

Liouville type theorem for hyperfunctions

and

its applications

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§0. Introduction

In his dissertation, R.Gay proved following theorem 1

Theorem 1. Let $f(z)$ be an entire function satisfying following conditions : for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$(1) \quad |f(z)| \leq C_\varepsilon \exp(\varepsilon |z|) \quad (z \in \mathbb{C}^n)$$

Namely, $f(z)$ is infra-exponential type.

(2) $f(m) = O(|m|^p)$ for all $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$ and some $p \in \mathbb{N}$. where $\mathbb{Z}^n$ denotes the lattice points and $\mathbb{N}$ is the set of natural numbers.

Then $f(z)$ is a polynomial of degree not exceeding p .

In the case of $n = 1$, this theorem is well known. (See Boas [2])

There is another similar type theorem due to S.Bernstein.

Theorem 2(S.Bernstein) Suppose that entire function $f(z)$ satisfies the following conditions: for any $\varepsilon > 0$, there exists a constant C_ε such that

$$(3) \quad |f(z)| \leq C_\varepsilon \exp(H_K(z) + \varepsilon |z|) \quad (z \in \mathbb{C})$$

where K is a real bounded closed interval and $H_K(z) = \sup_{j \in K} \operatorname{Re}(z_j)$.

$$(4) \quad f(x) = O(|x|^p) \quad (x \in \mathbb{R}).$$

Then $f(z)$ is a polynomial of degree not exceeding p .

Our aim in this paper is to unify these two theorems by making use of hyperfunction theory. R.Gay gave the proof of theorem 1 with the help of L.Schwartz distribution theory while our method is based on analytic function theory.

Following theorem 3 is our main result. Our proof is inspired by the technic in Boas's book [2].

Theorem 3. Suppose that entire function $f(z)$ satisfies the following conditions : for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$(5) \quad |f(z)| \leq C_\epsilon \exp(H_K(z) + \epsilon |z|) \quad (z \in \mathbb{C}^n)$$

where K is a real compact convex set in $\mathbb{R}^n$, $H_K(z) = \sup_{\zeta \in K} \operatorname{Re} \langle z, \zeta \rangle$.

$$(6) \quad f(m) = O(|m|^p) \quad \text{for all } m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n.$$

Then $f(z)$ is a polynomial of degree not exceeding p .

To prove theorem 3, we make use of Fourier-Laplace and Avanissian-Gay transforms of hyperfunctions with compact support. So in §1, we recall the definitions of these two transforms and their properties. Next in §2, we give the proof of theorem 3, and as corollaries of theorem 3 we obtain theorem 1 and 2. Finally, in §3, we will show some applications.

In what follows, K always denotes a real compact convex set and $\mathcal{B}[K]$ is the space of hyperfunctions with support contained in K .

§1. Fourier-Laplace and Avanissian-Gay transforms of hyperfunctions with compact support

For hyperfunction $T \in \mathcal{B}[K]$, we define its Fourier-Laplace transform $\tilde{T}(z)$ as follows:

$$\tilde{T}(z) = \langle T_\zeta, \exp(z\zeta) \rangle, \quad (z \in \mathbb{C}^n)$$

where $z\zeta = z_1\zeta_1 + z_2\zeta_2 + \dots + z_n\zeta_n$.

Following theorem 4 characterizes Fourier-Laplace transforms of $\mathcal{B}[K]$.

Theorem 4. Suppose that T belongs to $\mathcal{B}[K]$. Then its Fourier-Laplace transform $\tilde{T}(z)$ satisfies the following estimate: for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$|\tilde{T}(z)| \leq C_\epsilon \exp(H_K(z) + \epsilon |z|) \quad (z \in \mathbb{C}^n)$$

Conversely, if an entire function $f(z)$ satisfies the above estimate, then there exists a hyperfunction $T \in \mathcal{B}[K]$ such that $f(z) = \tilde{T}(z)$.

For the proof of theorem 4, we refer to [5].

Next we define Avanissian-Gay transforms $G_T(w)$ of $T \in \mathcal{B}[K]$ as follows:

$$G_T(w) = \left\langle T_{\zeta}, \prod_{i=1}^n (1 - w_i \exp(\zeta_i))^{-1} \right\rangle.$$

Remark that in general $G_T(w)$ is defined for $w = (w_1, \dots, w_n)$ in $\prod_{i=1}^n (\mathbb{C} \setminus \exp(-K_i))$, where K_i denotes i -th projection of K .

Now we enumerate some properties of $G_T(w)$.

Proposition 5. (Avanissian-Gay [1])

- (7) $G_T(w)$ is holomorphic in $\prod_{i=1}^n (\mathbb{C} \setminus \exp(-K_i))$.
 (8) $G_T(w) = (-1)^{\xi(w_1) + \dots + \xi(w_n)} \sum_{m \in (\mathbb{N} \cup 0)^n} \tilde{T}((-1)^{\xi(w_1)m_1, \dots, (-1)^{\xi(w_n)m_n}} \theta(w_1)^{m_1} \dots \theta(w_n)^{m_n})$

where $\theta(w) = w$ if $|w| < 1$ and w^{-1} if $|w| > 1$. $\xi(w) = 0$ if $|w| < 1$ and $\xi(w) = 1$ if $|w| > 1$.

From this development, we have that $\lim_{|w| \rightarrow \infty} G_T(w) = 0$.

(9) (Inversion formula)

$$\tilde{T}(z) = (2\pi i)^{-n} \int \prod_{i=1}^n \Gamma_i G_T(e^{-\zeta_i}) \exp(z \zeta_i) d\zeta_i$$

where Γ_i is a positively oriented contour surrounding K_i .

Remark that Avanissian-Gay transform is one-to-one because of this inversion formula.

§2. Proof of Theorem 3.

First of all, we prepare the following lemma

Lemma 6. Suppose that $T \in \mathcal{B}[K]$. If its Fourier-Laplace transform $\tilde{T}(z)$ satisfies following condition:

$$(10) \limsup_{|m| \rightarrow \infty} |\tilde{T}(m)|^{1/|m|} \leq 1.$$

Then T belongs to $\mathcal{B}[\{0\}]$. In another words, $\tilde{T}(z)$ is an entire function of infra-exponential type.

(proof) For the simplicity, we will confine ourselves to the cases of $n = 1$ and 2 .

(i) $n = 1$ By proposition 5-(8), we have following expansions :

$$G_T(w) = \sum_{n=0}^{\infty} \tilde{T}(n)w^n \quad (|w| < 1),$$

$$G_T(w) = - \sum_{n=1}^{\infty} \tilde{T}(-n)w^{-n} \quad (|w| > 1).$$

From the assumption (10), it is easy to see that $G_T(w)$ is holomorphic in $|w| < 1$ and $|w| > 1$. From this and proposition 5-(7), $G_T(w)$ is holomorphic in $\mathbb{C} \setminus \{1\}$. By means of Inversion formula(5-(9)), we have

$$\tilde{T}(z) = (2\pi i)^{-1} \int_{\Gamma} G_T(e^{-\zeta}) \exp(z\zeta) d\zeta$$

As $G_T(w)$ is holomorphic in $\mathbb{C} \setminus \{1\}$, $G_T(e^{-\zeta})$ is holomorphic in $\mathbb{C} \setminus \{0\}$. Therefore we can shrink Γ towards $\{0\}$ arbitrarily. Hence $\tilde{T}(z)$ is entire function of infra-exponential type.

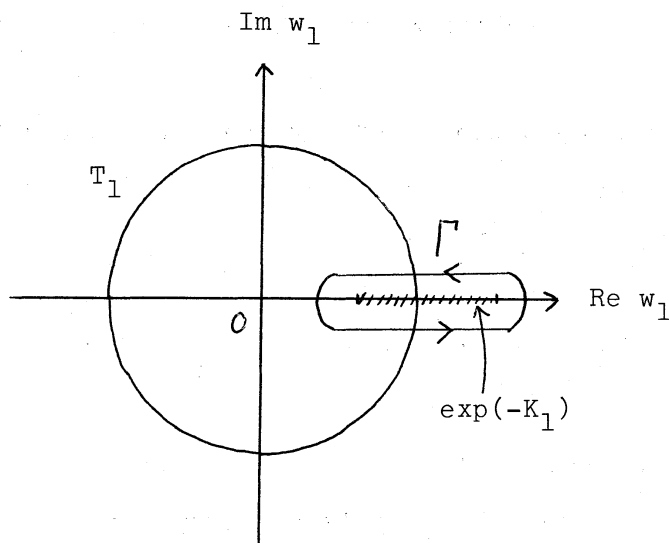
(ii) $n=2$ From 5-(7) and (8), it is easy to see that $G_T(w_1, w_2)$ is holomorphic in $\prod_{i=1}^2 (\mathbb{C} \setminus \exp(-K_i))$ and $\prod_{i=1}^2 (\mathbb{C} \setminus T_i)$, where T_i is unit circle.

To show that $G_T(w_1, w_2)$ is holomorphic in $(\mathbb{C} \setminus \{1\}) \times (\mathbb{C} \setminus \{1\})$, we consider following Cauchy integral :

$$G_T(w_1, w_2) = -(2\pi i)^{-1} \int_{\Gamma} G_T(t, w_2) (t - w_2)^{-1} dt$$

Γ is shown in the Figure 1.

Figure 1.



Let fix w_2 in $\mathbb{C} \setminus T_2$. Then we can deform Γ towards 1, since $G_T(t, w_2)$ is holomorphic in $\prod_{i=1}^2 (\mathbb{C} \setminus T_i)$. As $\lim_{|w_1| \rightarrow \infty} G_T(w_1, w_2) = 0$, $G_\Gamma(w_1, w_2)$ gives single valued analytic continuation of $G_T(w_1, w_2)$ up to $(\mathbb{C} \setminus \{1\}) \times (\mathbb{C} \setminus T_2)$. Interchanging the role of w_1 and w_2 , we can obtain analytic continuation of $G_T(w_1, w_2)$ up to $(\mathbb{C} \setminus T_1) \times (\mathbb{C} \setminus \{1\})$.

By elementary set operation, we have the following identity :

(A^c denotes the complement of A .)

$$\begin{aligned} & (A_1 \cap B_1)^c \times B_2^c \cup B_1^c \times (A_2 \cap B_2)^c \cup A_1^c \times A_2^c \\ &= (A_1 \cap B_1)^c \times (A_2 \cap B_2)^c. \end{aligned}$$

Putting $A_1 = \exp(-K_1)$ and $B_1 = T_1$, we see that $G_T(w_1, w_2)$ is holomorphic in $(\mathbb{C} \setminus \{1\}) \times (\mathbb{C} \setminus \{1\})$.

Like in the case of $n = 1$, we apply inversion fomula:

$$\tilde{T}(z_1, z_2) = (-2\pi i)^{-2} \int_{\Gamma_1 \times \Gamma_2} G_T(e^{-\zeta_1}, e^{-\zeta_2}) \exp(z_1 \zeta_1 + z_2 \zeta_2) d\zeta_1 d\zeta_2$$

Since $G_T(e^{-\zeta_1}, e^{-\zeta_2})$ is holomorphic in $(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, we can shrink the contours Γ_i ($i = 1, 2$) towards the origin.

Hence $\tilde{T}(z_1, z_2)$ is entire function of infra-exponential type.

This means that T belongs to $\mathcal{B}[\{0\}]$.

Now we pass to the proof of theorem 3. First we prove theorem 3 in the case of $n = 1$. Next we will treat the case of $n = 2$.

(proof of theorem 3) By means of theorem 4, there exists hyperfunction $T \in \mathcal{B}[K]$ such that $f(z) = \tilde{T}(z)$. By virtue of preceding Lemma, $f(z)$ is entire function of infra-exponential type.

(iii) $n=1$ We consider the following entire function $F(z)$.

$$F(z) = z^{-p-1} \left(f(z) - \sum_{n=0}^p a_n z^n \right) \quad (z \in \mathbb{C})$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is Maclaurin expansion of $f(z)$.

We put $H(z) = F^2(z)$. As is easily seen, $H(z)$ is entire function of infra-exponential type, and $H(m) = O(m^{-2})$. There exists a hyperfunction $S \in \mathcal{B}[\{0\}]$ such that $\tilde{S}(z) = H(z)$. We consider Avanissian-Gay transform $G_S(w)$ of S . $G_S(w)$ has following expansion:

$$G_S(w) = \sum_{n=0}^{\infty} \tilde{S}(n) w^n \quad (|w| < 1)$$

$$G_S(w) = - \sum_{n=1}^{\infty} \tilde{S}(-n) w^{-n} \quad (|w| > 1)$$

Since $\tilde{S}(m) = O(m^{-2})$, $G_S(w)$ is bounded in $\mathbb{C} \setminus \{0\}$. By virtue of celebrated Riemann's theorem concerning with removable singularity, $G_S(w)$ is entire function. Furthermore, as $\lim_{|w| \rightarrow \infty} G_S(w) = 0$, $G_S(w)$ vanishes identically. By means of the inversion formula, $H(z) = \tilde{S}(z)$ also vanishes identically.

Hence we obtain the following desired result:

$$f(z) = \sum_{n=0}^p a_n z^n.$$

(iv) $n=2$ We develop $f(z_1, z_2)$ as follows :

$$f(z_1, z_2) = \sum a_{n_1, n_2} z_1^{n_1} z_2^{n_2}$$

$$= \sum_{n_2} a_{n_2} (z_1) z_2^{n_2}$$

We put $F(z_2) = f(m_1, z_2)$ for fixed integer m_1 . Since $f(z_1, z_2)$ is entire function of infra-exponential type, $F(z_2)$ has same property.

Moreover $F(z_2)$ satisfies following estimate :

$$F(m_2) = O(|m_2|^p) \quad \text{for all } m_2 \in \mathbb{Z}.$$

Therefore by preceding result in (iii), $F(z_2)$ is polynomial of degree not exceeding p . This means that $f(m_1, z_2) = \sum_{n_2=0}^p a_{n_2}(m_1) z_2^{n_2}$

On the other hand, we have

$$f(m_1, z_2) = \sum_{n_2=0}^{\infty} a_{n_2}(m_1) z_2^{n_2}$$

Hence for $n_2 \geq p+1$,

$$a_{n_2}(m_1) = 0 \quad \text{for all } m_1 \in \mathbb{Z}.$$

Applying Cauchy's integral formula to $f(z_1, z_2)$, it is readily seen that $a_{n_2}(z_2)$ is entire function of infra-exponential type. From the result in (iii), we see that $a_{n_2}(z_1)$ vanishes identically for $n_2 \geq p+1$.

Therefore

$$f(z_1, z_2) = \sum_{n_2=0}^p a_{n_2}(z_1) z_2^{n_2}$$

Repeating the same argument, we obtain

$$f(z_1, z_2) = \sum_{\substack{n_1=0 \\ n_2=0}}^p a_{n_1, n_2} z_1^{n_1} z_2^{n_2}$$

By means of the assumption, $f(m_1, m_2) = O(|m|^p)$, we can conclude that $f(z_1, z_2)$ is polynomial of degree not exceeding p .

This is our desired result.

§3. Applications.

From theorem 3, immediately, we obtain following

Proposition 7. Suppose that T is a element of $\mathcal{B}[K]$. Then followings are equivalent.

(11) T is a finite linear combination of derivatives of order not exceeding p of Dirac's delta function.

(12) $\tilde{T}(z)$ is a polynomial of degree not exceeding p .

(13) $\tilde{T}(m) = O(|m|^p)$ for all $m \in \mathbb{Z}^n$.

In next three applications, we will show how to apply theorem 1.

First we treat the discriminant of Hill's equation. Following facts are well known ([4]).

(14) Discriminant $\Delta(E)$ is an entire function of order $1/2$.

(15) $\lim_{E \rightarrow \infty} E^{1/2} (\Delta(E) - 2\cos(\pi E^{1/2})) = 0$.

Now we put $f(E) = \Delta(E) - 2\cos(\pi E^{1/2})$. Then we have following

Proposition 8. If $f(E)$ have the following property:

$$f(-am) = O(|m|^p) \quad (m = 1, 2, 3, \dots)$$

for some positive number a , then potential $V(x)$ in Hill's equation vanishes identically.

(proof) First we remark that $f(E)$ is an entire function of infra-exponential type. So from the assumption, (15) and theorem 1, we can conclude that $f(E)$ is polynomial. Furthermore, $f(E)$ vanishes identically because of (15). Namely, $\Delta(E) = 2\cos(\pi E^{1/2})$. This means that there is no unstable intervals. Hence by virtue of Hochstadt's theorem ([4]), we can obtain desired result.

Next we pass to eigenvalue problem of trace class operator defined on a Hilbert space. Let A be a trace class operator. According to [7], we put

$$f(z) = \det(I + zA) \quad (z \in \mathbb{C}).$$

Then following two facts are known[7]:

(16) $f(z)$ is entire function of infra-exponential type.

(17) $f(z) = \prod_{j=1}^{N(A)} (1 + z \lambda_j(A))$

where $\lambda_j(A)$ ($j=1 \dots N(A)$) are the eigen values of A counted with algebraic multiplicity and $N(A)$ is finite or countably infinite.

Under these preparation, we have following

Proposition 9. If $f(m) = O(|m|^p)$ for all $m \in \mathbb{Z}$ and some natural number p , then operator A has at most p eigenvalues.

(proof) From the assumption and theorem 1, we will see that $f(z)$ is polynomial. By virtue of (17), there exist at most p eigen values.

Remark : If the assumption holds for $p=0$, then we can conclude that eigenvalue of A is just 0. For the details of this we refer to exercise 156 in [7].

Finally we give one method to construct analytic functionals without unique carrier. For the definition and some special terminologies of analytic functionals, we refer [5] and [6].

Proposition 10. Suppose that entire function $g(z)$ satisfies following conditions :

(18) $g(z)$ is even function,

(19) there exist constants A and B such that

$$|g(z)| \leq B \exp(A|z|) \quad (z \in \mathbb{C})$$

(20) $g(m) = O(|m|^p)$,

(21) $g(z)$ is not polynomial.

Put $T = FB^{-1}(g(\sqrt{z_1 z_2}))$, where FB^{-1} denotes inverse Fourier-Borel (sometime called Fourier-Laplace) transform. Then analytic functional T is carried by polydiscs $D_{a,b} = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| \leq aA, |z_2| \leq bA, ab = 1, a > 0, b > 0\}$. But T is not carried by the origin.

Remark : Intersection of $D_{a,b}$ is the origin.

(proof) From the assumption (18), $g(\sqrt{z_1 z_2})$ is entire function. By the elementary calculation and (19), $g(\sqrt{z_1 z_2})$ satisfies the following inequality : for any positive numbers a and b satisfying $ab=1$,

$$|g(\sqrt{z_1 z_2})| \leq B \exp(aA|z_1| + bA|z_2|) \quad ((z_1, z_2) \in \mathbb{C}^2)$$

This means that T is carried by the polydiscs $D_{a,b}$. Now we will show that T is not carried by the origin. If it were so, then $g(\sqrt{z_1 z_2})$ should be entire function of infra-exponential type.

Therefore $g(z)$ also have same property. Applying theorem 1 and (20), we obtain that $g(z)$ is polynomial. This contradicts (21).

To close this final section, we give three examples of function $g(z)$.

Example 1. $g(z) = z^{2p} \cos z$. ($p = 0, 1, \dots$)

$T = FB^{-1}(\cos(\sqrt{z_1 z_2}))$ is firstly proposed by L.Hörmander in [5].

Example 2. $g(z) = z^{2p-1} \sin z$. ($p = 0, 1, \dots$)

Example 3. $g(z) = z^{-p} J_p(z)$. ($J_p(z)$ is Bessel function of order p and $p = 0, 1, \dots$). $T = FB^{-1}(J_0(\sqrt{z_1 z_2}))$ is considered by A.Martineau with slight modification in [6].

References

- [1] V.Avanissian and R.Gay: Sur une transformations des fonctionnelles
et ses applications aux fonctions entières a plusieurs variables,
Bull. Soc. Math. France 103(1975) 341-384.
- [2] R.P.Boas: Entire Functions, Academic Press, New York(1954)
- [3] R.Gay: Doctor Thesis, Universite Louis Pasteur-Strasbourg I,
Department de Mathematique. 1976.
- [4] H.Hochstadt: The functions of Mathematical Physics, Wiley
Interscience, New York(1971)
- [5] L.Hörmander: An introduction to Complex Analysis in Several
Variables, Van North Holland (1966)
- [6] A.Martineau: Sur les fonctionnelles analytiques et la
transformation de Fourier-Borel, J.Anal.Jerusalem. t.11(1964)
1-164.
- [7] M.Reed and B.Simon: Analysis of Operator, Academic Press,
New York, San Francisco, London (1978).

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