $0 < R_0 \leq 1$, and fix it.

Lemma 4.1. Suppose that there exists a sufficiently small $\delta > 0$ such that

$$\lim_{r \downarrow 0} \left(\frac{1}{|B_r|} \iint_{Q_r^2(\bar{t}, \bar{x})} |Du|^p dt dx \right) < \delta \quad (4.1)$$

Then, taking $R_0 > 0$ sufficiently small, for $0 < \alpha < 1$, there exists a positive constant γ depending only on m, p, α, δ and $\iint_Q |Du|^p dt dx$ such that

$$\frac{1}{|Q_r^2|} \iint_{Q_r^2(t_0, x_0)} |Du|^p dt dx \leq \gamma r^{-\alpha p} \quad (4.2)$$

holds for any $(t_0, x_0) \in Q_{R_0/4}^2$ and all $0 < r < R_0/4$.

Proof. Let $Q_{4R}^2(t_0, x_0) \subset Q_{R_0}^2$ be fixed arbitrarily. Consider the Dirichlet problem:

$$\partial_t v^i - \operatorname{div}(|Dv|^{p-2} Dv^i) = 0 \quad \text{in } Q_R^\theta, \quad i = 1, \dots, n, \quad (4.3)$$

$$v = u \quad \text{on the parabolic boundary of } Q_R^\theta \quad (4.4)$$

where $\theta = 2 + \alpha(p - 2)$.

Existence of weak solutions to (4.3) in the sense of (1.4) and to (4.4) in the sense of traces of $W_p^1(Q_R^\theta)$ functions can be established by a straightforward adoption of Galerkin method as presented for example in [12].

Subtracting (1.1) by (4.3) and testing the resulting inequality by $v - u$ on Q_R^θ (note Remark after Lemma 2.4), we have

$$\frac{1}{2} \int_{B_R \times \{t_0\}} |v - u|^2 dx + \iint_{Q_R^\theta} |Dv - Du|^p dt dx \leq a \iint_{Q_R^\theta} |Du|^p |v - u| dt dx. \quad (4.5)$$

Noticing the maximum estimate of the solution to (4.3) and (4.4) (see [13]), from (4.5) we deduce two inequalities for $0 < r < R$:

$$\iint_{Q_R^\theta} |Dv|^p dt dx \leq \gamma \iint_{Q_R^\theta} |Du|^p dt dx, \quad (4.6)$$

$$\iint_{Q_R^\theta} |Du|^p dt dx \leq 2^{p-1} \iint_{Q_R^\theta} |Dv|^p dt dx + 2^{p-1} \iint_{Q_R^\theta} |Dv - Du|^p dt dx. \quad (4.7)$$

From (2.2) in Prop.2.2 and (4.6) we obtain for $0 < r < R$

$$\iint_{Q_r^\theta} |Dv|^p dt dx \leq \gamma \left(\frac{r}{R} \right)^{m+\theta-\alpha p} \left\{ \iint_{Q_R^\theta} |Du|^p dt dx + 1 \right\} \quad (4.8)$$

Combining (4.8) with (4.7) gives that

$$\iint_{Q_r^\theta} |Du|^p dt dx \leq \gamma \left(\frac{r}{R} \right)^{m+\theta-\alpha p} \left(\iint_{Q_R^\theta} |Du|^p dt dx + 1 \right) + \gamma \iint_{Q_r^\theta} |Du - Dv|^p dt dx. \quad (4.9)$$

Now we estimate $\iint_{Q_r^\theta} |Du - Dv|^p dt dx$. In the following ε is determined in Lemma 3.1. By Hölder inequality we have

$$\iint_{Q_R^\theta} |Du|^p |v - u| dt dx \leq \left(\iint_{Q_R^\theta} |Du|^{p+\varepsilon} dt dx \right)^{p/(p+\varepsilon)} \left(\iint_{Q_R^\theta} |v - u|^{(p+\varepsilon)/\varepsilon} dt dx \right)^{\varepsilon/(p+\varepsilon)} \quad (4.10)$$

Noting the boundedness of v , we obtain from Poincarè inequality and (4.5)

$$\iint_{Q_R^\theta} |v - u|^{(p+\varepsilon)/\varepsilon} dt dx \leq \gamma(M) \iint_{Q_R^\theta} |v - u|^p dt dx \leq \gamma R^p \iint_{Q_R^\theta} |Du|^p dt dx. \quad (4.11)$$

To estimate $\frac{1}{|Q_R^\theta|} \iint_{Q_R^\theta} |Du|^{p+\varepsilon} dt dx$ we use a partition argument (refer to [13]). Set, for a subset $\tilde{Q} \subset Q$

$$\begin{aligned} & f(\tilde{Q}) \\ &= \gamma \left\{ \left(\iint_{\tilde{Q}} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{p\alpha_1/\beta_1(p-2+2\alpha_1(1-\alpha_2))} \right. \\ & \quad \left. + \left(\iint_{\tilde{Q}} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{p/\tilde{\beta}_2(p-2\tilde{\alpha}_2)} \right\}^{m/(m+2)} \end{aligned}$$

where the parameters are determined in Lemma 3.1. We assume that r^θ/r^p is an integer where note $\theta \leq p$, and subdivide Q_r^θ into $s = r^{\theta-p}$ boxes with vertices $(t_0, x_0), \dots, (t_{s-1}, x_0)$. Then, from (3.1) in Lemma 3.1 we obtain

$$\begin{aligned} & \frac{1}{|Q_R^\theta|} \iint_{Q_R^\theta} |Du|^{p+\varepsilon} dt dx \leq \frac{R^p}{R^\theta} \sum_{i=0}^{s-1} \frac{1}{|Q_R^p|} \iint_{Q_R^p(t_i, x_0)} |Du|^{p+\varepsilon} dt dx \\ & \leq \gamma \frac{R^p}{R^\theta} \sum_{i=0}^{s-1} \left\{ \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}(t_i, x_0)} |Du|^p dt dx \right)^{\frac{p+\varepsilon}{p}} + (f(Q_{4R}(t_i, x_0)))^{p+\varepsilon} \right\} \\ & \leq \gamma \frac{R^p}{R^\theta} \sum_{i=0}^{s-1} \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}(t_i, x_0)} |Du|^p dt dx \right) \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}(t_i, x_0)} |Du|^p dt dx \right)^{\frac{\varepsilon}{p}} \\ & \quad + \gamma \frac{R^p}{R^\theta} \sum_{i=0}^{s-1} (f(Q_{4R}(t_i, x_0)))^{p+\varepsilon}. \end{aligned} \quad (4.12)$$

Taking $R_0 > 0$ sufficiently small we obtain from (4.1) and Lebesgue absolute continuous theorem

$$\frac{1}{|B_{4R}|} \iint_{Q_{4R}^\theta(t_i, x_0)} |Du|^p dt dx < \delta \quad \text{for } i = 0, 1, \dots, s-1. \quad (4.13)$$

Note that at most $([4^p] + 1)$ cylinders $Q_{4R}(t_i, x_0)$ ($i = 0, 1, \dots, s-1$) are overlapped with each $Q_{4R}(t_i, x_0)$ ($i = 0, 1, \dots, s-1$), so that we have

$$\sum_{i=0}^{s-1} \iint_{Q_{4R}(t_i, x_0)} |Du|^p dt dx \leq ([4^p] + 1) \iint_{Q_{4R, R^\theta + (4^p-1)R^p}(t_0, x_0)} |Du|^p dt dx. \quad (4.14)$$

From (4.13) and (4.14) we obtain

$$\begin{aligned} & \frac{R^p}{R^\theta} \sum_{i=0}^{s-1} \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}(t_i, x_0)} |Du|^p dt dx \right) \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}(t_i, x_0)} |Du|^p dt dx \right)^{\frac{\varepsilon}{p}} \\ & \leq \frac{4^\theta([4^p] + 1)}{4^p} (4R)^{-\varepsilon} \delta^{\varepsilon/p} \frac{1}{|Q_{4R}^\theta|} \iint_{Q_{4R, R^\theta + (4^p-1)R^p}(t_0, x_0)} |Du|^p dt dx. \end{aligned} \quad (4.15)$$

We also find that

$$\begin{aligned} \frac{R^p}{R^\theta} \sum_{i=0}^{s-1} (f(Q_{4R}(t_i, x_0)))^{p+\varepsilon} &\leq \frac{R^p}{R^\theta} s(f(Q_{4R, R^\theta + (4^p-1)R^p}(t_0, x_0)))^{p+\varepsilon} \\ &\leq (f(Q_{4R, R^\theta + (4^p-1)R^p}(t_0, x_0)))^{p+\varepsilon}. \end{aligned} \quad (4.16)$$

Here note that by taking $R_0 > 0$ sufficiently small, $R^\theta + (4^p - 1)R^p \leq (4R)^\theta$ holds for any $0 < R < R_0$. Combining (4.15) and (4.16) with (4.12) we have

$$\begin{aligned} &\frac{1}{|Q_R^\theta|} \iint_{Q_R^\theta} |Du|^{p+\varepsilon} dt dx \\ &\leq \gamma \frac{4^\theta([4^p] + 1)}{4^p} \delta^{\varepsilon/p} R^{-\varepsilon} \frac{1}{|Q_{4R}^\theta|} \iint_{Q_{4R, R^\theta + (4^p-1)R^p}(t_0, x_0)} |Du|^p dt dx + \gamma (f(Q_{4R}^\theta(t_0, x_0)))^{p+\varepsilon} \end{aligned} \quad (4.17)$$

Substituting (4.11) and (4.17) into (4.10) and noting that $0 < R < 1$ and $\theta \leq p$, we have

$$\begin{aligned} &\iint_{Q_R^\theta} |Du|^p |v - u| dt dx \leq \gamma \delta^{\frac{\varepsilon}{p+\varepsilon}} \iint_{Q_{4R}^\theta} |Du|^p dt dx \\ &+ \gamma |Q_R^\theta| \left(\frac{1}{|B_{4R}|} \iint_{Q_{4R}^\theta} |Du|^p dt dx \right)^{\varepsilon/(p+\varepsilon)} \left\{ \left(\iint_{Q_{4R}^\theta} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{\frac{p\alpha_1}{\beta_1(p-2+2\alpha_1(1-\alpha_2))}} \right. \\ &\left. + \left(\iint_{Q_{4R}^\theta} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{\frac{p}{\tilde{\beta}_2(p-2\tilde{\alpha}_2)}} \right\}^{mp/(m+2)} \end{aligned} \quad (4.18)$$

Combining (4.18) and (4.5) with (4.9) gives that

$$\begin{aligned} &\iint_{Q_r^\theta} |Du|^p dt dx \leq \gamma \left\{ \left(\frac{r}{R} \right)^{m+\theta-\alpha p} + \delta^{\frac{\varepsilon}{p+\varepsilon}} \right\} \left(\iint_{Q_{4R}^\theta} |Du|^p dt dx + 1 \right) \\ &+ \gamma |Q_R^\theta| \left(\frac{1}{|B_{4R}|} \iint_{Q_{4R}^\theta} |Du|^p dt dx \right)^{\varepsilon/(p+\varepsilon)} \left\{ \left(\iint_{Q_{4R}^\theta} |Du|^{2(1-\alpha_2)\beta_2} dt dx \right)^{\frac{p\alpha_1}{\beta_1(p-2+2\alpha_1(1-\alpha_2))}} \right. \\ &\left. + \left(\iint_{Q_{4R}^\theta} |Du|^{2(1-\tilde{\alpha}_2)\tilde{\beta}_2} dt dx \right)^{\frac{p}{\tilde{\beta}_2(p-2\tilde{\alpha}_2)}} \right\}^{mp/(m+2)}. \end{aligned} \quad (4.19)$$

Again noting (4.13) and iterating (4.19) similarly as Lemma 2.1 in [8], p86 (also see [9], p446) we have that for all $0 < \alpha < 1$, there exists a positive constant γ depending only on m, p, α and $\iint_Q |Du|^p dt dx$ such that

$$\iint_{Q_r^\theta(t_0, x_0)} (1 + |Du|^p) dt dx \leq \gamma r^{m+\theta-\alpha p} \quad (4.20)$$

holds for any $0 < r < R_0/4$ and $(t_0, x_0) \in Q_{R_0/4}^2$.

From a partition argument (see (4.12)) and (4.20), we obtain (4.1).

Proof of theorem. Let $(\bar{t}, \bar{x})$ satisfy (4.1). Exploiting Lemma 4.1 and estimating similarly as in the proof of Prop.3.3 in [13], pp118-120, we deduce that, for any $0 < \alpha < 1$ there exists a positive constant γ depending only on m, p, α and $\iint_Q |Du|^p dt dx$ such that

$$\frac{1}{|Q_r^2|} \iint_{Q_r^2(t_0, x_0)} |u - \bar{u}_{Q_r^2(t_0, x_0)}|^p dt dx \leq \gamma r^{p(1-\alpha)} \quad (4.21)$$

holds for all $(t_0, x_0) \in Q_{R_0/4}^2(\bar{t}, \bar{x})$ and any $0 < r < R_0/4$.

From (4.21) and Prop.2.1 with setting $Q = Q_{R_0/4}^2(\bar{t}, \bar{x})$, $\theta = 2$ and $\mu = 2(1 - \alpha)$ we conclude that $u \in C^{0,\beta}(Q_{R_0/4}^2)$ for any $0 < \beta < 1$.

To obtain the assertion of Theorem, we have only to recall Prop.3.2 in [9], p447 (also see [8]) and to note the L^q -estimate for $|Du|$ (Lemma 3.1).

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