

ある4次元の量指標のLに対応する保型形式について

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In [5], H. Maass showed that the dimension of a space of generalized Whittaker functions satisfying certain system of differential equations on Siegel's upper half space H_2 of degree 2 is three. First of all, we shall investigate the structure of a space of generalized Whittaker functions which are eigen functions for the algebra of invariant differential operators on H_2 . The theory of generalized Whittaker functions is discussed in Yamasita [12], [13], [14], [15] with full generality. We can get an outlook of the space of generalized Whittaker functions by using elementary calculus instead of representation theory of Lie groups. Generalized Whittaker functions naturally appear in the theory of indefinite theta function. We shall show commutation relations between the invariant differential operators on H_2 and those on the product $H_1 \times H_1$ of two copies of the upper half plane H_1 with respect to a theta function. The relations are analogues of commutation relations for Hecke operators in [1], [17], [16] and are proved in some cases for Laplacian in [8], [2]. We essentially use the result

in Nakajima [10] where the generators of the center of the universal enveloping algebra of $Sp(2, \mathbb{R})$ are explicitly given. By commutation relations we can construct an automorphic form F on H_2 corresponding to L-function with Grössencharacter of a certain biquadratic field. Generalized Whittaker functions investigated appear in the Fourier expansion of F with respect to translations in H_2 and so we can define a "Fourier" coefficient which is a constant by ratio the Fourier coefficient to a generalized Whittaker function. A certain "Fourier" coefficient of F is given in 2.

1. We denote, as usual, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field, and the complex number field. We denote the algebra of invariant differential operators on H_2 by $\mathcal{D}$ and the center of the universal enveloping algebra by $\mathcal{C}$. Nakajima calculated generators of $\mathcal{D}$, $\mathcal{C}$ in [9], [10]. The generators of $\mathcal{D}$ are

$$\Delta_1 = \sum_{i,j=1}^3 y_i y_j \partial_i \bar{\partial}_j - d(\partial_1 \bar{\partial}_3 + \bar{\partial}_1 \partial_3 - (1/2)\partial_2 \bar{\partial}_2)$$

and

$$\begin{aligned} \Delta_2 = & d^2(\partial_1 \partial_3 - (1/4)\partial_2^2)(\bar{\partial}_1 \bar{\partial}_3 - (1/4)\bar{\partial}_2^2) \\ & + \sqrt{-1}(1/4)d\left(\sum_{i=1}^3 y_i \partial_i\right)(\bar{\partial}_1 \bar{\partial}_3 - (1/4)\bar{\partial}_2^2) \\ & + \sqrt{-1}(1/4)d\left(\sum_{i=1}^3 y_i \bar{\partial}_i\right)(\partial_1 \partial_3 - (1/4)\partial_2^2) \end{aligned}$$

$$+(1/16)d(\partial_1\bar{\partial}_3 + \bar{\partial}_1\partial_3 - (1/2)\partial_2\bar{\partial}_2)$$

where we put

$$\begin{aligned} z_i &= x_i + \sqrt{-1}y_i \quad (1 \leq i \leq 3), \\ \partial_i &= \frac{\partial}{\partial z_i} = (1/2)\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \quad (1 \leq i \leq 3), \\ \bar{\partial}_i &= \frac{\partial}{\partial \bar{z}_i} = (1/2)\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) \quad (1 \leq i \leq 3), \end{aligned}$$

and $d = y_1 y_3 - y_2^2$ for $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in H_2$. For two complex numbers d_1, d_2 , we consider the space $\mathbb{W}$ formed by functions $f(Z) = g(Y)$ of $Z = X + iY \in H_2$, satisfying $\Delta_1 f = d_1 f$ and $\Delta_2 f = d_2 f$ with some functions g of Y , which we call generalized Whittaker functions. If $g(Y)e^{2\pi i \text{tr} X}$ belongs to $\mathbb{W}$, g has to satisfy

$$\begin{aligned} (1.1) \quad & \frac{1}{8}(4y_3 y_2 \frac{\partial^2}{\partial y_3 \partial y_2} + 4y_2^2 \frac{\partial^2}{\partial y_3 \partial y_1} + 2y_3^2 \frac{\partial^2}{\partial y_3^2} \\ & + 4y_2 y_1 \frac{\partial^2}{\partial y_2 \partial y_1} + y_3 y_1 \frac{\partial^2}{\partial y_2^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + 2y_1^2 \frac{\partial^2}{\partial y_1^2} \\ & - 8\pi^2 y_3^2 - 16\pi^2 y_2^2 - 8\pi^2 y_1^2) g = d_1 g \end{aligned}$$

and

$$\begin{aligned} (1.2) \quad & - \frac{d}{256} \left(2(16d\pi^2 + 1) \frac{\partial^2}{\partial y_2^2} - 32(8d\pi^2 - 1)\pi^2 \right. \\ & + d(8 \frac{\partial^4}{\partial y_1 \partial y_2^2 \partial y_3} - 16 \frac{\partial^4}{\partial y_1^2 \partial y_3^2} + 64\pi^2 \frac{\partial^2}{\partial y_1^2} \\ & \left. - \frac{\partial^4}{\partial y_2^4} + 64\pi^2 \frac{\partial^2}{\partial y_3^2} \right) \end{aligned}$$

$$\begin{aligned}
& + 16y_2 \frac{\partial^3}{\partial y_1 \partial y_2 \partial y_3} - 4y_1 \frac{\partial^3}{\partial y_1 \partial y_2^2} + 16y_3 \frac{\partial^3}{\partial y_1 \partial y_3^2} \\
& - 8 \frac{\partial^2}{\partial y_1 \partial y_3} + 16y_1 \frac{\partial^3}{\partial y_1^2 \partial y_3} - 64\pi^2 y_3 \frac{\partial}{\partial y_1} \\
& - 4y_2 \frac{\partial^3}{\partial y_2^3} - 4y_3 \frac{\partial^3}{\partial y_2^2 \partial y_3} + 64\pi^2 y_2 \frac{\partial}{\partial y_2} \\
& - 64\pi^2 y_1 \frac{\partial}{\partial y_1} \Big) g = d_2 g
\end{aligned}$$

with $d = y_1 y_3 - y_2^2$. Put

$$\begin{aligned}
(1.3) \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \\
g(Y) &= h(t_1, t_2, \theta).
\end{aligned}$$

Then h is periodic with respect to θ , so that h has Fourier expansion

$$(1.4) \quad h(t_1, t_2, \theta) = \sum_{n \in \mathbb{Z}} B_n(t_1, t_2) e^{ni\theta}$$

where $B_n(t_1, t_2)$ is a solution of differential equations

$$\begin{aligned}
(1.5) \quad & -4^{-1}(t_1 - t_2)^{-2} \left(4(t_1^2 + t_2^2)(t_1 - t_2)^2 \pi^2 + 2n^2 t_1 t_2 \right. \\
& \left. - (t_1 - t_2)^2 \left(t_1^2 \frac{\partial^2}{\partial t_1^2} + t_2^2 \frac{\partial^2}{\partial t_2^2} \right) \right. \\
& \left. - t_1 t_2 (t_1 - t_2) \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right) \right) B = d_1 B
\end{aligned}$$

and

$$\begin{aligned}
(1.6) \quad & \frac{t_1 t_2}{64(t_1 - t_2)^4} \left(2 \left(16(t_1 - t_2)^2 \pi^2 t_1 t_2 + 5t_1^2 - 18t_1 t_2 + 5t_2^2 \right) n^2 \right. \\
& + 8(8\pi^2 t_1 t_2 - 1)(t_1 - t_2)^4 \pi^2 + 4n^4 t_1 t_2 \\
& - (t_1 - t_2) \left(4(2t_1 - t_2)n^2 t_2 - 16(t_1 - t_2)^2 \pi^2 t_1^2 - \right. \\
& \quad \left. \left. - 3t_1^2 + 8t_1 t_2 - 3t_2^2 \right) \frac{\partial}{\partial t_2} \right. \\
& - (t_1 - t_2)^2 t_2 \left(16(t_1 - t_2)^2 \pi^2 t_1 + 3t_1 - 2t_2 \right) \frac{\partial^2}{\partial t_2^2} \\
& - (t_1 - t_2) \left(16(t_1 - t_2)^2 \pi^2 t_2^2 + 4(t_1 - 2t_2)n^2 t_1 + 3t_1^2 \right. \\
& \quad \left. - 8t_1 t_2 + 3t_2^2 \right) \frac{\partial}{\partial t_1} \\
& - t_1 (t_1 - t_2)^2 \left(16(t_1 - t_2)^2 \pi^2 t_2^2 - 2t_1 + 3t_2 \right) \frac{\partial^2}{\partial t_1^2} \\
& + 4t_2^2 (t_1 - t_2)^3 \frac{\partial^3}{\partial t_1 \partial t_2^2} - 4t_1^2 (t_1 - t_2)^3 \frac{\partial^3}{\partial y_1^2 \partial y_2} \\
& + 2(4n^2 + 1)t_1 t_2 (t_1 - t_2)^2 \frac{\partial^2}{\partial t_1 \partial t_2} \\
& \left. + 4t_1 t_2 (t_1 - t_2)^4 \frac{\partial^4}{\partial t_1^2 \partial t_2^2} \right) B_n = d_2 B_n.
\end{aligned}$$

Now we introduce variables $x = t_1 - t_2$, $y = t_1 + t_2$ and put

$$(1.7) \quad B_n(t_1, t_2) = C_n(x, y)$$

where $y > 0$, $-y < x < y$. Then the differential equations for B_n yield the condition for C_n

$$\begin{aligned}
(1.8.1) \quad & -\frac{1}{8x^2} \left(4\pi^2 x^2 (x^2 + y^2) + (y^2 - x^2)n^2 - x^2 (x^2 + y^2) \frac{\partial^2}{\partial y^2} - x^2 (x^2 + y^2) \frac{\partial^2}{\partial x^2} \right. \\
& \left. - x(y^2 - x^2) \frac{\partial}{\partial x} - 4x^3 y \frac{\partial^2}{\partial x \partial y} \right) C_n = d_1 C_n,
\end{aligned}$$

$$\begin{aligned}
(1.8.2) \quad & \frac{(y^2-x^2)}{256x^4} \left((n^4+16\pi^4x^4)(y^2-x^2) + 2(4\pi^2x^2y^2-4\pi^2x^4-2y^2+7x^2)n^2 \right. \\
& - 8\pi^2x^4+4x^3y \left(x \frac{\partial^3}{\partial y \partial x^2} + \frac{\partial^2}{\partial x \partial y} - x \frac{\partial^3}{\partial y^3} \right) \\
& + x^4(y^2-x^2) \left(\frac{\partial^4}{\partial y^4} - 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial x^4} \right) \\
& - 2x^3(y^2+x^2) \left(\frac{\partial^3}{\partial x \partial y^2} - \frac{\partial^3}{\partial x^3} \right) \\
& - x^2(2n^2y^2-2n^2x^2+8\pi^2y^2x^2-8\pi^2x^4+y^2-3x^2) \frac{\partial^2}{\partial x^2} \\
& + x(2n^2y^2-6n^2x^2-8\pi^2x^2y^2-8\pi^2x^4+y^2-7x^2) \frac{\partial}{\partial x} \\
& + 2x^2(n^2y^2-n^2x^2-4\pi^2x^2y^2+4\pi^2x^4+x^2) \frac{\partial^2}{\partial y^2} \\
& \left. - 4x^2y(n^2+4\pi^2x^2) \frac{\partial}{\partial y} \right) C_n = d_2 C_n.
\end{aligned}$$

Put $C_n(x,y) = (x^2-y^2) \sum_{l=-\infty}^{\infty} a_l(y)x^l$ where $a_l(y)=0$ if $l \leq m$ with an integer $m \geq 0$ and $a_m(y) \neq 0$. Then the equation (1.8.1) implies

$$\begin{aligned}
(1.9.1) \quad & \frac{1}{8} \left((1^2+21+n^2-4\pi^2y^2-2)a_1 + ((1+2)^2-n^2)y^2a_{1+2} - 4\pi^2a_{1-2} \right. \\
& \left. + a_{1-2}'' + y^2a_1'' + (41+1)ya_1' \right) = d_1a_1,
\end{aligned}$$

especially $(m^2-n^2)y^2a_m=0$, $((m+1)^2-n^2)y^2a_{m+1}=0$ which result $m=n$, $a_{m+1}=0$ and therefore $a_l=0$ unless $l \equiv m \pmod{2}$. The equation (1.8.2) implies

$$\begin{aligned}
(1.9.2) \quad & - \frac{1}{256} \left(2(1+n+2)(1-n+2)y^3(ya_{1+2}'' + 2a_{1+2}') \right. \\
& - (1^4 - 21^2n^2 + 161^2\pi^2y^2 - 61^2 + 161\pi^2y^2 \\
& - 161 + n^4 - 16n^2\pi^2y^2 - 10n^2 + 16\pi^4y^4 + 40\pi^2y^2 + 12)a_1 \\
& - 2(21^2 + 21 - 2n^2 - 4\pi^2y^2 - 1)y^2a_1'' \\
& \left. + 2(1^2 + 21 - n^2 + 4\pi^2y^2 + 7) \right)
\end{aligned}$$

$$\begin{aligned}
& (1+n+2)(1-n+2)y^2 a_{1+2} \\
& + 8(1^2 - 21 - n^2 + 4\pi^2 y^2 - 3)\pi^2 a_{1-2} \\
& + 2(1^2 - 21 - n^2 - 8\pi^2 y^2 + 7)a_{1-2}'' \\
& - 4(1^2 - 81 - n^2 - 4\pi^2 y^2 - 8)ya_1' \\
& - (1+n+4)(1+n+2)(1-n+4)(1-n+2)y^4 a_{1+4} \\
& - a_{1-4}^{(4)} + 8\pi^2 a_{1-4}'' + 2y^2 a_{1-2}^{(4)} + 4ya_{1-2}^{(3)} - 16\pi^2 ya_{1-2}' \\
& - y^4 a_1^{(4)} - 4y^3 a_1^{(3)} - 16\pi^4 a_{1-4}) = d_2 a_1
\end{aligned}$$

for all integers l . Since we especially have

$$\begin{aligned}
(1.10) \quad a_{n+2} &= \frac{1}{4(n+1)y^2} \left(2(4d_1 - n^2 - n + 2\pi^2 y^2 + 1)a_n - 4(n+1)ya_n' - y^2 a_n'' \right), \\
a_{n+4} &= \frac{1}{8(n+1)y^2} \left(2(4d_1 - n^2 - 3n + 2\pi^2 y^2 - 3)a_{n+2} - 4(n+3)ya_{n+2}' \right. \\
& \quad \left. - y^2 a_{n+2}'' - a_n'' + 4\pi^2 a_n \right),
\end{aligned}$$

and

$$\begin{aligned}
(1.11) \quad & 4(64d_2 + 4n^2 - 4n\pi^2 y^2 + 4n - 4\pi^4 y^4 - 10\pi^2 y^2 - 3)a_n \\
& + 8y^2(2n + 4\pi^2 y^2 + 7)(n+1)a_{n+2} + 16(2n + \pi^2 y^2 + 2)ya_n' \\
& - 2(2n - 4\pi^2 y^2 - 1)y^2 a_n'' - 32(n+2)(n+1)a_{n+4}y^4 \\
& + 8y^3(n+1)(ya_{n+2}' + 2a_{n+2}') - y^4 a_n^{(4)} - 4y^3 a_n^{(3)} = 0,
\end{aligned}$$

by (1.9.1), (1.9.2), we obtain an ordinary differential equation

$$\begin{aligned}
(1.12) \quad & (8(n^2 + n + 1)d_1 - 16d_1^2 + 64d_2 - n^4 - 2n^3 + n^2 \\
& + 2n + 8\pi^2 y^2)a_n \\
& + 4(4(n+1)d_1 - n^3 - 3n^2 - 2n + 4\pi^2 y^2)ya_n'
\end{aligned}$$

$$\begin{aligned}
& + 2(4d_1 - 3n^2 - 9n + 2\pi^2 y^2 - 6)y^2 a_n'' \\
& - 4(n+2)y^3 a_n^{(3)} - y^4 a_n^{(4)} = 0,
\end{aligned}$$

for a_n . It is more convenient to introduce parameters λ_1, λ_2 defined by

$$(1.13) \quad d_1 = \frac{\lambda_1 + \lambda_2 - 2}{8}, \quad d_2 = \frac{(\lambda_1 - \lambda_2)^2}{256} - \frac{\lambda_1 + \lambda_2}{32} + \frac{3}{64}$$

to describe the solutions of (1.12). With these λ_1, λ_2 (1.12) becomes

$$\begin{aligned}
(1.14) \quad & - (\lambda_1 \lambda_2 - n(n+1)\lambda_1 - n(n+1)\lambda_2 \\
& + n^4 + 2n^3 + n^2 - 8\pi^2 y^2) a_n \\
& + 2((\lambda_1 + \lambda_2)(n+1) - 2n^3 - 6n^2 - 6n + 8\pi^2 y^2 - 2) y a_n' \\
& + (\lambda_1 + \lambda_2 - 6n^2 - 18n + 4\pi^2 y^2 - 14) y^2 a_n'' \\
& - 4(n+2)y^3 a_n^{(3)} - y^4 a_n^{(4)} = 0.
\end{aligned}$$

For $v \in \mathbb{C}$, $m \in \mathbb{Z}$, denote the associated Legendre function of the first kind by $P_v^m(z)$ and that of the second kind by $Q_v^m(z)$ as usual. Then we have

$$\begin{aligned}
(1.15) \quad P_v^m(z) &= \frac{\Gamma(v+m+1)}{\pi \Gamma(v+1)} \int_0^\pi \left(z + (z^2 - 1)^{1/2} \cos t \right)^v \cos mt \, dt, \\
Q_v^m(z) &= (-1)^m \frac{\Gamma(v+m+1)}{2^{v+1} \Gamma(v+1)} (z^2 - 1)^{m/2} \int_{-1}^1 \frac{(1-t^2)^v}{(z-t)^{v+m+1}} \, dt
\end{aligned}$$

for z not on the real axis between 1 and ∞ where we assume

$w^\mu = e^{\mu \log w}$, $\log w = \log |w| + i \arg w$, $-\pi < \arg w < \pi$ for w , $\mu \in \mathbb{C}$. We put

$$(1.16) \quad \begin{aligned} P_\nu^m(x) &= \lim_{y \rightarrow +0} e^{3m\pi i/2} P_\nu^m(x+iy) = \lim_{y \rightarrow +0} e^{m\pi i/2} P_\nu^m(x-iy), \\ Q_\nu^m(x) &= \frac{1}{2} \lim_{y \rightarrow +0} \left(e^{-m\pi i/2} Q_\nu^m(x+iy) - e^{m\pi i/2} Q_\nu^m(x-iy) \right) \end{aligned}$$

for $-1 < x < 1$. $P_\nu^m(z)$ and $Q_\nu^m(z)$ are independent solutions of Legendre's differential equation

$$(1.17) \quad \frac{d}{dz} \left((1-z^2) \frac{d}{dz} \right) u + \nu(\nu+1)u - \frac{m^2}{1-z^2} u = 0.$$

Put

$$(1.18) \quad \begin{aligned} c_{11} &= -Q_{\nu_2}^m(0) = \sqrt{\pi} 2^{m-1} e^{m\pi i} \\ &\quad \sin\left(\frac{\nu_2 - m}{2}\right) \Gamma\left(\frac{\nu_2 + m + 1}{2}\right) / \Gamma\left(\frac{\nu_2 - m}{2} + 1\right), \\ c_{12} &= P_{\nu_2}^m(0) = \sqrt{\pi} 2^m e^{m\pi i} / \left(\Gamma\left(\frac{\nu_2 - m}{2} + 1\right) \Gamma\left(\frac{-\nu_2 - m + 1}{2}\right) \right), \\ c_{21} &= -\frac{d}{dz} Q_{\nu_2}^m(0) = -\sqrt{\pi} 2^m e^{m\pi i} \\ &\quad \cos\left(\frac{\nu_2 + m}{2}\right) \Gamma\left(\frac{\nu_2 + m}{2} + 1\right) / \Gamma\left(\frac{\nu_2 - m + 1}{2}\right), \\ c_{22} &= \frac{d}{dz} P_{\nu_2}^m(0) = 2^{m+1} e^{m\pi i} \\ &\quad \sin\left(\frac{\nu_2 + m}{2} \pi\right) \Gamma\left(\frac{\nu_2 + m}{2} + 1\right) / \left(\sqrt{\pi} \Gamma\left(\frac{\nu_2 - m + 1}{2}\right) \right). \end{aligned}$$

Then $c_{12} \neq 0$, $c_{22} \neq 0$ for $-1 < \Re \nu_2 < 0$. Put

$$(1.19) \quad R_{\nu_2}^m(z) = c_{11} P_{\nu_2}^m(z) + c_{12} Q_{\nu_2}^m(z),$$

$$S_{v_2}^m(z) = c_{21} P_{v_2}^m(z) + c_{22} Q_{v_2}^m(z).$$

Then we obtain

Proposition 1. Put $v_1 = \frac{-1+\sqrt{1-4\lambda_1}}{2}$, $v_2 = \frac{-1+\sqrt{1-4\lambda_1}}{2}$, and assume $-1 < \operatorname{Re} v_1 < 0$, $-1 < \operatorname{Re} v_2 < 0$. Then there exist polynomials h_1, h_2 of y^{-1} of degree $n-1, n$ such that

$$\begin{aligned} & \int_0^\infty \left(\int_1^\infty P_{v_1}^0(z_1) R_{v_2}^n(z_2) (-2\pi y z_2)^{-n} \right. \\ & \quad \left. (z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1 = h_1, \\ & \int_0^\infty \left(\int_1^\infty P_{v_1}^0(z_1) S_{v_2}^n(z_2) (-2\pi y z_2)^{-n} \right. \\ & \quad \left. (z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1 = h_2, \\ & \int_1^\infty \int_1^\infty P_{v_1}^n(z_1) P_{v_2}^n(z_2) (z_1^2 - 1)^{n/2} \\ & \quad (z_2^2 - 1)^{n/2} e^{-2\pi z_1 z_2 y} dz_1 dz_2, \end{aligned}$$

are linearly independent solutions of the equation (1.14).

Note that polynomials h_1, h_2 can be given explicitly. By the recurrence relation (1.10) we obtain especially

Theorem 1. The functions of x, y

$$\begin{aligned} C_n(x, y) &= (x^2 - y^2) \int_1^\infty \int_1^\infty P_{v_1}^n(z_1) P_{v_2}^n(z_2) \\ & \quad J_n \left(2\pi i (z_1^2 - 1)^{1/2} (z_2^2 - 1)^{1/2} x \right) e^{-2\pi z_1 z_2 y} dz_1 dz_2, \end{aligned}$$

are solutions of the equations (1.8.1), (1.8.2) where J_n denotes the Bessel function of first kind.

2. Here we introduce a certain theta function and show the commutation relations of invariant differential operators and explain their connection with the generalized Whittaker functions. we deal with L-function of biquadratic fields with Grössencharacter at the end.

For $(g_1, g_2) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, put

$$(2.1) \quad \rho(g_1, g_2) = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} g_1 & -1 & & \\ & & & \\ \hline & & g_1 & -1 \\ & & & \end{array} \right) \left(\begin{array}{cc|cc} a_2 E & c_2 E & & \\ & & & \\ \hline b_2 E & d_2 E & & \\ & & & \end{array} \right) \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right)$$

with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then $\rho(g_1, g_2)$ is an element in the orthogonal group for $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. For an odd squarefree integer N , we define a lattice $\mathcal{L}$ by

$$(2.2) \quad \mathcal{L} = \left\{ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \\ m_{41} & m_{42} \end{pmatrix} \mid \begin{array}{l} m_{i,j} \in \mathbb{Z} \text{ for } (i,j) \neq (1,2), \\ Nm_{1,2} \in \mathbb{Z} \end{array} \right\}.$$

For $Z = X + iY \in \mathbb{H}_2$ and a Dirichlet character χ modulo N , put

$$(2.3) \quad \begin{aligned} \theta_\chi(Z, (g_1, g_2)) &= |Y| \sum_{\substack{m = \begin{pmatrix} * & m_{12} \\ * & * \\ * & * \\ * & * \end{pmatrix} \in \mathcal{L}}} \chi(Nm_{12}) e^{\pi i N \mathrm{tr} \left(X \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} [\rho(g_1, g_2)m] \right)} \\ &\quad e^{-\pi N \mathrm{tr} \left(Y \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} [\rho(g_1, g_2)m] \right)} \end{aligned}$$

where $S[T]$ means tTST , which is usually called theta function.

Put $g_{x+iy} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ for $x+iy \in H_1$, the upper half plane and put

$$(2.4) \quad \theta(Z, z_1, z_2) = \theta_\chi(Z, (g_{z_1}, g_{z_2}))$$

then we have

Theorem 2. For $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

$$\begin{aligned} \Delta_1 \theta(Z, z_1, z_2) &= 1/8 \left(y_1^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) + y_2^2 \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right) \right) \theta(Z, z_1, z_2), \\ \Delta_2 \theta(Z, z_1, z_2) &= \left(\frac{1}{256} \left(y_1^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) - y_2^2 \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right) \right)^2 \right. \\ &\quad - \frac{1}{32} \left(y_1^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) + y_2^2 \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right) \right) \\ &\quad \left. + \frac{3}{64} \right) \theta(Z, z_1, z_2). \end{aligned}$$

Note that Δ_1, Δ_2 defined in 1 are differential operators with

respect to Z not to z_1, z_2 . Let φ_1, φ_2 be Maass wave cusp forms

with the character χ satisfying $y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi_1(z) = \lambda_1 \varphi_1(z)$, $y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi_2(z) = \lambda_2 \varphi_2(z)$.

Put $d_0 z = y^{-2} dx dy$ for $z = x + iy$, and

define

$$(2.5) \quad F_{\varphi_1, \varphi_2}(Z) = \int_{\Gamma \backslash H_1} \int_{\Gamma \backslash H_1} \theta(Z, z_1, z_2) \varphi_1(z_1) \varphi_2(z_2) d_0 z_1 d_0 z_2.$$

with $\Gamma = \Gamma_0(N)$. Then we have

$$(2.6) \quad \Delta_1 F_{\varphi_1, \varphi_2}(Z) = d_1 F_{\varphi_1, \varphi_2}(Z), \quad \Delta_2 F_{\varphi_1, \varphi_2}(Z) = d_2 F_{\varphi_1, \varphi_2}(Z),$$

with d_1, d_2 defined by the equalities (1.5) for λ_1, λ_2 , and

$$(2.7) \quad F_{\varphi_1, \varphi_2}(\sigma Z) = \chi(d) F_{\varphi_1, \varphi_2}(Z),$$

hold for $\sigma = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & a & b \\ * & * & c & d \end{pmatrix} \in \Gamma_2$ where

$$\Gamma_2 = \left\{ \sigma = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \mid \begin{array}{l} \sigma \in \text{Sp}(2, \mathbb{Q}), \\ a_{21}, a_{31}, a_{32}, a_{41}, a_{42} \in \mathbb{N}Z, \\ Na_{13} \in \mathbb{Z}, \text{ other } a_{ij} \in \mathbb{Z} \end{array} \right\}.$$

There exist a lattice $\mathcal{T}$ in $\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Q} \right\}$ containing $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ such that $F_{\varphi_1, \varphi_2}(Z)$ is expanded as follows:

$$(2.8) \quad F_{\varphi_1, \varphi_2}(Z) = \sum_{T \in \mathcal{T}} A(T, Y) e^{2\pi i \text{tr}(TX)}.$$

Hence $W(Y) = A\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1 \end{pmatrix} Y \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1 \end{pmatrix}\right)$ is a generalized Whittaker function. A direct calculation shows that $W(Y)$ is equal to

$$\sum_{n \in \mathbb{Z}} b_n B_n(t_1, t_2) e^{ni\theta} \text{ and } b_0 \text{ is equal to}$$

$$(2.9) \quad \sum_{j=1}^h \chi(a_j) \varphi_1 \left(\frac{-1}{\frac{c_j N + \sqrt{-N}}{a_j}} \right) \varphi_2 \left(\frac{-1}{\frac{c_j N + \sqrt{-N}}{a_j}} \right)$$

up to a constant multiple, where $\mathfrak{U}_j = a_j \mathbb{Z} + (c_j N + \sqrt{-N}) \mathbb{Z}$ runs over the full set of representatives of the ideal class group of $\mathbb{Q}(\sqrt{-N})$ with class number h and $B_n(t_1, t_2)$ is defined by (1.3), (1.7) with $C_n(x, y)$ in Theorem 1.

Note that the corresponding L-function (the spinor L-function) of $F_{\varphi_1, \varphi_2}(Z)$ is the product $L(s, \varphi_1) L(s, \varphi_2)$ of L-functions of φ_1 and φ_2 . Let $K_i = \mathbb{Q}(\sqrt{d_i})$ be a real quadratic field with discriminant d_i for $i=1, 2, 3$. Let $d_3 = d_1 d_2$ and assume the class number of $K = K_3$ is one. Let $\mathfrak{o}$ be the ring of integers in K and E_+ the group of all totally positive units in $\mathfrak{o}$. Put

$$(2.10) \quad g_1(z, \xi_m) = \sum_{\substack{\mu \in \mathfrak{o}/E_+ \\ \mu \neq 0}} \xi_m(\mu) \psi(\mu) y^{1/2} K_{im\kappa}(2\pi |N_{K/\mathbb{Q}} \mu| y) \cos(2\pi x N_{K/\mathbb{Q}} \mu),$$

$$g_2(z, \xi_m) = \sum_{\substack{\mu \in \mathfrak{o}/E_+ \\ \mu \neq 0}} \xi_m(\mu) \psi(\mu)^2 y^{1/2} K_{im\kappa}(2\pi |N_{K/\mathbb{Q}} \mu| y) \cos(2\pi x N_{K/\mathbb{Q}} \mu)$$

where $\xi_m(\mu) = |\mu/\mu'|^{im\kappa}$, $\kappa = 2\pi/\log \varepsilon$, $\psi(\mu) = \left(\frac{d_1}{N_{K/\mathbb{Q}} \mu} \right)$, ε the fundamental unit, $K_{im\kappa}$ the modified Bessel function. Then g_1, g_2 are Maass waveforms and we can take g_1, g_2 as φ_1, φ_2 in (2.5). In this case L-function corresponding to F_{φ_1, φ_2} is

$$\sum_{\alpha} \xi(N_{F/k}\alpha) N_{F/Q}\alpha^{-s}$$

where α runs over all integral ideals in $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $(\alpha, d_1 d_2) = 1$.

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