

GRAPH-THEORETIC APPROACHES
TO
LINEAR ELECTRICAL NETWORK ANALYSIS

BY
TAKAO OZAWA

MARCH 1975

FACULTY OF ENGINEERING
KYOTO UNIVERSITY
JAPAN



GRAPH-THEORETIC APPROACHES
TO
LINEAR ELECTRICAL NETWORK ANALYSIS

BY
TAKAO OZAWA

MARCH 1975

FACULTY OF ENGINEERING
KYOTO UNIVERSITY
JAPAN

DOC
1975
14
電気系

CONTENTS

CHAPTER 1. ELECTRICAL NETWORKS AND GRAPHS	
1.1	Introduction 1
1.2	Network Restrictions 3
1.3	Derivation of Voltage and Current Graphs 6
CHAPTER 2. PROPERTIES OF MULTICOLORED-BRANCH GRAPHS	
2.1	Introduction 8
2.2	Notations and Basic Properties 8
2.3	Degree of Interference of Cutsets and Loops 11
2.4	Concluding Remarks 18
CHAPTER 3. MIXED ANALYSIS OF LINEAR NETWORKS	
3.1	Introduction 20
3.2	Graphs Consisting of a Pair of Disjoint Trees ... 21
3.3	Graph-Partitioning for the Mixed Analysis 28
3.4	Concluding Remarks 33
CHAPTER 4. SOLVABILITY OF LINEAR NETWORKS	
4.1	Introduction 34
4.2	Common Trees of Voltage and Current Graphs 35
4.3	Voltage and Current Graphs without Common Tree .. 41
4.4	Partition of Voltage and Current Graphs 52
4.5	Classification of Network Solvability 59
4.6	Concluding Remarks 63
CHAPTER 5. STATE EQUATIONS AND ORDER OF COMPLEXITY OF LINEAR NETWORKS	
5.1	Introduction 65
5.2	Order of Complexity of Linear Active Networks ... 66

5.3 State Equations of Linear Active Networks	72
5.4 Concluding Remarks	86
AKNOWLEDGEMENT	87
REFERENCES	88

CHAPTER 1 ELECTRICAL NETWORKS AND GRAPHS

1.1 INTRODUCTION

The application of linear graph theory to electrical network analysis can be dated back to the middle of the nineteenth century, when Kirchhoff applied topological concepts to formulate the loop-basis network equations. Although in the first half of the twentieth century Kuratowski and Whitney made important contributions in the development of graph theory by introducing many useful concepts, the application of topological methods to network analysis had not made significant advancement until the beginning of the second half of the century.

In 1950's and 1960's a great number of so-called topological formulas were given for both passive and active networks. Many elaborate topological techniques were presented to handle networks containing mutual inductors, transformers, gyrators, controlled sources and so on. During that period the state-space approaches to network analysis also began to attract wide attention of network theorists. The formulation of the state equations for an RLC network was well established making full use of topological properties of the network. Extensive efforts were made to derive the state equations for various types of networks.

In recent years the application of digital computers to network analysis accelerated the need of systematic method. Graph-theoretic approaches have been found useful to the systematic formulation of the network equations and the state equations, to the reduction of computation time and memory size, or to the

handling of ill behaviors encountered in computing.

The purpose of this dissertation is not to present the full scope of computer-aided network analysis, but to give fundamental investigations to the network topology, which offer underlying principles to solve the fundamental problems arising in computer application to network analysis.

In the remaining sections of this chapter the restrictions on the networks considered in this dissertation are explained, and a brief description of a graphical method, called the 2-graph method, is given.

Some of the fundamental properties of multi-colored-branch graphs, which can represent electrical networks in a very convenient way, are discussed in Chapter 2. The results derived in the chapter are used in the following chapters.

The networks considered in Chapter 3 are RLC networks, which contain no mutually coupling elements. The topological properties useful for the mixed analysis of such networks are derived.

So-called active networks are considered in Chapters 4 and 5. Although the adjective "active" is originally concerned with the generation of energy in the networks, now it is often used to signify the existence of mutually coupling elements, especially controlled sources, which may or may not generate energy.

The problem of solvability, one of the most fundamental and important problems in electrical network analysis, is studied in Chapter 4, and a solution to this problem is given. The theorems and algorithms obtained in the chapter provide a fundamental contribution in electrical network analysis.

In Chapter 5 the state equations and the order of complexity

of active networks are studied from a topological point of view. The way how the network topology is related to the final form of the state equations is clarified, and a graphical solution to the problem of the order of complexity is given in a very compact form.

1.2 NETWORK RESTRICTIONS

The graph of a network is obtained by replacing all its elements by simple line segments. We denote the network to be considered by N^* and its graph by G^* . The asterisk on the shoulder of N indicates that the network contains independent sources. The network obtained from N^* by contracting independent voltage sources and deleting independent current sources is denoted by N . The network N is a so-called free-network. The graph of N is denoted by G .

For the simplicity of the presentation of the graph-theoretic approaches which will be discussed in this dissertation, we impose the following restrictions on N^* .

(i) Restrictions on the network elements.

The network N^* contains only those elements that are listed below. The symbols indicating the elements and the numbers of the elements are also listed.

In Chapter 3:

elements	symbols	numbers of elements
independent voltage sources	e	n_e
independent current sources	j	n_j
resistors	r	n_r

inductors	l	n_l
capacitors	c	n_c

In Chapters 4 and 5, in addition to the above:

elements	symbols	numbers of elements
current-controlled voltage sources	α, β	n_β
voltage-controlled current sources	γ, δ	n_δ

A current-controlled voltage source[voltage-controlled current source][@] is a twoport element consisting of a current sensor [voltage sensor] and a controlled voltage source[current source], as shown in Fig.1.1[Fig.1.2]. The voltage vs current relation of the element is given by (1.1)[(1.2)].

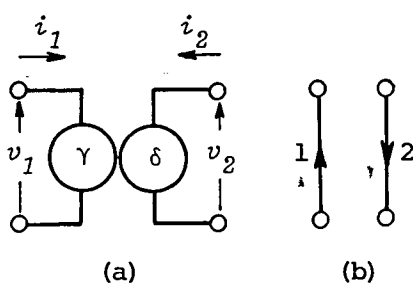
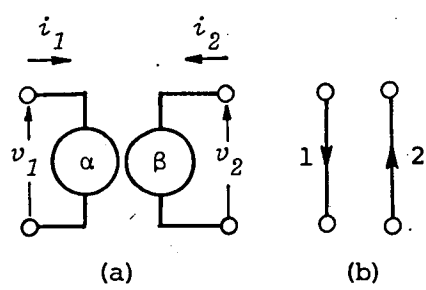


Fig.1.1 (a) Current-controlled voltage source. α : current sensor, β : controlled voltage source. (b) Graphical representation.

Fig.1.2 (a) Voltage-controlled current source. γ : voltage sensor, δ : controlled current source. (b) Graphical representation.

$$\left. \begin{aligned} v_1 &= 0 \\ v_2 &= \beta_{12} i_1 \end{aligned} \right\} \quad (1.1)$$

$$\left. \begin{aligned} i_1 &= 0 \\ i_2 &= \delta_{12} v_1 \end{aligned} \right\} \quad (1.2)$$

[@] The repetition of the dual sentence is avoided. The words in [] are to replace their preceding words in the dual sentence.

The other elements which may appear in a usual active network, such as mutual inductors, transformers, gyrators and impedance convertors, can be replaced by their equivalent controlled-source representations. [1.1] A current-controlled current source can be replaced by a cascade connection of a current-controlled voltage source and a voltage-controlled current source. [1.2] A voltage-controlled voltage source can be replaced similarly. Thus in spite of the above restrictions we suffer no loss of generality of the network which we are considering.

(ii) Restrictions on the network topology.

(ii-1) There is no loop consisting of independent voltage sources and/or current sensors only, nor cutset consisting of independent current sources and/or voltage sensors only.

(ii-2) A controlled source is controlled by a sensor.

The first restriction (ii-1) is imposed so that the definitions of the independent sources are consistent with Kirchhoff's voltage and current laws. A current sensor or a voltage sensor is essentially a short- or an open-circuit element respectively. Then a loop consisting of current sensors or a cutset consisting of voltage sensors has no meaning in a usual electrical network.

The second restriction (ii-2) is imposed so as to be the network topology consistent with the restriction (i) on the network elements and the definitions of the controlled sources as given in Fig.1.1 and Fig.1.2. Thus if, for instance, many voltage sources are controlled by a current through an element, as many current sensors as the sources are inserted in series with the element. The existence of a current[voltage] sensor which is not in series[parallel] with an element, is allowed.

Therefore we are considering quite wide varieties of network topology.

1.3 DERIVATION OF VOLTAGE AND CURRENT GRAPHS

In Chapters 4 and 5 we will use the 2-graph method. [1.3]

The 2-graphs, or the voltage and current graphs, denoted by G_v and G_i , are the graphs of network N_v and N_i respectively, which are derived from N^* in either of the following two ways.

To derive $N_v[N_i]$ from N^* : [1.1]

- (i) Obtain $N_v^*[N_i^*]$, which represents the voltage[current] relations in N^* , by deleting[contracting] independent and controlled current sources[voltage sources], and contracting[deleting] current sensors[voltage sensors]. Then contract[delete] independent voltage sources[current sources].
- (ii) Obtain N from N^* , and then delete[contract] controlled current sources[voltage sources] and contract[delete] current sensors[voltage sensors].

From the derivation of N_v and N_i , it can be easily seen that an element in N appears as a branch in G_v and G_i . Thus a branch in G_v and a branch in G_i have a one-to-one correspondence to each other. The corresponding branches in G_v and G_i are often regarded as the same branch appearing in different graphs, and are usually given the same identification. Of course the numbers of branches of G_v and G_i are equal. An example of G_v and G_i for a given network is shown in Fig.1.3. The suffixes of the symbols indicate the corresponding sensors and controlled sources.

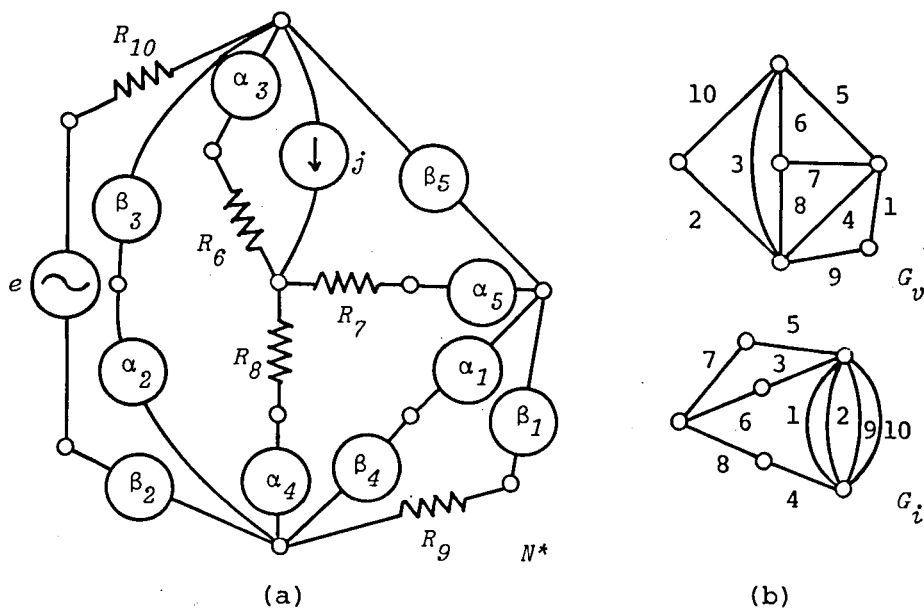


Fig.1.3 Example 1.1. (a) Network N^* . (b) Voltage and current graphs, G_v and G_i .

The above procedure to derive the 2-graphs is applicable to networks containing controlled sources. The derivation of 2-graphs from active network model using norators and nullators [1.4] is as follows.

To derive $N_v[N_i]$ from N^* :

(i) Obtain $N_v^*[N_i^*]$ by deleting[contracting] independent current sources[voltage sources]. Then delete[contract] norators and contract[delete] nullators.

or:

(ii) Obtain N from N^* and then delete[contract] norators and contract[delete] nullators.

The 2-graphs derived from a network using norators and nullators contain the branches corresponding to resistors, inductors and capacitors only.

CHAPTER 2 PROPERTIES OF MULTICOLORED-BRANCH GRAPHS

2.1 INTRODUCTION

A multicolored-branch graph is such a linear graph that the branches of the graph are partitioned into several sets, and a certain color is assigned to the branches belonging to each of the sets to signify the nature of the branches. For instance, if the graph represents an electrical network, the sets of branches may correspond to resistors, capacitors and inductors, or may correspond to passive elements and active elements. Two-colored-branch graphs were introduced by Reza to investigate the order of complexity of electrical networks. [2.1] Hattori presented a theory of multicolored-branch graphs and used it in the derivation of the state equations of electrical networks. [2.2] [2.3] He introduced the concept of the degree of interference of loops or cutsets in multicolored-branch graphs, as is defined in section 2.2. The order of complexity of a linear passive or active network was given in terms of the degree of interference of loops or cutsets. [2.2] [2.3]

2.2 NOTATIONS AND BASIC PROPERTIES

Let us first give some notations concerning multicolored-branch graphs. Given an arbitrary graph G , we partition the branches of G into sets $A_1, A_2, \dots, A_m$. A coloring of branches of G , or simply a coloring, is an assignment of colors to the branches of G . In other words, a coloring is a specification of the partition of the branches into sets $A_1, A_2, \dots, A_m$. For

simplicity, we denote the colors of the branches in sets $A_1, A_2, \dots$, and A_m also by $A_1, A_2, \dots$, and A_m , respectively. With a particular partition of branches or coloring the graph is denoted by $G(A_1, A_2, \dots, A_m)$. In the following we often delete or contract all the branches in a set. If, for instance, all the branches in set A_1 are deleted or contracted, the graph obtained by this operation is denoted by $G(0, A_2, \dots, A_m)$ or $G(1, A_2, \dots, A_m)$ respectively. Thus $A_k (k=1, 2, \dots, m)$ are regarded as variables taking one of $A_k, 0$ and 1 . Furthermore, we often pay attention to a particular set of branches only, and delete or contract the rest of the branches. In such cases the partition of the rest of the branches is insignificant and indicating it merely makes the notation unnecessarily complex. Thus if we delete or contract all the branches except those in a set, say A_k , we denote the derived graph by $G(A_k; 0)$ or $G(A_k; 1)$ respectively. The rank and nullity of $G(A_1, A_2, \dots, A_m)$ are denoted by $\rho(A_1, A_2, \dots, A_m)$ and $\mu(A_1, A_2, \dots, A_m)$ respectively.

The complementary set of A is denoted by $\bar{A}$, and the number of elements of A , by $|A|$.

Now we give some basic properties of a 2-colored-branch graph $G(A, B)$. It is obvious that^[2.2]

$$\rho(A, B) \geq \rho(A, 0) \geq \rho(A, 1) \geq 0 \quad (2.1)$$

$$\mu(A, B) \geq \mu(A, 1) \geq \mu(A, 0) \geq 0. \quad (2.2)$$

The following theorem is most fundamental.

Theorem 2.1

$$\max(\text{number of tree-branches of color } A) = \rho(A, 0) \quad (2.3)$$

$$\min(\text{number of tree-branches of color } A) = \rho(A, 1) \quad (2.4)$$

$$\max(\text{number of chords of color } A) = \mu(A, 1) \quad (2.5)$$

$$\min(\text{number of chords of color } A) = \mu(A, 0) \quad (2.6)$$

$$\max(\text{number of tree-branches of color } B) = \rho(0, B) \quad (2.7)$$

$$\min(\text{number of tree-branches of color } B) = \rho(1, B) \quad (2.8)$$

$$\max(\text{number of chords of color } B) = \mu(1, B) \quad (2.9)$$

$$\min(\text{number of chords of color } B) = \mu(0, B) \quad (2.10)$$

where the maximum or the minimum in the above equations is taken over all trees of $G(A, B)$.

Proof: We notice that a tree[cotree] is a subgraph which contains no loop[cutset], and that deletion[contraction] of branches yields no loop[cutset]. Hence the deletion[contraction] of branches of color B leaves the tree-branches[chords] of color A to form a tree or a part of a tree[a cotree or a part of a cotree] in $G(A, 0)[G(A, 1)]$. It is obvious that there is a tree[cotree] of $G(A, B)$ which contains the branches of a tree[cotree] of $G(A, 0)[G(A, 1)]$. Thus we have (2.3)[(2.5)]. If the number of tree-branches[chords] of color A is maximum, the number of chords[tree-branches] of color A is minimum. Hence we have (2.6) from (2.3)[(2.4) from (2.5)]. We can get (2.7)-(2.10) from (2.3)-(2.6) by merely replacing color A with color B .

Q.E.D.

Theorem 2.2^[2.2][2.4]

$$\rho(A, B) = \rho(A, 0) + \rho(1, B) = \rho(A, 1) + \rho(0, B) \quad (2.11)$$

$$\mu(A, B) = \mu(A, 0) + \mu(1, B) = \mu(A, 1) + \mu(0, B). \quad (2.12)$$

Equations in (2.11) and (2.12) can be interpreted in terms of

the numbers of tree-branches or chords of special trees of $G(A, B)$. From (2.3) and (2.8) we see that the first equation in (2.11) corresponds to a tree of $G(A, B)$ which contains a maximum of branches of color A and accordingly a minimum of branches of color B . The other equations in (2.11) and (2.12) can be interpreted likewise.

2.3 DEGREE OF INTERFERENCE OF CUTSETS AND LOOPS

A loop or cutset in $G(A_1, A_2, \dots, A_m)$ is called k -chromatic if the total number of colors assigned to the branches in the loop or cutset is exactly k . The degree of interference of cutsets[loops] in $G(A_1, A_2, \dots, A_m)$, denoted by $\bar{\rho}(A_1, A_2, \dots, A_m)$ [$\bar{\mu}(A_1, A_2, \dots, A_m)$], is the minimum number of m -chromatic cutsets[loops] in a set of $\rho(A_1, A_2, \dots, A_m)$ [$\mu(A_1, A_2, \dots, A_m)$] independent cutsets[loops], the minimum being taken over all possible such sets.

For a 2-colored-branch graph the degree of interference of cutsets or loops is given as follows. [2.2]

$$\bar{\rho}(A, B) = \rho(A, B) - \rho(A, 1) - \rho(1, B) = \rho(A, 0) - \rho(A, 1) \quad (2.13)$$

$$\bar{\mu}(A, B) = \mu(A, B) - \mu(A, 0) - \mu(0, B) = \mu(A, 1) - \mu(A, 0). \quad (2.14)$$

Since the number of branches of color A is,

$$|A| = \rho(A, 0) + \mu(A, 0) = \rho(A, 1) + \mu(A, 1), \quad (2.15)$$

the degree of interference of cutsets and that of loops in $G(A, B)$ are equal. Hence, both of them are called simply the

degree of interference in $G(A, B)$, and are denoted by $v(A, B)$, that is,

$$\bar{\rho}(A, B) = \bar{\nu}(A, B) = v(A, B). \quad (2.16)$$

Now the distance between a pair of trees is defined to be the number of branches contained in one of the trees but not in the other. For trees T_A and T_B , the distance between them is denoted by $\theta(T_A, T_B)$. The degree of interference can be related to the distance between a pair of trees of $G(A, B)$ as follows. [2.5][2.6]

Theorem 2.3

For a given graph $G(A, B)$ there is a pair of trees such that the distance between them is equal to the degree of interference.

Proof: Let $T_A[T_B]$ be a tree of $G(A, B)$ containing a maximum of branches of color $A[B]$. The number of branches of color A and B in $T_A[T_B]$ are, from (2.3) and (2.8), $\rho(A, 0)$ and $\rho(1, B)$ [from (2.4) and (2.7), $\rho(A, 1)$ and $\rho(0, B)$] respectively. In general there are many choices for T_A and T_B , but we choose a special pair T_A^* and T_B^* which has as many common branches as possible. Such a pair can be obtained as follows. Let a tree of $G(1, B)$ and $G(A, 1)$ be T_{1B} and T_{A1} respectively. Then choose a tree of $G(A, 0)$ which contains T_{A1} and denote it by T_{A0} . Likewise let T_{0B} be a tree of $G(0, B)$ containing T_{1B} . Then construct $T_A^* = T_{1B} \cup T_{A0}$ and $T_B^* = T_{A1} \cup T_{0B}$. Since T_A^* and T_B^* have T_{A1} and T_{1B} in common, the distance between T_A^* and T_B^* is given as,

$$\theta(T_A^*, T_B^*) = \rho(A, 0) - \rho(A, 1) = \rho(0, B) - \rho(1, B) = v(A, B). \quad (2.17)$$

and the theorem holds.

Q.E.D.

From the procedure in the above proof we also see that

$$\theta(T_A^*, T_B^*) \geq v(A, B), \quad (2.18)$$

since T_A^* and T_B^* have the maximum number of common branches.

Next let us consider the fundamental cutset matrix, Q_f , defined by T_A . We can write Q_f in the form

$$Q_f = [U \quad Q_p] = \begin{bmatrix} U_A & 0 & Q_{AB} & Q_{AA} \\ 0 & U_B & Q_{BB} & 0 \end{bmatrix} \quad (2.19)$$

where U is a unit matrix and Q_p is called the principal part of Q_f . U_A and U_B are unit matrices corresponding to the tree-branches of color A and color B respectively. Q_{AA} , Q_{AB} and Q_{BB} are submatrices of Q_p corresponding to the tree-branches of color specified by the first subscript, and to the chords of color specified by the second subscript. The lower rows of (2.19) correspond to the monochromatic cutsets of color B , and thus we have a zero-submatrix at the right-lower corner. We have: [2.5] [2.6]

Theorem 2.4

The rank of Q_{AB} is equal to the degree of interference in $G(A, B)$.

Proof: Consider a graph, denoted by G_A , which is obtained from G by contracting all the tree-branches of color B . The fundamental cutset matrix of G_A corresponds to the upper rows of (2.19). Since there are $\rho(A, 1)$ independent monochromatic

cutsets of color A in G_A , the upper rows of (2.19) can be converted to the form

$$\left[\begin{array}{c|c|c|c} Q_A' & 0 & 0 & Q_{AA}' \\ \hline & & Q_{AB}' & \end{array} \right]$$

by proper additions or subtractions among the rows. The zero-submatrix above Q_{AB}' has $\rho(A, 1)$ rows and thus the rank of Q_{AB} is not more than $\rho(A, 0) - \rho(A, 1) = v(A, B)$. Now deleting the branches of color A in G_A yields a graph consisting of the chords of color B of T_A . Noting that the number of tree-branches of color B in T_A is $\rho(1, B)$, we see that a tree of this graph contains $\rho(0, B) - \rho(1, B) = v(A, B)$ branches. As the columns of Q_{AB} contain the columns corresponding to the branches of such a tree, the rank of Q_{AB} is no less than $v(A, B)$. Hence we have the theorem.

Q.E.D.

Corollary 2.1

The rank of the principal part Q_p of a fundamental cutset matrix is equal to the rank of the graph obtained by deleting all the tree-branches, or is equal to the nullity of the graph obtained by contracting all the chords.

Proof: Assign color A to the tree and color B to the cotree. Then Q_{AB} in (2.19) is the principal part of the fundamental cutset matrix. Moreover $\rho(1, B) = 0$ and $\mu(A, 0) = 0$. Thus $v(A, B) = \rho(0, B) = \mu(A, 1)$, and the result follows from Theorem 2.3.

Q.E.D.

Dual discussions concerning the fundamental loop matrix can

also be given. Especially we have:

Corollary 2.2

The rank of the principal part of a fundamental loop matrix is equal to the rank of the graph obtained by deleting all the tree-branches, or is equal to the nullity of the graph obtained by contracting all the chords.

Corollary 2.2 also follows from the fact that the ranks of these two submatrices are equal.

The preceding discussions are concerned with the degree of interference in a graph with a given, or fixed, coloring. Let us now consider various coloring of branches. In contrast with Theorem 2.3 we have:

Theorem 2.5

For a given pair of trees, T_1 and T_2 , there is a coloring of branches such that the degree of interference is equal to the distance between the tree.

Proof: Assign color A to the branches of $T_1 - T_2$ and color B , to those of $T_2 - T_1$. The coloring of the common tree-branches and that of the common chords of T_1 and T_2 are arbitrary. The number of branches in $T_1 - T_2$ is $\theta(T_1, T_2)$. The numbers of common tree-branches and common chords of color A is no less than $\rho(A, 1)$ and $\mu(A, 0)$ respectively from (2.4) and (2.6). Thus from (2.15) we have $\rho(A, 0) + \mu(A, 0) \geq \theta(T_1, T_2) + \rho(A, 1) + \mu(A, 0)$. Then from (2.13) and (2.16)

$$v(A, B) \geq \theta(T_1, T_2) \quad (2.20)$$

A coloring which gives the equality in (2.20) can be obtained, for example, by assigning color B to all the common tree-branches and the common chords as well as to the branches of $T_2 - T_1$.

Q.E.D.

For a given graph G , the degree of interference varies depending on the coloring of branches, and there must be a maximum of the degree taken over all possible colorings. We denote such a maximum of the degree of interference by v_{max} , and the maximum of the distance between a pair of trees by θ_{max} . [2.7] Then:

Theorem 2.6

$$v_{max} = \theta_{max} \quad (2.21)$$

Proof: From (2.18) $v_{max} \leq \theta_{max}$, and from (2.20) $v_{max} \geq \theta_{max}$. Hence (2.21) must hold.

Q.E.D.

If we apply the proofs of Theorems 2.3 and 2.5 to the special cases where a coloring to give v_{max} is given and where a pair of maximally distant trees is given, we get the following theorem.

Theorem 2.7

If a coloring to give v_{max} is given, a maximally distant tree pair can be obtained by any procedure to obtain T_A and T_B in the

proof of Theorem 2.3. The branches of $T_A - T_B [T_B - T_A]$ are of color $A[B]$. Conversely if a pair of maximally distant trees T_1 and T_2 is given, a coloring to give v_{max} is obtained by assigning color A to $T_1 - T_2$ and color B to $T_2 - T_1$. The color of a common tree-branch or a common chord is arbitrary.

The coloring of a common tree-branch determines a monochromatic cutset, and that of a common chord, a monochromatic loop. All the monochromatic loops and cutsets are included in G_1 and G_2 , respectively, where G_1 is the principal subgraph with respect to common chords and G_2 is the principal subgraph with respect to common tree-branches. [2.7] The principal subgraph of disjoint trees G_0 consists of a pair of trees of different colors. An algorithm to obtain a pair of maximally distant trees is given in reference [2.7], and that to obtain a coloring to give v_{max} is given in reference [2.6].

As an application of the above discussions let us consider an LC network. For an LC network the number of non-zero natural frequencies is twice the degree of interference, if the inductor branches and capacitor branches are regarded as branches of color A and B respectively. [2.2][2.3] Then with the given network topology the maximum of the number of non-zero natural frequencies can be obtained by assigning inductors and capacitors according to a coloring to give v_{max} . Moreover, suppose the LC network has no zero-natural frequency. Then the number of branches of the graph representing the network is equal to the number of non-zero natural frequencies if, and only if, the graph consists of a pair of disjoint trees of different colors,

that is, a tree of inductors and a tree of capacitors. It follows that the graph of an LC network which realizes a rational function with the minimum number of elements for the given order, must, when the two terminals are properly open- or short-circuited so that the number of the natural frequencies agrees with the order, consist of a pair of disjoint trees, plus a branch corresponding to the zero-natural frequency, if any. Realizations in Foster's or Cauer's form clearly satisfy this condition. We can expect to construct different canonical forms using the graphs consisting of a pair of disjoint trees. Similar discussions can be given for LR or CR networks.

2.4 CONCLUDING REMARKS

The degree of interference of loops or cutsets in a multi-colored-branch graphs has not been sufficiently studied yet. In general the degree of interference of loops in $G(A_1, A_2, \dots, A_m)$ may or may not equal to the degree of interference of cutsets if $m \geq 3$. Moreover it can not be given in terms of the ranks and nullities of the graphs obtained by deleting or contracting all the branches belonging to some of the partitioned sets. The above statement can be verified by giving two graphs with different degrees of interference and showing that the ranks and nullities of every pair of the graphs which are derived from the two graphs by deleting or contracting all the branches in one or more of the partitioned sets, are equal. We can, however, give an upper bound and a lower bound on the degree of interference of loops or cutsets in terms of these ranks and nullities. [2.6]

The degree of interference is defined in the derivation of the state equations for an electrical network. Under certain conditions it agrees with the function which is defined by Tutte to show Menger's theorem for matroid, [2.8] and which is closely related to the connectivity of a graph. It is expected that the investigations made from quite different points of view can be combined together to develop new results.

CHAPTER 3 MIXED ANALYSIS OF LINEAR NETWORKS

3.1 INTRODUCTION

Since the computation time to solve a set of simultaneous equations increases rapidly as the number of equations increases, it is desirable, specially in handling large scale networks, to obtain as small a set of equations as possible. It has been shown that the required number of equilibrium equations for the nodal or loop analysis where the variables in the equations are either all voltages or all currents respectively, can, in certain cases, be reduced by choosing a suitable set of variables containing both voltages and currents. The analysis using such variables is called mixed analysis. The minimum number of equations required is equal to the topological degree of freedom of the network.^[3.1] For the mixed analysis the graph representing the given electrical network is partitioned to two branch-disjoint subgraphs associated with the voltage variables and the current variables respectively. The optimal partitionings (denoted P_d for brevity) which lead to the minimum number of equilibrium equations are closely related to maximally distant trees and the principal partition of the graph.^[2.7]

There are three subgraphs uniquely determined in the principal partition, though some of which may be an empty graph. In order to get a P_d the principal subgraph with respect to common chords of maximally distant trees should be included in the subgraph associated with voltage variables, and the principal subgraph with respect to common tree-branches should be in the subgraph associated with current variables.^[3.1] The third sub-

graph of the principal partition is called the principal subgraph of disjoint trees. In general it is subdivided into smaller subgraphs corresponding to the minimal graphs consisting of a pair of disjoint trees (abbreviated as PDT) and a partial ordering can be given to these smaller subgraphs together with the other two principal subgraphs.^[3.2] As for achieving a P_d , these smaller subgraphs may belong to the subgraph associated with voltage variables or to that associated with current variables, provided a certain restriction arising from the partial ordering is observed.^[3.3] Thus there may be more than one P_d for the given graph and to choose one of them some criterion other than the number of equations may be introduced.

The above-mentioned properties of a graph consisting of a PDT are discussed in section 3.2, and a method which enables us easily to choose the optimal P_d for some criterion is given in section 3.3.

3.2 GRAPHS CONSISTING OF A PAIR OF DISJOINT TREES

We first define the minimal graphs among the graphs consisting of a PDT, as those which contain no proper subgraph consisting of a PDT. The minimality can be determined independent of the PDT of the graphs. The minimal graphs for the number of nodes less than or equal to seven are shown in Fig.3.1.

Consider a connected graph G consisting of a PDT. If G is not minimal, it may contain one or more minimal subgraphs. For these subgraphs we get the following theorems.

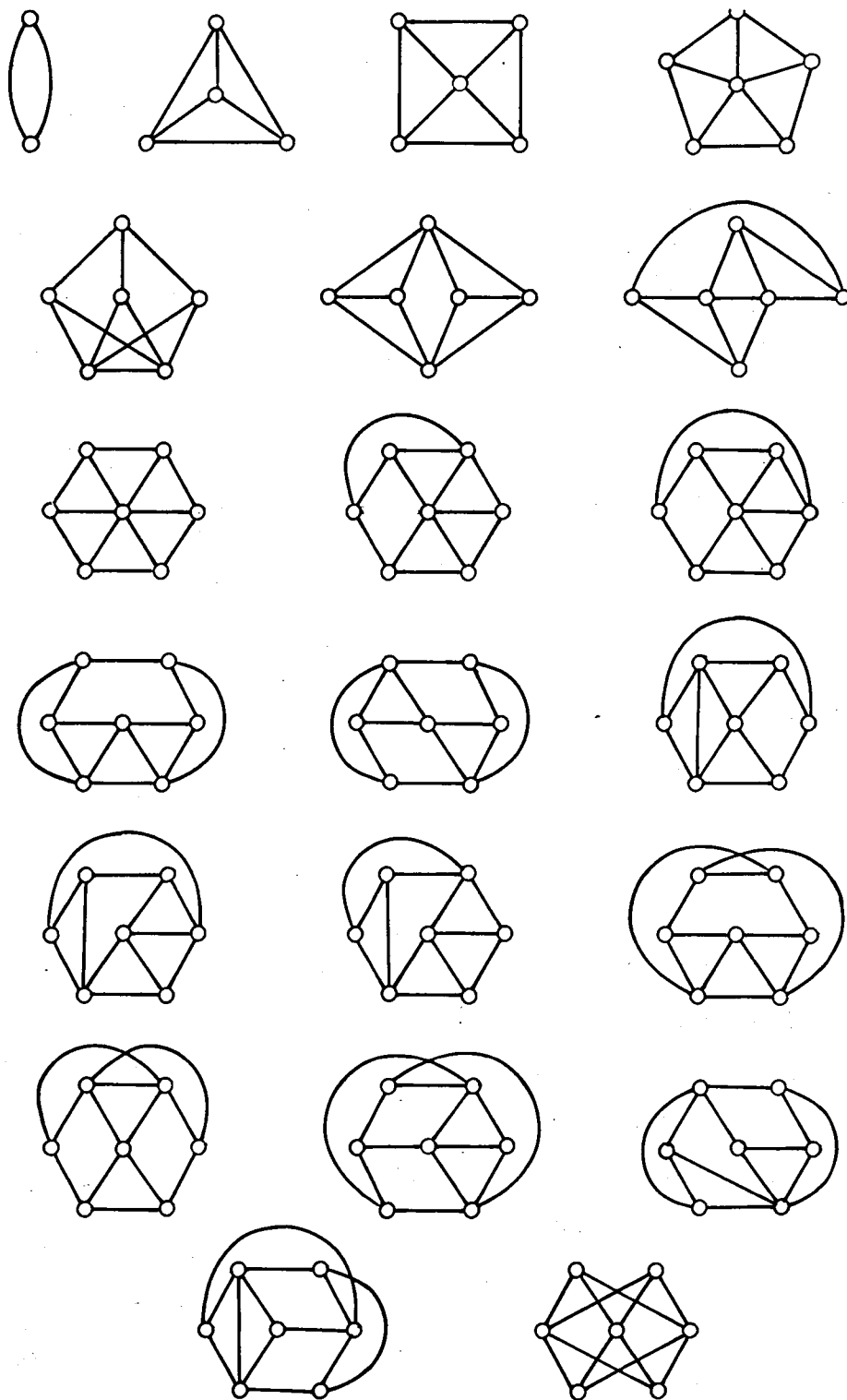


Fig.3.1 Minimal graphs with the number of nodes less than or equal to 7.

Theorem 3.1

Any pair of the minimal subgraphs in G has neither branches nor more than one node in common.

Proof: Let G_k and G_m be the pair of minimal subgraphs in G . Consider subgraphs $G_x = G_k \cup G_m$ and $G_y = G_k \cap G_m$. Noting that G consists of a PDT, we have $b_x = b_k + b_m - b_y$, $a_x = a_k + a_m - a_y$, $b_k = 2(a_k - 1)$, $b_m = 2(a_m - 1)$, $b_x \leq 2(a_x - 1)$ and $b_y \leq 2(a_y - 1)$, where b_k , b_m , b_x and b_y [a_k , a_m , a_x and a_y] are the numbers of branches [nodes] of G_k , G_m , G_x and G_y respectively. From the first five relations we get $b_y \geq 2(a_y - 1)$ and hence $b_y = 2(a_y - 1)$ from the last relation. Then b_y must be zero if $G_y \neq \emptyset$, since G_k and G_m are minimal subgraphs. It follows that a_y is one or zero ($G_y = \emptyset$).

Q.E.D.

Suppose a set of minimal subgraphs $G_{s1}, G_{s2}, \dots, G_{sk-1}, G_{sk}, G_{sk+1}, \dots$, and G_{sm} in G such that G_{sk} for $k=2, \dots, m-1$ has exactly one common node with each of G_{sk-1} and G_{sk+1} . Then we call the set a chain of minimal subgraphs. A chain is closed if G_{s1} and G_{sm} also have a node in common.

Theorem 3.2

No closed chain of the minimal subgraphs exists in G .

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and hence is omitted.

Theorem 3.3

The dual of a planar graph consisting of a PDT is also a graph consisting of a PDT.

Proof: Consider one of the PDT in the original graph. There is no cutset consisting only of the tree-branches. Hence there is, in the dual graph, no loop consisting of the branches which correspond to the tree-branches in the original graph. Therefore the PDT in the original graph corresponds to a PDT in the dual graph.

Q.E.D.

Next let us consider the graph denoted by $G^{(1)}$, which is derived from G by contracting all the branches of the minimal subgraphs in G . If G consists of the minimal subgraphs only, $G^{(1)}$ is a single node. Otherwise it can be shown that $G^{(1)}$ also consists of a PDT. It may be minimal or contains one or more minimal subgraphs. We can repeat the same branch-contracting operations on $G^{(1)}$ and determine $G^{(2)}$, and so on until $G^{(k)}$ is a single node. In this case we say that G is k -compound in the sense of the minimal graphs consisting of a PDT. We can give a partial ordering to the minimal subgraphs in the above procedure according to the order of appearance. An example is shown in Fig.3.2.

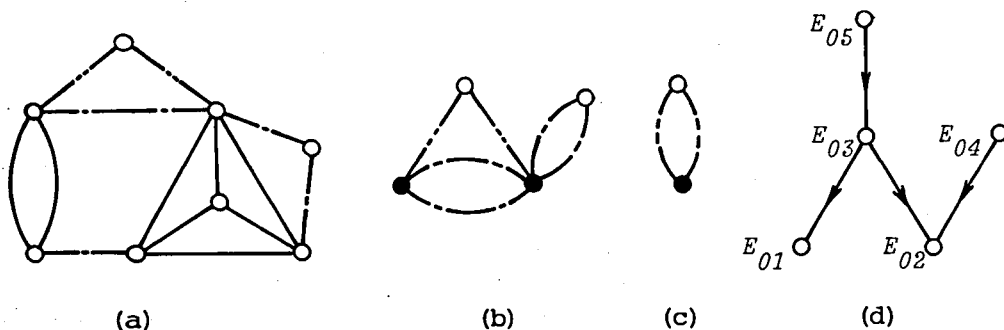


Fig.3.2 Example 3.1 (a) G . (b) $G^{(1)}$. (c) $G^{(2)}$. (d) Partial ordering of the sets of branches.

Theorem 3.4

The dual of a planar minimal graph consisting of a PDT is also minimal.

Proof: Suppose the dual graph is not minimal and contains a minimal subgraph. Consider the graph obtained by contracting the branches of this minimal subgraph. Corresponding to the graph there exists a subgraph consisting of a PDT in the original graph, which is a contradiction to the original graph being minimal.

Q.E.D.

In order to obtain those minimal subgraphs in $G, G^{(1)}, \dots$, and $G^{(k-1)}$ and the relation between them, we adopt the following algorithms based on the algorithms given in references [2.7] and [3.1]. We assume a pair of maximally distant trees (T_1, T_2) has already been obtained. By Algorithm 3.1 and Algorithm 3.2 given below we are going to determine the sets of branches denoted by L_x and C_y respectively.

Algorithm 3.1

- step 1. Choose a chord, say x , of T_1 . Set $L_x = \{x\}$. Find the fundamental loop defined by x and T_1 . Add to L_x the branches of the loop besides x .
- step 2. Find the fundamental loops defined by the newly added branches and T_2 . Add to L_x the branches in the loops which are not in L_x yet. If no new branch is added, stop. Otherwise go to step 3.
- step 3. Find the fundamental loops defined by the newly added branches and T_1 . Add to L_x the branches in the loops

which are not in L_x yet. If no new branch is added, stop. Otherwise go to step 2.

END

Algorithm 3.2

step 1. Choose a tree-branch, say y , of T_2 . Set $C_y = \{y\}$.

Find the fundamental cutset defined by y and the cotree of T_2 . Add to C_y the branches of the cutset other than y .

step 2. Find the fundamental cutsets defined by the newly added branches and the cotree of T_1 . Add to C_y the branches in the cutsets which are not in C_y yet. If no new branch is added, stop. Otherwise go to step 3.

step 3. Find the fundamental cutsets defined by the newly added branches and the cotree of T_2 . Add to C_y the branches in the cutsets which are not in C_y yet. If no new branch is added, stop. Otherwise go to step 2.

END

Suppose we obtain sets of branches L_x , L_y and L_z [C_x , C_y and C_z] by starting Algorithm 3.1 [Algorithm 3.2] from branches x , y and z respectively. We can easily see that they satisfy the axioms of a partially ordered set.

Theorem 3.5

- (i) If $L_x \supseteq L_y$ and $L_y \supseteq L_x$, then $L_x = L_y$.
- (ii) If $L_x \supseteq L_y$ and $L_y \supseteq L_z$, then $L_x \supseteq L_z$.

The dual theorem for C_x , C_y and C_z also holds, but is omitted here.

Theorem 3.6

$C_y \supseteq C_x$ if and only if $L_x \supseteq L_y$.

Proof: If $L_x \supseteq L_y$, then there must be a string of branches $x=x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_m=y$ such that x_{k+1} is included in the fundamental loop defined by x_k , and thus x_k is included in the fundamental cutset defined by x_{k+1} . Tracing the string backward by Algorithm 3.2, we see $C_y \supseteq C_x$. The proof for the converse is similar.

Q.E.D.

Now let $L_x \cap C_x = S_x$, $L_y \cap C_y = S_y$ and $L_z \cap C_z = S_z$. Then:

Theorem 3.7

$L_x = L_y$ and $C_x = C_y$ if $y \in S_x$ or $x \in S_y$. The converse also holds.

Proof: If $y \in S_x$, then $L_x \supseteq L_y$ and $C_x \supseteq C_y$. From Theorem 3.6, then, $C_y \supseteq C_x$ and $L_y \supseteq L_x$, and the result follows from Theorem 3.5. The converse is obvious.

Q.E.D.

Corollary 3.1

$C_y \supset C_x$ if and only if $L_x \supset L_y$.

Proof: Since $L_x \supset L_y$, $y \notin S_x$. Thus $y \notin C_x$ and $C_y \supset C_x$.

Q.E.D.

Corollary 3.2

$L_x = L_y$ if and only if $C_x = C_y$.

Corollary 3.3

If $y \notin S_x$ or $x \notin S_y$, then S_x and S_y are mutually disjoint.

Proof: If S_x and S_y have a common branch z , that is, $S_x \ni z$ and $S_y \ni z$, then $S_x = S_z$ and $S_y = S_z$ from Theorem 3.7. This obviously

contradicts $y \notin S_x$ or $x \notin S_y$.

Q.E.D.

From Corollary 3.3 we see that the branches of the graph can be partitioned into several sets. First we choose a branch x and determine S_x . Then we choose $y \notin S_x$ and determine S_y , and then S_z for $z \notin S_x \cup S_y$, and so on. A partial ordering can be given to these sets by use of Theorem 3.5.

It is not difficult to see that each of these sets S_x , S_y , S_z and so on, corresponds to one of the minimal subgraphs appearing in G , $G^{(1)}$, $G^{(2)}$ and so on. To see this we notice that the graph formed by the branches of S_x is a graph consisting of a PDT. Moreover, the number of its branches is minimum among the graphs consisting of a PDT and containing branch x . Then the locally least sets in the partial ordering correspond to the minimal subgraphs in G , and the next sets to the least, to the minimal subgraphs in $G^{(1)}$, and so on. The Hasse diagram, like that shown in Fig.3.2 (d), of the partial ordering of the sets of branches, represents the structure of the graph in the sense of minimal subgraphs consisting of a PDT.

3.3 GRAPH-PARTITIONING FOR THE MIXED ANALYSIS

The partial ordering of the sets of branches defined in section 3.2 is not necessarily restricted to the graphs consisting of a PDT. In general, the principal subgraph with respect to common chords is further partitioned into nonseparable components. The principal subgraph with respect to common tree-branches is partitioned likewise. In this case, however, the

principal subgraph is partitioned according to the nonseparable components of the graph which is obtained by contracting all the branches not belonging to the subgraph. The sets of branches in these nonseparable components are added to the partial ordering of the sets of branches which is derived from the partition of the principal subgraph of disjoint trees.

In order to determine the partial ordering of the sets, Algorithms 3.1 and 3.2 need a minor modification in handling the common chords and common tree-branches. If Algorithm 3.1 is started from a branch in T_2 , the common chords are not included in L_x . Therefore they must be added to L_x , if any of the branches in the nonseparable component to which they belong, is included in L_x . The common tree-branches must be excluded in steps 2 and 3, since it is not possible to find the fundamental loops defined by them. Similar modification is necessary for Algorithm 3.2. An example of a graph-partitioning and the partial ordering of sets of branches are shown in Fig. 3.3.

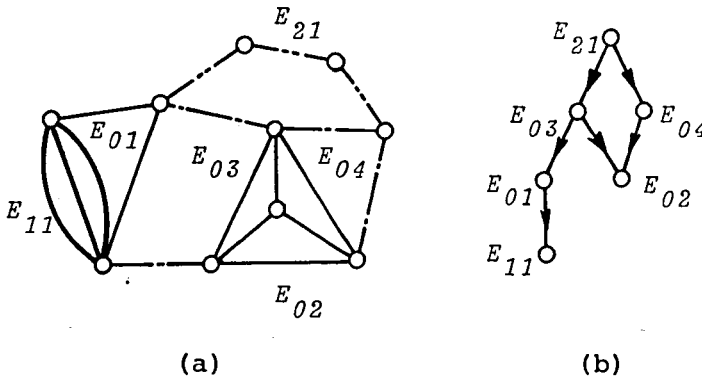


Fig.3.3 Example 3.2 (a) Graph-partitioning.
(b) Partial ordering of the set of branches.

The sets of branches in the principal subgraph with respect

to common chords[common tree-branches] are always the locally least[greatest] sets in the partial ordering.

Now let us denote the subgraphs associated with voltage and current variables in the mixed analysis by G_V and G_I respectively. The sets of branches belonging to G_V and G_I are denoted by V and I respectively, so that the whole graph is written as $G(V, I)$. The number of equations to be solved in the mixed analysis is equal to the hybrid rank of $G(V, I)$, defined as^[3.1]

$$\pi(V, I) = \rho(V, 0) + \mu(1, I). \quad (3.1)$$

The minimum of the hybrid rank taken over all possible partitions of G is called the topological degree of freedom.^[3.1] It is denoted by π_{min} . An optimal partition P_d to achieve π_{min} is given by a cut of the Hasse diagram showing the partial ordering of the sets of branches, that is:

Theorem 3.8

Every P_d is given by a cut of the Hasse diagram satisfying the following conditions.

- (i) $G_V[G_I]$ includes the principal subgraph with respect to common chords[common tree-branches].
- (ii) The cut traverses the lines of the diagram in the same direction.

Proof: We only give a proof for the condition (ii) here. For the condition (i) see references [2.7] and [3.1]. Contracting the branches of the principal subgraph with respect to common chords and deleting those of the principal subgraph with respect to tree-branches, we get a graph consisting of a PDT. Now (3.1)

is rewritten as

$$\pi(V, I) = \mu(V, I) + \rho(V, 0) - \mu(V, 0) \quad (3.2)$$

Since the rank of a minimal subgraph consisting of a PDT is equal to the nullity, the addition of such a subgraph to G_V has no effect on $\pi(V, I)$, as is seen from (3.2). Contracting the minimal subgraph, we again have a graph consisting of a PDT. Repeating the process, we see that the cut specified in (ii) gives a P_d . Since no subgraph of a minimal graph has the property that the rank is less than or equal to the nullity, there is no finer graph partitioning to achieve π_{min} other than those specified by (ii).

Q.E.D.

An example of a P_d is shown in Fig.3.4 for the Hasse diagram in Fig.3.3 (b).

Now if the diagram is such that all the sets are on a straight vertical line, all the possibilities for P_d can be easily seen. In order to bring a Hasse diagram to such a form we use the following algorithms and get two new diagrams which have less lines than the original one.

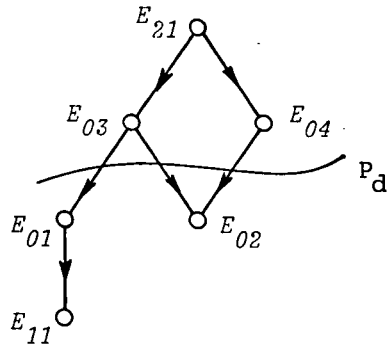


Fig.3.4 Optimal partitioning for the mixed analysis.

Algorithm 3.3

step 1. Choose a set in the diagram, to which the greatest number of lines are incident.

step 2. Draw a new diagram for each of the following two cases.

- (i) The chosen set and all the sets which are less, in the partial ordering, than the chosen set, are included in V . To draw the diagram for this case, contract all the directed paths from the chosen set to the sets of branches in G_1 .
- (ii) The chosen set and all the sets which are greater, in the partial ordering, than the chosen set, are included in I . To draw the diagram for this case, contract all the directed paths from the sets of branches in G_2 to the chosen set.

step 3. If any line appears in the new diagrams which connects two sets with some other longer path between them, delete the line.

END

We repeatedly apply the algorithm until the derived diagrams become simple enough for us to make a decision. For example, we pick E_{03} in Fig.3.3 (b), and if we include E_{03} in V , we get Fig.3.5 (a), and if in I , we get Fig.3.5 (b).

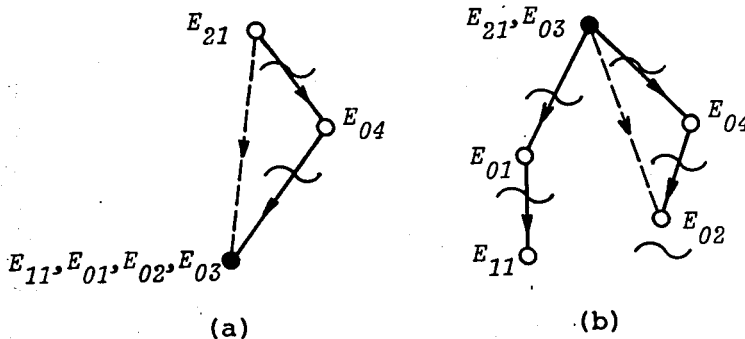


Fig.3.5 Possibilities for the optimal partitioning.

The dotted lines shown in Fig.3.5 are deleted in step 3. As

shown in Fig.3.5 (a) there are two possibilities for P_d in this case. In Fig.3.5 (b) we see $2 \times 3 = 6$ possibilities, and thus there are $2 + 6 = 8$ possibilities in all. To choose one of them we may introduce some other criterions than the number of equations.

3.4 CONCLUDING REMARKS

The graph partitioning discussed in section 3.2 has strong similarity to the partition of a bipartite graph, [3.4] [3.5] of a matrix, [3.6] of a matroid [3.7] and also of 2-graphs which will be discussed in section 4.4. The structure of a graph defined in sections 3.2 and 3.3 seems to be a fundamental concept, which has wide application to various objects.

An optimal partitioning P_d is much easier to see on the Hasse diagram showing the partial ordering of the sets of branches than the original graph. The operation on the diagram in Algorithm 3.3 is essentially a foldant algorithm. It is easy to perform by computers as well as by hands.

CHAPTER 4 SOLVABILITY OF LINEAR NETWORKS

4.1 INTRODUCTION

The problem of solvability of a linear active network has been approached from various points of view depending on the restrictions imposed on network elements and network topology. [4.1]-[4.5] The problem is especially important in computer-aided network analysis when a numerical solution does not converge. The divergence due to improper numerical integration and the divergence due to lack of existence of a unique solution must be distinguished.

Although the problem involves the values of the network elements and is therefore an algebraic one, the topological approach assuming the values are independent to each other, or the values are arbitrary, is important to clarify the cause and the type of network singularity. When the 2-graph method is used, it can be said, as will be shown in section 4.2, that the topological condition for the solvability is the existence of a common tree[@] of the voltage and current graphs. If there is no common tree, the network shows singularity, different from the examples given in reference [4.6], without any special element values or any special relations among them. Changing the values of network elements has no effect on this type of network instability.

A necessary and sufficient condition for the existence of a common tree of the 2-graphs is derived in section 4.3. If there

@ A common tree is a tree in both the voltage and current graphs.

is no common tree, subgraphs which are the cause of the non-existence can be distinguished, and a partition of 2-graphs can be defined. The partition has similar properties to the principal partition of a graph^{[2.7][3.1][3.2]} or the canonical form of a bipartite graph,^{[3.4][3.5]} but it is much more complicated, since 2-graphs whose topology may be quite different from each other's, are dealt with. An algorithm to obtain the partition and a tree of one of the graphs which has as many common branches as possible with a tree of the other, is given. Such a tree must be a common tree, if a common tree exists. Therefore the algorithm also gives an efficient way to obtain a common tree.

In section 4.4 it is shown that a certain structure of 2-graphs which is similar to that discussed in section 3.2, and which is represented by a partial ordering of sets of branches, can be defined. Algorithms to determine the structure are also given.

4.2 COMMON TREES OF VOLTAGE AND CURRENT GRAPHS

The restrictions on the network considered in this section and the procedure to obtain the voltage and current graphs from the network are given in sections 1.2 and 1.3 respectively. The notations of multicolored-branch graphs defined in section 2.2 are used as needed.

Now we classify the unknown variables in the network into two sets as follows.

Set 1.

- (i) The voltages across the resistors, inductors, capacitors,

controlled voltage sources and voltage sensors. Their voltage vector is denoted by v_1 . (ii) The currents through the resistors, inductors, capacitors, current sensors and controlled current sources. Their current vector is denoted by i_1 .

Set 2.

(i) The voltages across the independent and controlled current sources. Their voltage vector is denoted by v_2 . (ii) The currents through the independent and controlled voltage sources. Their current vector is denoted by i_2 .

For the variables in Set 1 we have

$$\begin{bmatrix} D_y & -D_z \\ B & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -B_e v_e \\ -Q_j i_j \end{bmatrix} \quad (4.1)$$

where

$$D_y = \begin{bmatrix} U_r & & & & \\ & U_L & & & \\ & & pD_c & & \\ & & & U_\beta & \\ 0 & & & & D_\delta \end{bmatrix} \quad (4.2)$$

and

$$D_z = \begin{bmatrix} D_r & & & & \\ & pD_L & & & \\ & & U_c & & \\ & & & D_\beta & \\ 0 & & & & U_\delta \end{bmatrix} \quad (4.3)$$

U 's are unit matrices, and D_r , D_L , D_c , D_β and D_δ are diagonal matrices of element values. $B[Q]$ is a fundamental loop[cutset]

matrix obtained from $G_v[G_i]$; $-B_e v_e$ and $-Q_j i_j$ are the voltage and current vectors due to the independent voltage and current sources respectively. The rank and nullity of $G_v[G_i]$ are denoted by ρ_v and $\mu_v[\rho_i$ and $\mu_i]$ respectively. Then $\mu_v[\rho_i]$ is derived from G and $G(E_\delta; 1)[G(E_\beta; 0)]$, where $E_\delta[E_\beta]$ is the set of branches corresponding to the controlled current[voltage] sources. [4.7] The rank>nullity of G is denoted by $\rho[\mu]$.

Lemma 4.1

$$\mu_v = \mu - \mu(E_\delta; 1) \quad (4.4)$$

$$\rho_i = \rho - \rho(E_\beta; 0). \quad (4.5)$$

Proof: Consider $G(E_\alpha, \bar{E}_\alpha)$, where E_α is the set of branches corresponding to the current sensors. Because of the topological restriction (ii) in section 1.2, $\mu(1, E_\alpha)$ is equal to μ , but it is also equal to the sum of μ_v and $\mu(E_\delta; 1)$ from (2.4). Thus we have (4.4). Equation (4.5) can be proved dually.

Q.E.D.

Now $\rho + \mu = n_1 + n_\beta + n_\delta$ where $n_1 = n_r + n_l + n_c + n_\beta + n_\delta$, $\rho_v + \mu_v = \rho_i + \mu_i = n_1$, $\rho(E_\delta; 1) + \mu(E_\delta; 1) = n_\delta$ and $\rho(E_\beta; 0) + \mu(E_\beta; 0) = n_\beta$. Then we get

$$\mu_v - \mu_i = \rho_i - \rho_v = \mu(E_\beta; 0) + \rho(E_\delta; 1) \quad (4.6)$$

$$\mu_v + \rho_i = n_1 + \mu(E_\beta; 0) + \rho(E_\delta; 1). \quad (4.7)$$

From these equations we immediately have the following theorem.

Theorem 4.1 [4.7]

If and only if, there is no voltage-source-only loop nor current-source-only cutset in G^* , the ranks>nullities of G_v

and G_i are equal, and the number of equations in (4.1) is equal to the number of unknown variables in (4.1).

Proof: $\mu(E_\beta; 0)$ and $\rho(E_\beta; 1)$ in (4.6) and (4.7) should be zero.

Q.E.D.

Note that the condition in Theorem 4.1 is a necessary condition for the existence of a common tree, since there can be no common tree if the ranks of the voltage and current graphs are different.

If the condition of Theorem 4.1 is satisfied, the coefficient matrix of (4.1) is square and its determinant is expanded as the sum of common-tree-immittance products as follows. [4.8][4.9]

$$\begin{vmatrix} D_y & -D_z \\ B & 0 \\ 0 & Q \end{vmatrix} = \sum_{all\ T} (sign\ T) \Pi_{immit} \quad (4.8)$$

where T denotes a common tree of G_v and G_i , and "sign T " is the sign permutation of T . Π_{immit} signifies the common-tree-immittance products. (The detail of Π_{immit} will be shown in section 5.2.) From (4.8) we see that the determinant becomes zero if there is no common tree of G_v and G_i , or if there are common trees but there exist special relations among the values of network elements so that all the common-tree-immittance products are canceled out. If the condition in Theorem 4.1 is not satisfied there exists no common tree. Then whether the condition is satisfied or not, it can be said that the topological condition for the solvability is the existence of a common tree of G_v and G_i . The network shown in Fig.1.3 is actually an

example of a network without common tree. The network is singular even if there is no special relations among the values of the network elements. The singularity of this network is caused by the network topology only.

We add one more theorem for the existence of a common tree of G_v and G_i . [4.7]

Theorem 4.2

A sufficient condition for the existence of a common tree of G_v and G_i is that there exists no loop consisting of voltage sources and current sensors nor cutset consisting of current sources and voltage sensors, beside the condition in Theorem 4.1 that there is no voltage-source-only loop nor current-source-only cutset, in G^* .

Proof: If the condition is satisfied, there is a tree of G^* which contains all the voltage sources and current sensors but no current sources or voltage sensors. Corresponding to this tree, a common tree can be obtained.

Q.E.D.

The determinant in (4.8) can be evaluated by generating all the common trees of G_v and G_i . We give an algorithm for the generation of all the common trees which is based on the Minty's algorithm for generation of all the trees of one graph. [4.10] The existence of a common tree can, of course, be checked by the algorithm.

Algorithm 4.1

step 1. Set $k=0$, $G_u=G_v$ and $G_h=G_i$.

step 2. Check if there is a self-loop in G_u or G_h . If there is one, go to step 3. Otherwise go to step 4.

step 3. If the corresponding branch in G_h or G_u to the self-loop in G_u or G_h , respectively, is a bridge[@], go to step 6. Otherwise delete the self-loop and its corresponding branch to obtain a new pair of graphs. Let the new pair be G_u and G_h , and go to step 2.

step 4. Check if there is a bridge in G_u or G_h . If there is one, contract it and its corresponding branch to obtain a new pair of graphs. Let the new pair be G_u and G_h , and go to step 2. If there is no bridge go to step 5.

step 5. If G_u and G_h are null graphs, print out a common tree formed by the contracted branches, and go to step 6. Otherwise go to step 7.

step 6. Set $k=k+1$. Go to step 8.

step 7. Set $k=k+1$. Set $G_{vk}=G_u$, $G_{ik}=G_h$, and $m_k=0$. Choose a branch say b_k , in G_{vk} . Contract b_k and its corresponding branch in G_{ik} to obtain a new pair of graphs. Let the new pair be G_u and G_h , and go to step 2.

step 8. If $k=1$, stop. Otherwise go to step 9.

step 9. Set $k=k-1$. If $m_k=0$, go to step 10. If $m_k=1$, go to step 8.

step 10. Set $m_k=1$. Delete b_k and its corresponding branch in G_{vk} and G_{ik} respectively to obtain a new pair of graphs. Let the new pair be G_u and G_h , and go to step 4.

END

@ A branch is a bridge if there is no other path connecting its end points than the branch itself.

The graphs G_{vk} and G_{ik} ($k=1, 2, \dots$) in step 7 must be stored until they are reused in step 10. The number of graph pairs to be stored is at most the rank of the original 2-graphs. The contracted and deleted branches to derive G_{vk} and G_{ik} are also to be recorded.

4.3 VOLTAGE AND CURRENT GRAPHS WITHOUT COMMON TREE^[4.11]

First we consider all possible partitions of branches into two sets A and B , or in other words, all possible $G_v(A, B)$ and $G_i(A, B)$.

Lemma 4.2

A necessary condition for the existence of a common tree of G_v and G_i is that there exists no partition of branches such that

$$\rho_v(A, 1) > \rho_i(A, 0) \quad (4.9)$$

or

$$\rho_i(A, 1) > \rho_v(A, 0). \quad (4.10)$$

Proof: The minimum number of branches of A that should be in a tree of G_v is $\rho_v(A, 1)$ by (2.4), and the maximum number of branches of A that can be in a tree of G_i is $\rho_i(A, 0)$ by (2.3). Thus if there is a partition satisfying (4.9), there can be no common tree. The proof for (4.10) is similar.

Q.E.D.

Next we consider 2-graphs without common tree. For any tree,

say T_u , of G_v , there is, in G_i , at least one loop consisting only of the tree-branches of T_u , and/or one cutset, only of the chords of T_u . The branches belonging to such loops and cutsets are denoted by R_u and Q_u respectively, and the numbers of independent such loops and cutsets are denoted by ξ_u and η_u respectively. A tree corresponding to T_u is defined to be a tree of G_i which is derived from T_u by removing proper ξ_u branches in R_u and adding proper η_u branches in Q_u . We have

$$\xi_u + \rho_i = \eta_u + \rho_v. \quad (4.11)$$

Let T_h be a tree of G_i corresponding to T_u . There are, in G_v , loops consisting of the tree-branches of T_h and/or cutsets, of the chords of T_h . The sets of branches belonging to such loops and cutsets are denoted by R_h and Q_h respectively. Now assume that a subset A of branches is given. Let ξ_{uA} and η_{uA} be the number of branches which are in A and also among the removed and added branches, respectively, to obtain T_h from T_u , and let t_{uA} and t_{hA} be the number of tree-branches which are in A and also in T_u and T_h respectively. Then we have

$$\xi_u \geq \xi_{uA} = t_{uA} - t_{hA} + \eta_{uA} \geq t_{uA} - t_{hA} \geq \rho_v(A, 1) - \rho_i(A, 0). \quad (4.12)$$

The last relation in (4.12) can be derived by use of (2.4) and (2.3). Now if ξ_u is minimum for all trees of G_v , T_u is called a maximally-common tree (abbreviated as MCT) of G_v . It is easy to see from (4.11) that ξ_u is minimum if and only if η_u is minimum. A tree corresponding to an MCT of G_v is called an MCT of G_i .

Theorem 4.3

Choose two MCT's, T_u and T_w , of G_v . Then

$$R_u \cap Q_w = \emptyset \quad (4.13)$$

$$R_w \cap Q_u = \emptyset. \quad (4.14)$$

Proof: It is obvious that R_u and Q_u are disjoint. If $R_u = R_w$, nothing more to prove. If $R_u \neq R_w$, choose a branch, say x , in R_w which is not in T_u , and then find, in G_v , a branch, say y , which is in the fundamental cutset defined by x with respect to T_w . By an elementary tree transformation[@] we get a new tree of G_v , $T_{w+1} = T_w \cup y - x$. Since T_w is an MCT and x is in R_w , T_{w+1} must be an MCT, and y should be in R_{w+1} . Besides, $Q_{w+1} = Q_w$, and thus $(R_w \cup R_{w+1}) \cap Q_w = \emptyset$. Repeating the operations for m times and obtaining T_{w+m} , we see that

$$Q_{w+m} = Q_w \quad \text{and} \quad \left(\bigcup_{k=0}^m R_{w+k} \right) \cap Q_w = \emptyset.$$

Tree T_{w+m} has m more common tree-branches with T_u than T_w , and thus there must be a finite m such that $R_{w+m} = R_u$. Hence we have (4.13). Equation (4.14) can be proved similarly.

Q.E.D.

Corollary 4.1

$$\Omega \cap \Gamma = \emptyset \quad (4.15)$$

@ An elementary tree transformation is a transformation from one tree to another with the distance between the two trees being exactly one.

where

$$\Omega = \begin{matrix} U \\ \text{all MCT's of } G_v \end{matrix} R_u \quad (4.16)$$

$$\Gamma = \begin{matrix} U \\ \text{all MCT's of } G_v \end{matrix} Q_u. \quad (4.17)$$

The set of branches which are neither in Ω nor in Γ is denoted by Λ (Λ may be empty.). Then the branches of the 2-graphs are partitioned into three subsets Ω , Γ and Λ . An example of such a partition is given in Fig.4.1. For this

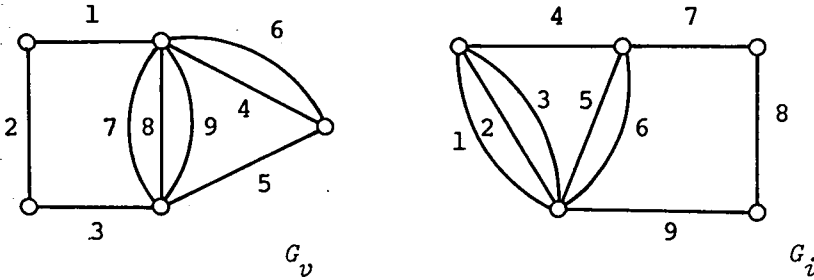


Fig.4.1 2-graphs for which no common tree exists.

example an MCT of G_v is obtained by choosing two branches from branches 1, 2 and 3, a branch from branches 4, 5 and 6, and a branch from branches 7, 8 and 9. For instance $T_u = \{1, 2, 4, 7\}$ is an MCT of G_v , and then $R_u = \{1, 2\}$, $Q_u = \{8, 9\}$, $\Omega = \{1, 2, 3\}$, $\Gamma = \{7, 8, 9\}$ and $\Lambda = \{4, 5, 6\}$. A corresponding MCT of G_i to T_u is $T_h = \{1, 4, 7, 8\}$.

For a corresponding pair of MCT's, T_u of G_v and T_h of G_i , we have the following theorem.

Theorem 4.4

(i) [(ii)]. The fundamental loops [cutsets] in G_i defined by the branches in $\bar{T}_h [T_h]$ and also in $\Omega [\Gamma]$ consist only of branches in $\Omega [\Gamma]$.

(iii)[(iv)] The fundamental cutsets[loops] in G_v defined by the branches in $T_u[\bar{T}_u]$ and also in $\Omega[\Gamma]$ consist only of branches in $\Omega[\Gamma]$.

Proof: (i) Suppose the fundamental loop defined by branch x in Ω contains a branch not in Ω . We consider a sequence of elementary tree transformations of MCT's of G_v , as was described in the proof of Theorem 4.3, to obtain, from T_u , such an MCT, T_w , of G_v , that x is in R_w . If the tree-branches in the fundamental loop under consideration remain as the tree-branches of T_w , the loop contains a branch not in Ω , and R_w contains a branch not in Ω , which is a contradiction to the definition of Ω . In the other case, suppose a tree-branch, say y , in the loop becomes a chord after a certain elementary tree transformation, and a chord, say z , becomes a tree-branch. Before the transformation y belongs, in G_v , to a loop consisting of tree-branches of an MCT of G_v . We see after the transformation the fundamental loop defined by x is altered to contain, instead of y , a part of the loop of tree-branches, but that it remains to contain the branch not in Ω . See Fig.4.2 (a). To ascertain these we have to check two more cases; an elementary tree transformation such that the branch not in Ω is bypassed by a part of the loop of tree-branches containing y , as shown in Fig.4.2 (b), and an elementary tree transformation such that the fundamental loop defined by x is altered to contain some other path of tree-branches in Ω , which is formed by the addition of z as a tree-branch, as shown in Fig.4.2 (c). In such cases the loops of tree-branches contain a branch not in Ω before or after the transformation. Thus either case is impossible. In any stage

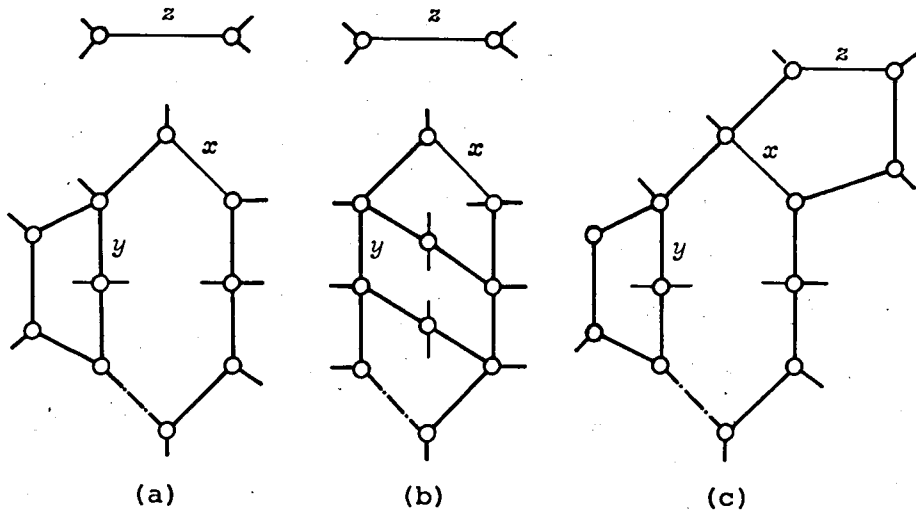


Fig.4.2 Elementary tree transformation exchanging y and z . The thick lines indicate the tree-branches in Ω , and the dotted lines indicate the branch not in Ω . (a) Possible case. (b) and (c) Impossible cases.

of the sequence, therefore, the fundamental loop defined by x remains to contain a branch not in Ω , and we eventually come up to a contradiction. Thus we must have the result as given in the theorem.

(iii) Suppose the fundamental cutset defined by a branch, say x , in Ω contains a branch not in Ω . We again consider the same kind of elementary tree transformations described above. If an MCT, T_w , of G_v such that R_w contains x , is obtained without changing the fundamental cutset defined by x , a contradiction can be caused by performing one more elementary tree transformation removing x from, and adding the branch not in Ω to, T_w . In the other case if the fundamental cutset defined by x is altered not to contain the branch not in Ω after an elementary tree transformation with a branch, say y , in the cutset becoming a tree-branch, then the fundamental cutset defined by y contains

the branch not in Ω . Hence we still come up to a contradiction, and we must have the result as given in the theorem.

(ii) and (iv). Dual proofs to those for (i) and (iii) can be given.

Q.E.D.

Theorem 4.5

The sets of branches defined by (4.16) and (4.17) are identical to the sets of branches Ω and Γ which satisfies the following Conditions 1 and 2, respectively, with respect to a pair of corresponding MCT's, T_u of G_v , and T_h of G_i .

Condition 1.

- (i) $G_i(\Omega; 0)$ contains R_u .
- (ii) $T_h \cap G_i(\Omega; 0)$ is a forest of $G_i(\Omega; 0)$.
- (iii) $G_v(\Omega; 1)$ contains Q_h .
- (iv) $T_u \cap G_v(\Omega; 1)$ is a forest of $G_v(\Omega; 1)$.
- (v) The set of branches Ω is a minimal one satisfying (i)-(iv).

Condition 2.

- (i) $G_i(\Gamma; 1)$ contains Q_u .
- (ii) $T_h \cap G_i(\Gamma; 1)$ is a forest of $G_i(\Gamma; 1)$.
- (iii) $G_v(\Gamma; 0)$ contains R_h .
- (iv) $T_u \cap G_v(\Gamma; 0)$ is a forest of $G_v(\Gamma; 0)$.
- (v) The set of branches Γ is a minimal one satisfying (i)-(iv).

Proof: The set defined by (4.16) is denoted by Ω' in this proof to distinguish the two definitions. First, from Theorem 4.4 we see that (i)-(iv) in Condition 1 hold for Ω' . Thus Ω which also satisfies (v) in Condition 1 must be included in Ω' .

Second, let us regard Ω as set A in (4.12) and the corresponding MCT's as T_u and T_h in (4.12). Then $\xi_u = \xi_{u\Omega}$ from (i) in Condition 1. Besides, since Ω' does not include Q_u from Theorem 4.3, Ω does not include Q_u . Thus $\eta_{u\Omega} = 0$. Furthermore $t_{u\Omega} = \rho_v(\Omega; 1)$ and $t_{h\Omega} = \rho_i(\Omega; 0)$ from (iv) and (ii) in Condition 1. Consequently all the equalities in (4.12) hold for Ω , and we have

$$\xi_u = \rho_v(\Omega; 1) - \rho_i(\Omega; 0). \quad (4.18)$$

Next let T_w and T_k be an arbitrary pair of corresponding MCT's. If we regard Ω as A in (4.12) and T_w and T_k as T_u and T_h in (4.12) respectively, we see that all the equalities in (4.12) must hold, since $\xi_w = \xi_u$. Thus $\xi_w = \xi_{w\Omega}$ and the ξ_w branches which are removed from T_w to obtain T_k are all in Ω . But the ξ_w branches can be chosen to include an arbitrary branch in R_w , and thus R_w must be in Ω . It follows, then, from (4.16) that Ω' must be included in Ω . We have seen that the latter is included in the former, and therefore, they are identical to each other. The fact that the set of branches defined by (4.17) is identical to that satisfying Condition 2, can be shown dually.

Q.E.D.

From Condition 2 we get

$$\eta_u = \rho_i(\Gamma; 1) - \rho_v(\Gamma; 0) \quad (4.19)$$

for the corresponding MCT's. We also have that

- (ii)' $T_h \cap G_i(\bar{\Omega}; 1)$ is a forest of $G_i(\bar{\Omega}; 1)$,
- (iv)' $T_u \cap G_v(\bar{\Omega}; 0)$ is a forest of $G_v(\bar{\Omega}; 0)$,
- (ii)" $T_h \cap G_i(\bar{\Gamma}; 0)$ is a forest of $G_i(\bar{\Gamma}; 0)$,
- (iv)" $T_u \cap G_v(\bar{\Gamma}; 1)$ is a forest of $G_v(\bar{\Gamma}; 1)$,

and therefore

$$\eta_u = \rho_i(\bar{\Omega}; 1) - \rho_v(\bar{\Omega}; 0) \quad (4.20)$$

$$\xi_u = \rho_v(\bar{\Gamma}; 1) - \rho_i(\bar{\Gamma}; 0). \quad (4.21)$$

These equations can also be obtained from (4.18) and (4.19) by use of (2.11) and (4.11).

Since (4.18)-(4.21) hold for 2-graphs without common tree, we see that the condition in Lemma 4.2 is also sufficient for the existence of a common tree, and we have the following fundamental theorem.

Theorem 4.6

A necessary and sufficient condition for the existence of a common tree of G_v and G_i is that there exists no partition satisfying (4.9) or (4.10).

Proof: The necessity follows from Lemma 4.2. If there is no common tree, $\xi_u \neq 0$ or $\eta_u \neq 0$. Ω or Γ satisfies (4.9) or (4.10). Thus the condition is also sufficient.

Q.E.D.

We also see that Ω is a set of branches which gives the maximum of the far-right-hand side of (4.12).

From Theorem 4.5 we recognize that the subgraphs in G_v and G_i formed by the branches in $\Omega[\Gamma]$ have strong similarity to $G_2[G_1]$ and $G_1[G_2]$, respectively, of the principal partition of a graph, [2.7] and also that an MCT resembles an extremal tree. [3.1]

Thus we can extend the known algorithm to obtain the principal partition and an extremal tree, to those to obtain Ω , Γ and an MCT. In the following algorithm we first find the set of

branches satisfying Condition 1 with respect to an arbitrary tree T_u of G_v . Then if T_u is not an MCT, a tree of G_v having one more common branch with a tree of G_i is obtained. The new tree is used to find the set of branches satisfying Condition 1. This operation is repeated until an MCT of G_v and the set of branches satisfying Condition 1 with respect to the MCT, that is, set Ω , are obtained.

Algorithm 4.2

- step 1. Set $k=1$, $G_{vk}=G_v$, $G_{ik}=G_i$, and $T_{uk}=T_u$. Go to step 2.
- step 2. Find R_{uk} in G_{ik} . If $R_{uk} \neq \emptyset$, go to step 3. Otherwise go to step 7.
- step 3. Find, in G_{vk} , the fundamental cutsets defined with respect to T_{uk} by the branches of R_{uk} . Let the set of the branches in the cutsets be F_{uk} . If F_{uk} contains no branch in Q_u , go to step 4. Otherwise go to step 5.
- step 4. Delete and contract the branches of F_{uk} from G_{vk} and G_{ik} , respectively, to obtain G_{vk+1} and G_{ik+1} . Obtain also T_{uk+1} from T_{uk} by contracting the branches of R_{uk} . Set $k=k+1$ and return to step 2.
- step 5. Choose a branch, say x , which is in Q_u and also in F_{uk} . Choose a branch, say y , which is in R_{uk} and also in the fundamental loop defined by x with respect to T_u . Obtain $T_w = T_u \cup x - y$ by an elementary tree transformation. If $k=1$, set $T_u = T_w$ and return to step 1. If $k>1$, go to step 6.
- step 6. $Q_w \cap F_{uk-1} \neq \emptyset$ in this case. Set $T_u = T_w$, $Q_u = Q_w$ and $k=k-1$. Return to step 5.

step 7. Set $\Omega = F_{u1} \cup F_{u2} \cup \dots \cup F_{uk-1}$, and stop.

END

Example of Algorithm 1.

Let $T_u = \{1, 2, 3, 4\}$ for the graphs in Fig.4.1. Then $Q_u = \{7, 8, 9\}$. step 1. $T_{u1} = \{1, 2, 3, 4\}$. step 2. $R_{u1} = \{1, 2, 3\}$. step 3. $F_{u1} = \{1, 2, 3, 5, 7, 8, 9\}$. step 5. Let $x=7$ and $y=1$. Then $T_w = \{2, 3, 4, 7\}$. step 1. $Q_u = \{8, 9\}$. $T_{u1} = \{2, 3, 4, 7\}$. step 2. $R_{u1} = \{2, 3\}$. step 3. $F_{u1} = \{1, 2, 3\}$. step 4. $T_{u2} = \{4, 7\}$. step 2. $R_{u2} = \emptyset$. step 7. $\Omega = \{1, 2, 3\}$.

Theorem 4.7

The tree obtained at the end of Algorithm 4.2 is an MCT of G_v .

Proof: Note that Ω obtained by Algorithm 4.2 satisfies Condition 1 with respect to T_u . Regard Ω as A in (4.12). Then for the tree T_u at the end of the algorithm, $\xi_u = \xi_{u\Omega}$, $\eta_{u\Omega} = 0$, $t_{u\Omega} = \rho_v(T; 1)$ and $t_{h\Omega} = \rho_i(T; 0)$. Therefore all the equalities in (4.12) hold. Hence ξ_u is minimum, since only equalities or inequalities hold for an arbitrary tree of G_v .

Q.E.D.

If there exists a common tree, an MCT must be a common tree. Thus we can obtain a common tree by Algorithm 4.2, if one exists. If the MCT obtained by Algorithm 4.2 is not a common tree, there is no common tree.

From the definitions (4.16) and (4.17) we see that the sets of branches Ω and Γ are uniquely determined for a given pair of graphs. Then by Theorem 4.5 and 4.7 we see that Ω can be obtained by Algorithm 4.2. We can get Γ by the dual algorithm

which is obtained from Algorithm 4.2 by exchanging " G_v " and " G_i " and replacing " u " by " h " and " Ω " by " Γ ". Let Λ be the set of branches not belonging to Ω or Γ . Then we get a partition of 2-graphs, $G_v(\Omega, \Gamma, \Lambda)$ and $G_i(\Omega, \Gamma, \Lambda)$. At least a common tree exists for $G_v(0, 1, \Lambda)$ and $G_i(1, 0, \Lambda)$.

4.4 PARTITION OF VOLTAGE AND CURRENT GRAPHS

Similarly to the fact that a graph consisting of a pair of disjoint trees may have a finer structure, [3.2] 2-graphs for which a common tree exists, may have a finer structure that goes along the conditions described in the previous section. The branches of 2-graphs can be partitioned into several sets and a partial ordering can be given to these sets. To see this we first give algorithms to obtain sets of branches H_x and K_x satisfying the following Conditions 3 and 4, respectively, for a branch x with respect to a common tree T , which we assume has been found.

Condition 3.

- (i) $x \in H_x$.
- (ii) $T \cap G_i(H_x; 0)$ is a forest of $G_i(H_x; 0)$.
- (iii) $T \cap G_v(H_x; 1)$ is a forest of $G_v(H_x; 1)$.
- (iv) H_x is a minimal set satisfying (i)-(iii).

Condition 4.

- (i) $x \in K_x$.
- (ii) $T \cap G_i(K_x; 1)$ is a forest of $G_i(K_x; 1)$.
- (iii) $T \cap G_v(K_x; 0)$ is a forest of $G_v(K_x; 0)$.

(iv) K_x is a minimal set satisfying (i)-(iii).

Algorithm 4.3-1

step 1. $H_x = \emptyset$. Suppose x is a chord in G_i . Add x to H_x .

step 2. Find the fundamental loops in G_i defined by the branches which are newly added to H_x . If some of the branches in the loops have not been in H_x yet, add them to H_x , and go to step 3. Otherwise stop.

step 3. Find the fundamental cutsets in G_v defined by the branches which are newly added to H_x . If some of the branches in the cutsets have not been in H_x yet, add them to H_x , and go to step 2. Otherwise stop.

END

Algorithm 4.3-2

Exchange " G_v " and " G_i " and replace " H " by " K " in Algorithm 4.3-1.

END

Algorithm 4.3-3

Exchange "loops" and "cutsets" and replace "chord" by "tree-branches" and " H " by " K " in Algorithm 4.3-1.

END

Algorithm 4.3-4

Exchange further " G_v " and " G_i " and replace " K " by " H " in Algorithm 4.3-3.

END

From Conditions 3 and 4 we see that the following equations hold for H_x and K_x respectively.

$$\rho_v(H_x; 1) = \rho_i(H_x; 0) \quad (4.22)$$

$$\rho_v(K_x; 0) = \rho_i(K_x; 1). \quad (4.23)$$

Now we denote the sets of branches obtained by the above algorithms for branches x , y and z by H_x , K_x , H_y , K_y , H_z and K_z respectively. Then they satisfy the axioms of a partially ordered set.

Theorem 4.8

- (i) If $H_x \supseteq H_y$ and $H_y \supseteq H_x$, then $H_x = H_y$.
- (ii) If $H_x \supseteq H_y$ and $H_y \supseteq H_z$, then $H_x \supseteq H_z$.

These relations also hold for K_x , K_y and K_z .

Theorem 4.9

$H_x \supseteq H_y$ if and only if $K_y \supseteq K_x$.

Proof: If $H_x \supseteq H_y$ then there is a string of branches $x=x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_m=y$ determined by Algorithm 4.3-1 or Algorithm 4.3-4, such that x_{k+1} is in the fundamental loop or cutset defined by x_k . This string can be traced back by Algorithm 4.3-2 or Algorithm 4.3-3.

Q.E.D.

Now let $H_x \cap K_x = M_x$, $H_y \cap K_y = M_y$ and $H_z \cap K_z = M_z$.

Theorem 4.10

$H_x = H_y$ and $K_x = K_y$, if $y \in M_x$ or $x \in M_y$. The converse also holds.

Corollary 4.2

$K_y \supset K_x$ if and only if $H_x \supset H_y$.

Corollary 4.3

If $y \notin M_x$ or $x \notin M_y$, then M_x and M_y are mutually disjoint.

The proofs for Theorem 4.10, Corollary 4.2 and Corollary 4.3 are the same as these for Theorem 3.7, Corollary 3.1 and Corollary 3.3 respectively.

From Theorem 4.10 we see that the branches of the 2-graphs can be partitioned into several sets in the same way as was discussed in section 3.2. We can give a partial ordering to these sets and define a structure of 2-graphs, which is represented by the Hasse diagram showing the partial ordering.

The structure of 2-graphs having at least a common tree can be extended to 2-graphs without common tree. In general Ω and Γ are also partitioned into the subsets which are determined by the following algorithms. Consider $G_v(\Omega; 1)$ and $G_i(\Omega; 0)$. For a branch, say x in Ω , we get Θ_x , as follows.

Algorithm 4.4

step 1. $\Theta_x = \emptyset$. Add x to Θ_x .

step 2. Find all the nonseparable parts of $G_v(\Omega; 1)$ which include any of the branches newly added to Θ_x . If, in the nonseparable parts, there are branches which have not been in Θ_x yet, add them to Θ_x and go to step 3. Otherwise stop.

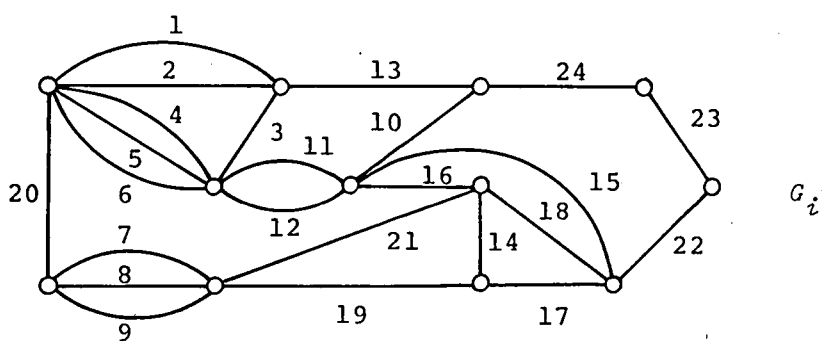
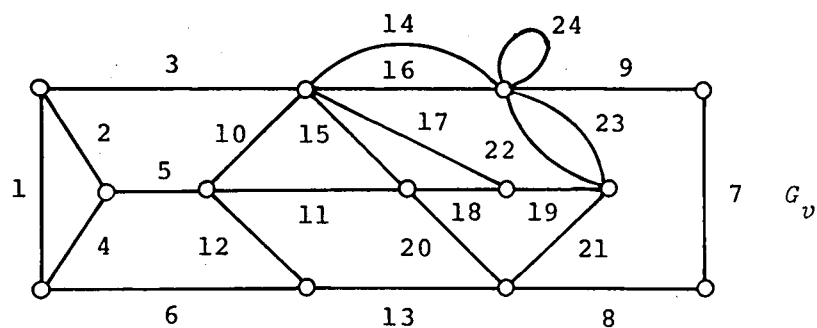
step 3. Find all the nonseparable parts of $G_i(\Omega; 0)$ which

include any of the branches newly added to Θ_x . If, in the nonseparable parts, there are branches which have not been in Θ_x yet, add them to Θ_x , and go to step 2. Otherwise stop.

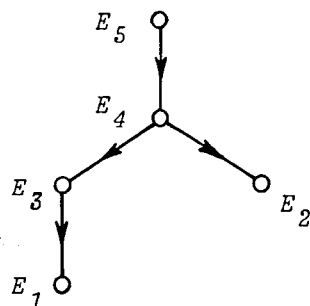
END

The subgraph formed by the branches of Θ_x is called a common nonseparable part of $G_v(\Omega; 1)$ and $G_i(\Omega; 0)$. We denote the sets obtained by Algorithm 4.4 for branches x, y, z etc. by $\Theta_x, \Theta_y, \Theta_z$, etc. respectively. Ω is partitioned as follows. We first find Θ_x for an arbitrary branch x in Ω , and next Θ_y for $y \notin \Theta_x$, and then Θ_z for $z \notin \Theta_x \cup \Theta_y$ and so on. The partition of Γ can be determined by the dual algorithm which is obtained from Algorithm 4.4 by replacing " Ω " by " Γ " and exchanging " G_v " and " G_i ".

The structure of 2-graphs without common tree is represented by the partial ordering of the partitioned subsets in Ω, Γ and Λ . In order to determine the partial ordering, Algorithm 4.3-1 or Algorithm 4.3-4 must be modified as follows. First, instead of a common tree T , a pair of corresponding MCT's of G_v and G_i are used. Secondly, if a branch of a common nonseparable part is included in H_x , all the branches in the nonseparable part are added to H_x . The partial ordering is determined according to the inclusive relation among the subsets of branches H_x, H_y, H_z , etc. An example of 2-graphs and the Hasse diagram showing their structure are given in Fig.4.3. For these 2-graphs we get, for instance, $H_{15} = E_1 \cup E_2 \cup E_3 \cup E_4$, $K_{15} = E_4 \cup E_5$ and $M_{15} = E_4$; $H_7 = E_2$, $K_7 = E_2 \cup E_4 \cup E_5$ and $M_7 = E_2$; $\Omega = E_1 \cup E_2$, $\Gamma = E_5$, and $\Lambda = E_3 \cup E_4$.



(a)



(b)

$E_1 = \{1, 2, 3, 4, 5, 6\}$, $E_2 = \{7, 8, 9\}$, $E_3 = \{10, 11, 12, 13\}$, $E_4 = \{14, 15, 16, 17, 18, 19, 20, 21\}$, $E_5 = \{22, 23, 24\}$.

Fig.4.3 (a) Example of 2-graphs. (b) Partial ordering of the sets of branches.

We can also discuss the partition of 2-graphs and the solvability of (4.1) in terms of the canonical form of the coefficient matrix. Obtain the fundamental loop and cutset matrices B and Q with respect to a corresponding pair of MCT's of G_v and G_i , and arrange the rows and columns of $\begin{bmatrix} B \\ Q \end{bmatrix}$ according to the partial ordering of sets of branches in the same way as a (0-1) matrix. [1.1] [3.4] [3.5] Then we get a canonical form as shown in Fig.4.4. The nonzero elements of the matrix are all contained in the shaded region. We see that if there is no common tree, there is no balance between the numbers of variables and equations. The termrank of the matrix is obviously less than the dimension of the matrix. Fundamental loop and cutset matrices with respect to trees of G_v and G_i , which are not MCT's may yield a coefficient matrix with its termrank equal to its dimension.

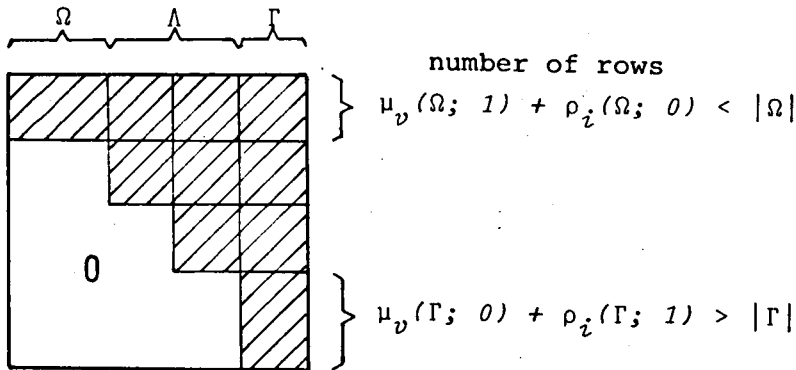


Fig.4.4 Canonical form of the coefficient matrix.

The relation between the structure of 2-graphs and that of a graph discussed in section 3.3 is schematically shown in Fig.4.5. The solid arrows show the likeness and the dotted arrows, the duality among the subgraphs.

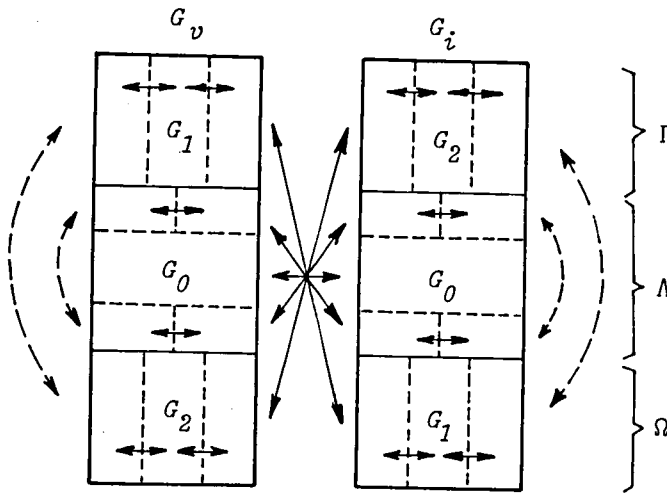


Fig.4.5 Structure of 2-graphs in connection with the structure of a graph.

4.5 CLASSIFICATION OF NETWORK SOLVABILITY

On the basis of the results obtained in the previous sections we give a classification of network solvability in this section. We are first concerned with the solvability of (4.1). For the brevity of the description we use the following notations.

CTE: the cases where at least a common tree of G_v and G_i exists.

CTN: the cases where no common tree of G_v and G_i exists.

SE: the cases where the unique solution of (4.1) exists.

SNP: the cases where no solution of (4.1) is possible for non-zero independent source voltages or currents.

SND: the cases where some of the unknown variables in (4.1) are not determinate.

RE: the cases where there exist special relations among the element values.

RN: the case where no special relation among the element values exists.

Case 1. CTE. Usually we have Case 1-a.

Case 1-a. RN and then it follows SE.

Case 1-b-1. RE and SNP. An example is given in Fig.4.6.

(Example 4.1.)

Case 1-b-2. RE and SND. Example 4.2 in Fig.4.7.

Case 1-b-3. RE and SE. The relations among the element values may have no effect on solvability, and we have this case.

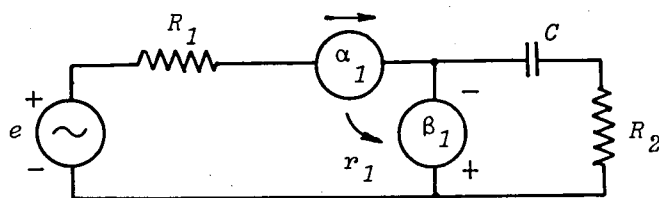


Fig.4.6 Example 4.1. $R_1 = r_1$.

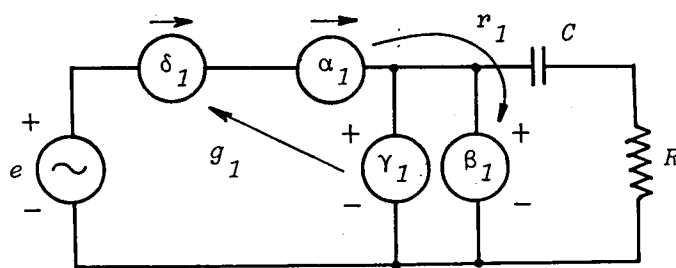


Fig.4.7 Example 4.2. $r_1 g_1 = 1$.

Case 2. The condition in Theorem 4.1 is satisfied but CTN.

Usually we have Case 2-a-1.

Case 2-a-1. RN, then it follows SNP and SND. See Fig.1.3, Example 1.1.

Case 2-a-2. RE and SND. Example 4.3 in Fig.4.8. Voltage v_{β_1} is indeterminate.

Case 2-b-1. RN and SND. Example 4.4 in Fig.4.9. Voltage v_{β_3} is indeterminate.

Case 2-b-2. RE, SNP and SND. This case follows from Case 2-a-1 if the relations have no effect on solvability.

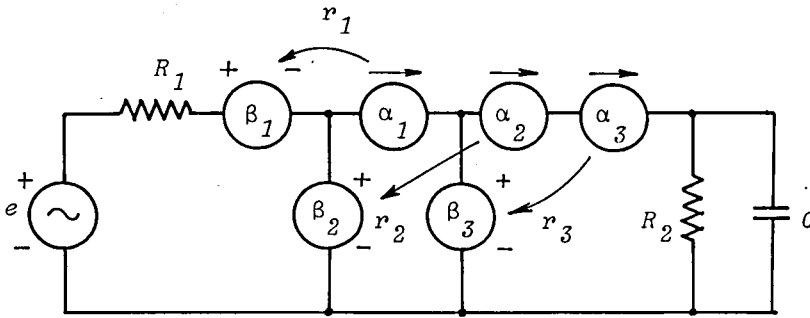


Fig.4.8 Example 4.3.

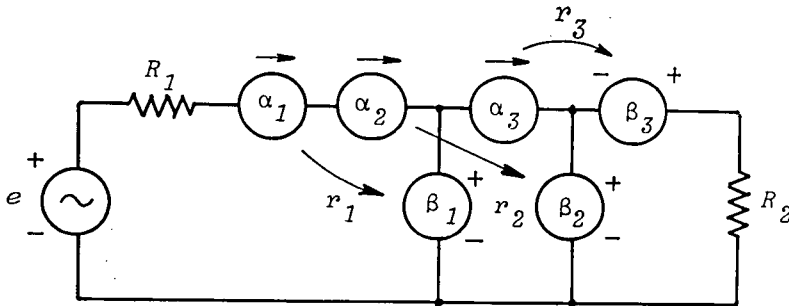


Fig.4.9 Example 4.4. $r_1 = r_2$.

Case 3. The condition in Theorem 4.1 is not satisfied. CTN.

Usually we have Case 3-a-1.

Case 3-a-1. RN and SNP. Example 4.5 in Fig.4.10.

Case 3-a-2. RN and SND. An example can be obtained by

combining Example 4.3 in Fig.4.8 and Example 4.5 in Fig.4.10.

Case 3-a-3. RN and SE. Example 4.6 in Fig.4.11. Another example can be derived from Example 4.3 in Fig.4.8 by contracting resistor R_1 .

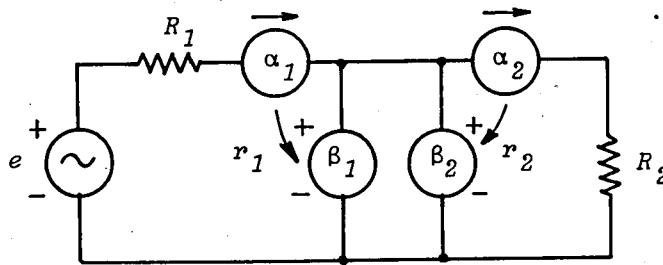


Fig.4.10 Example 4.5.

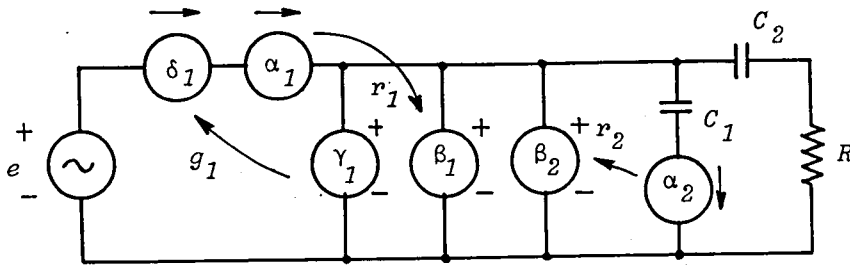


Fig.4.11 Example 4.6.

Case 3-b-1. RE and SNP. This case follows from Case 3-a-1 if the relations have no effect on solvability.

Case 3-b-2. RE and SND. Example 4.7 in Fig.4.12. If $r_1 g_1 = 1$ and $r_1 = r_2$, v_{β_1} is indeterminate.

Case 3-b-3. RE and SE. Example 4.5 in Fig.4.10 with $r_2 = R_2$.

No other case than are given above, is possible.

Now we consider the variables in Set 2 which is specified

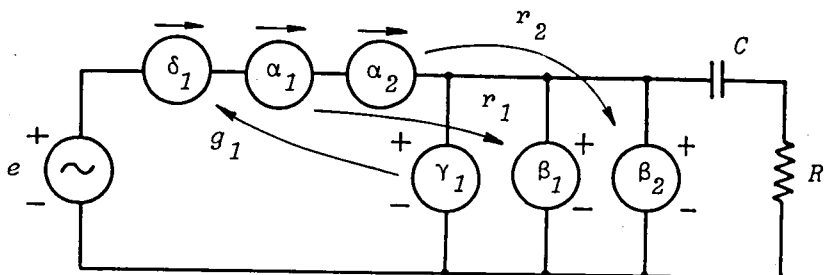


Fig.4.12 Example 4.7. $r_1 g_1 = 1$ and $r_1 = r_2$.

in section 4.2. If the condition in Theorem 4.1 is satisfied, there is a tree of G^* containing all the voltage sources, independent or controlled, but no current sources. Thus, if v_1 and i_1 are obtained, the unknown variables in Set 2 can be determined from Kirchhoff's laws. Next let us consider the cases where the condition in Theorem 4.1 is violated. Choosing a tree of G^* which contains a maximum of voltage sources and a minimum of current sources, we get $n_2 - u(E_\beta; 0) - \rho(E_\delta; 1)$ equations for the variables in Set 2, where n_2 is the number of these variables. Therefore, some of the variables are indeterminate, even if v_1 and i_1 are obtained. In general v_1 and i_1 themselves can be obtained only with certain special relations among the element values, as is previously indicated.

4.6 CONCLUDING REMARKS

The condition in Theorem 4.1 is equivalent to the condition 2) in Theorem 1 of reference [4.3] or the condition 1) in Theorem of reference [4.4]. The necessary and sufficient condition (4.9) and (4.10) for the existence of a common tree seems

to correspond to the famous Hall's condition in the matching theory. [1.1] [4.12] Equation (4.18) is a min=max relation like König's theorem. [1.1] Algorithm 4.2 is similar to the Hungarian method to obtain a maximal matching. [4.13] It is an efficient algorithm to obtain a common tree, if one exists, or an MCT, if no common tree exists.

Although the existence of a common tree is often guaranteed by checking the condition of Theorem 4.2, it is not so for an active network model using norators and nullators. The discussions in this chapter are, of course, applicable to the 2-graphs derived from such a network model, giving topological conditions and algorithms to check the solvability.

CHAPTER 5 STATE EQUATIONS AND ORDER OF COMPLEXITY OF LINEAR NETWORKS

5.1 INTRODUCTION

The state equations, as one of the important ways of describing a linear network, are written in a very compact form which only shows the relation between the first derivatives of the state variables and the state variables plus external forces. All the other variables are eliminated in the process of deriving the state equations from the network equations. In general, this process is very complicated in the case of an active network, and the relation between the state equations and the original network is difficult to observe. [4.2] [4.3] [5.1] [5.2]

The dimension of the state space, or in other words, the order of complexity of an active network depends not only on the network topology but also on the element values.

In this chapter we investigate how the topological properties of the network and the properties of the network elements are combined to formulate the state equations. An upper bound on the order of complexity which can be determined from the network topology only, is discussed in section 5.2. If there is no special relation among the element values, the network topology which gives the upper bound is to correspond to the state space.

We derive the state equations for an active network under the restriction imposed in section 1.2, trying to make equations which appear in the process of derivation, as simple as possible so that the relation between the relevant equations and the network topology can be observed. There are networks, however,

to which no unified procedure, except completely algebraic ones, to derive the state equations is applicable. We illustrate the difficulties encountered in the process of derivation by giving several examples.

5.2 ORDER OF COMPLEXITY OF LINEAR ACTIVE NETWORKS

On the basis of the 2-graph method Tow^[4.8] showed that an upper bound on the order of complexity[@] can be stated with respect to some particular common tree, which he called a maximal tree. The condition for a maximal tree seems to be rather complex. We give a much simpler common tree with respect to which an upper bound on the order of complexity of a general active network is given.^[5.3]

Unlike Tow we replaced a current-controlled current source by a cascade connection of a current-controlled voltage source and a voltage-controlled current source. A voltage-controlled voltage source was replaced in a similar way. Hence D_δ and D_β in (4.2) and (4.3), respectively, are constant diagonal matrices and " p "'s in the common-tree-immittance products Π_{immitt} arise only from the capacitors in the common tree and the inductors in its cotree. Let T_σ be such a common tree that the sum of the numbers of the capacitors in the common tree and the inductors in its cotree is σ . Then we can rewrite (4.8) as follows.

@ Here the order of complexity is the number of natural frequencies including the zero frequency. It is the same as the dynamical degree of freedom.^[1.1]

$$\begin{vmatrix} D_y & -D_z \\ B & 0 \\ 0 & Q \end{vmatrix} = \pm \sum_{\sigma=0}^{\sigma_{max}} \sum_{T_\sigma} (sign T_\sigma) \Pi_{c\sigma} \Pi_{\delta\sigma} \Pi_{L\sigma} \Pi_{r\sigma} \Pi_{\beta\sigma} p^\sigma \quad (5.1)$$

where $(sign T_\sigma)$ is the sign permutation for T_σ , and $\Pi_{c\sigma}$, $\Pi_{\delta\sigma}$, $\Pi_{L\sigma}$, $\Pi_{r\sigma}$ and $\Pi_{\beta\sigma}$ are the product of the capacitances in T_σ , the product of the mutual conductances of the voltage-controlled current sources in T_σ , the product of the inductances in $\bar{T}_\sigma$, the product of the resistances in $\bar{T}_\sigma$, and the product of the mutual resistances of current-controlled voltage sources in $\bar{T}_\sigma$, respectively. We define a normal common tree as a common tree that contains a maximum of capacitors and a minimum of inductors. Then we have:

Theorem 5.1

An upper bound on the order of complexity is equal to the sum of the numbers of the capacitors in a normal common tree and that of the inductors in the cotree of the normal common tree. The upper bound is the lowest possible if the network topology only is known.

The upper bound is equal to the order of complexity if there is only one normal common tree or if there are more than one normal common tree but there exists no special relation among the values of elements specified by the normal common trees.

Now let E_c and E_l be the sets of the capacitor branches and inductor branches respectively, and let T_c be a maximal common

subtree (a maximal set of branches which contains no loop in either of 2-graphs) of $G_v(E_c; 0)$ and $G_i(E_c; 0)$. T_c is the common part of a corresponding pair of MCT's of $G_v(E_c; 0)$ and $G_i(E_c; 0)$. Dually let P_l be a maximal common subcotree (a maximal set of branches which contains no cutset in either of 2-graphs) of $G_v(E_l; 1)$ and $G_i(E_l; 1)$. Contract [delete] the branches of $T_c[P_l]$ in $G_v(E_c; 0)$ and $G_i(E_c; 0)$ [$G_v(E_l; 1)$ and $G_i(E_l; 1)$]. Then any of the branches in $E_c - T_c[E_l - P_l]$ becomes a self-loop [bridge] in either or both of the derived graphs, because if a branch is not a self-loop [bridge] in either of the derived graphs, it can be added to $T_c[P_l]$, which contradicts $T_c[P_l]$ being maximal. Let $P_c[T_l]$ be the set of branches which are self-loops [bridges] in both of the derived graphs. Then we have:

Theorem 5.2

If a common tree T_R exists for the 2-graphs derived from G_v and G_i by contracting the branches of $T_c \cup T_l$ and deleting $P_l \cup P_c$, then $T = T_c \cup T_l \cup T_R$ is a normal common tree.

Proof: The subtree consisting of capacitors of a normal common tree of G_v and G_i also forms a subtree in $G_v(E_c; 0)$ and $G_i(E_c; 0)$. Thus the number of branches of the subtree is no more than the number of branches in T_c . The dual statement is valid for the number of inductors in T .

Q.E.D.

Note that T_R contains no capacitor, and $\bar{T}_R$, no inductor.

Now let us consider the transformation of common trees, or the common-tree transformation, in 2-graphs, as an extension of the tree transformation in one graph. Contrary to the fact

that an elementary tree transformation in one graph is always possible, an elementary common tree transformation may or may not be possible. Suppose T is a common tree of G_v and G_i , and x is a chord of T .

Theorem 5.3

If there is a sequence of branches $x=x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, x_{2m}=x$ that satisfies the following conditions;

- (i) x_{2k+1} is in the fundamental loop in $G_v[G_i]$ defined with respect to T by x_{2k} , $k=0, 1, \dots, m-1$, and x_{2k} is in the fundamental cutset in $G_i[G_v]$ defined with respect to T by x_{2k-1} , $k=1, 2, \dots, m$,
- (ii) there exists no sequence which is properly contained in the sequence and satisfies condition (i),

then, the set $\{x_1, x_3, \dots, x_{2k+1}, \dots, x_{2m-1}\}$ contains no cutset in $G_v[G_i]$, and the set $\{x_0, x_2, \dots, x_{2k}, \dots, x_{2m-2}\}$ contains no loop in $G_i[G_v]$.

Proof: Suppose that the set $\{x_1, x_3, \dots, x_{2k+1}, \dots, x_{2h+1}, \dots, x_{2m-1}\}$ contains a cutset in $G_v[G_i]$, and that the cutset contains x_{2k+1} . Branch x_{2k+1} belongs to the fundamental loop defined by x_{2k} . Since a loop has an even number of common branches with a cutset, the fundamental loop must contain one more branch, say x_{2h+1} , in the cutset. Let $2k+1 < 2h+1$, then there is a sequence of branches $x_0, x_1, \dots, x_{2k}, x_{2h+1}, \dots, x_{2m}$, which satisfies condition (i) and is properly contained in $x_0, x_1, \dots, x_{2k}, x_{2k+1}, \dots, x_{2h+1}, \dots, x_{2m}$. This contradicts condition (ii). The proof for the rest of the theorem is similar.

Q.E.D.

Theorem 5.4

If there is a sequence of branches $x=x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, x_{2m}=x$ that satisfies the conditions in Theorem 5.3, then x_{2k+1} , $k=1, 2, \dots, m-1$, is not contained in the fundamental loops defined by $x_0, x_2, \dots, x_{2k-2}$ in $G_v[G_i]$, and x_{2k} , $k=2, 3, \dots, m$, is not contained in the fundamental cutsets defined by $x_1, x_3, \dots, x_{2k-3}$ in $G_i[G_v]$.

Proof: Suppose x_{2k+1} is contained in the fundamental loop defined by x_{2h} ($h < k$). Then there is a sequence $x_0, x_1, \dots, x_{2h}, x_{2k+1}, \dots, x_{2m}$ which satisfies condition (i) and is properly contained in the sequence $x_0, x_1, \dots, x_{2h}, \dots, x_{2k}, x_{2k+1}, \dots, x_{2m}$. This contradicts condition (ii). The proof for the rest of the theorem is similar.

Q.E.D.

Theorem 5.5

If there is a sequence of branches $x=x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, x_{2m}=x$ such that satisfies the conditions in Theorem 5.3, then

$$T' = T \cup \left(\bigcup_{k=0}^{m-1} \{x_{2k}\} \right) - \left(\bigcup_{k=1}^m \{x_{2k-1}\} \right) \quad (5.2)$$

is a new common tree of G_v and G_i .

Proof: We prove that T' is a tree of G_v . By an elementary tree transformation we can get a tree of G_v , $T_1 = T \cup x_0 - x_1$. Since x_{2k+1} , $k=1, 2, \dots, m-1$, is not contained in the fundamental loops defined by x_0 with respect to T from Theorem 5.4, and thus the fundamental cutset defined by x_{2k+1} , $k=1, 2, \dots, m-1$, with respect to T does not contain x_0 , the fundamental cutset defined

by x_{2k+1} , $k=1,2,\dots,m-1$, with respect to T_1 which now includes x_0 as a tree-branch, remains unchanged and still contains x_{2k} , that is, the fundamental loop defined by x_{2k} with respect to T_1 still contains x_{2k+1} . Thus another tree transformation $T_2 = T_1 \cup x_2 - x_3$ is possible. Furthermore x_{2k+1} , $k=2,\dots,m-1$, is not included in the fundamental loops defined by $x_2, \dots, x_{2k-2}$ with respect to T_1 . Repeating the process, we get a tree of G_v . By the dual discussion to the above we can show $\bar{T}'$ is a cotree of G_i , and hence T' is a tree of G_i . It follows that T' is a common tree of G_v and G_i .

Q.E.D.

We can give the dual discussions to the above concerning a common tree T and a sequence of branches starting with a tree-branch of T instead of a chord.

Next we apply the above theorem to a normal common tree of G_v and G_i . Let T be a common tree. We assign a weight to each branch as follows. Weight ω_k of branch x_k is 1, if x_k represents a capacitor in $\bar{T}$ or an inductor in T , -1, if x_k represents a capacitor in T or an inductor in $\bar{T}$, and 0 otherwise.

Theorem 5.6

If there is a sequence of branches $x=x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, x_{2m}=x$ that satisfies the conditions in Theorem 5.3 for a normal common tree T , then the sum of the weights assigned to the branches in the sequence is non-positive, that is,

$$\omega_0 + \omega_1 + \dots + \omega_{2k-1} + \omega_{2k} + \omega_{2k+1} + \dots + \omega_{2m-1} \leq 0. \quad (5.3)$$

Proof: If the sum of the weights is positive, the sum of the

numbers of capacitors in T and inductors in $\bar{T}$ can be increased by performing the common-tree transformation specified in Theorem 5.5, which contradicts T being a normal common tree.

Q.E.D.

Corollary 5.1

There is no sequence of branches $x=x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, x_{2m}=x$ that satisfies condition (i) in Theorem 5.3 (not necessarily condition (ii)) for a normal common tree T with $\omega_0=1$, and $\omega_1=\omega_2=\dots=\omega_{2m-1}=0$.

Proof: If a sequence satisfying only condition (i) existed, T' obtained by (5.2) may or may not be a common tree, but the sequence contains a subsequence which also satisfies conditions (i) and (ii). A contradiction can be shown for this subsequence, since it contains x_0 whose weight $\omega_0=1$.

Q.E.D.

An efficient algorithm to obtain a normal common tree is now available. [5.4]

Although the voltages of the capacitors in a normal common tree and the currents through the inductors in the cotree of the normal common tree are to constitute a set of the state variables, derivation of the state equations for these variables sometimes needs a complex process.

5.3 STATE EQUATIONS OF LINEAR ACTIVE NETWORKS

In spite of the simple statement on the order of complexity in terms of a normal common tree, we encounter subtle difficulty

in deriving the state equations if we use a normal common tree. Here we discuss derivation of state equations using an over-normal tree. [1.2] [5.2] [5.5]

We find an overnormal tree of G^* , denoted by T , having the following properties.

1. T contains all the independent voltage sources and current sensors.
2. T contains a maximum of controlled voltage sources consistent with property 1.
3. T contains a maximum of capacitors consistent with properties 1 and 2.
4. T contains no independent current sources nor voltage sensors.
5. T contains a minimum of controlled sources consistent with property 4.
6. T contains a minimum of inductors consistent with properties 4 and 5.

A tree having the above properties can always be found in G^* with the topological restrictions (ii) specified in section 1.2. Now we assume that T contains all the controlled voltage sources but no controlled current source. The elements in T , that is, the independent voltage sources, current sensors, controlled voltage sources, capacitors, resistors and inductors in T are indicated by e , α , β , c , g and f , respectively. The elements in the cotree of T , that is, the independent current sources, voltage sensors, controlled current sources, inductors, resistors, and capacitors in $\bar{T}$ are indicated by j , γ , δ , l , r and s , respectively.

Now in order to make the form of the network equations simpler we replace resistors in T and $\bar{T}$ by their controlled source equivalents as shown in Fig.5.1 (a) and (b) respectively.

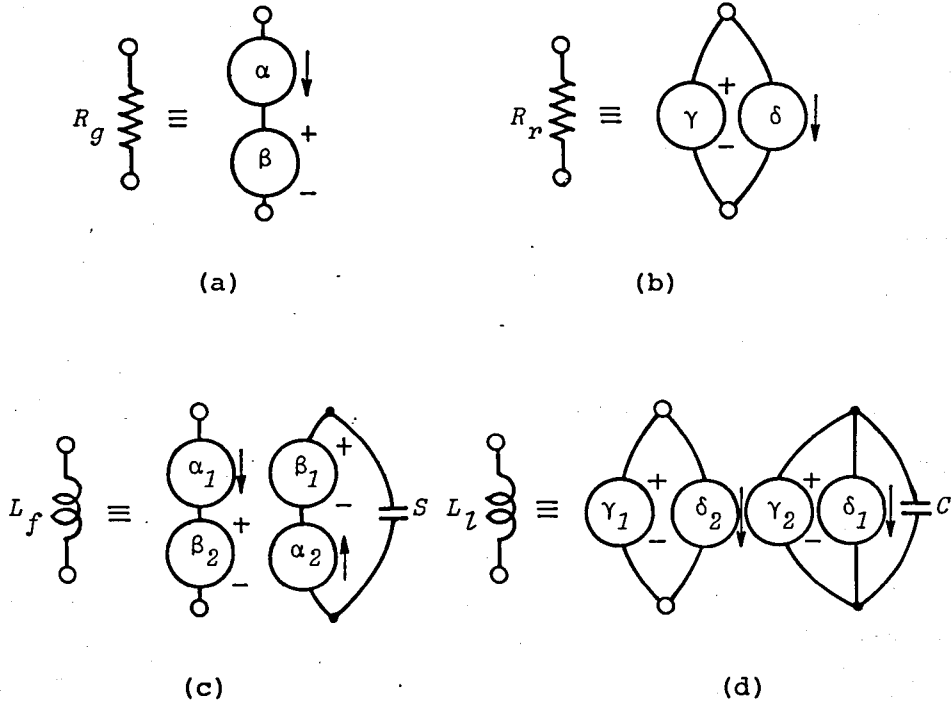


Fig.5.1 (a) Equivalent circuit for a resistor in T . (b) Equivalent circuit for a resistor in $\bar{T}$. (c) Equivalent circuit for an inductor in T . (d) Equivalent circuit for an inductor in $\bar{T}$.

The network equations are, then,

$$\begin{bmatrix} v_\gamma \\ v_l \\ v_s \end{bmatrix} = - \begin{bmatrix} B_{\gamma f} & B_{\gamma c} & B_{\gamma \beta} & B_{\gamma e} \\ B_{l f} & B_{l c} & B_{l \beta} & B_{l e} \\ 0 & B_{s c} & B_{s \beta} & B_{s e} \end{bmatrix} \begin{bmatrix} v_f \\ v_c \\ v_\beta \\ v_e \end{bmatrix} \quad (5.4)$$

$$\begin{bmatrix} i_\alpha \\ i_c \\ i_f \end{bmatrix} = - \begin{bmatrix} Q_{\alpha s} & Q_{\alpha l} & Q_{\alpha \delta} & Q_{\alpha j} \\ Q_{cs} & Q_{cl} & Q_{c\delta} & Q_{cj} \\ 0 & Q_{fl} & Q_{f\delta} & Q_{fj} \end{bmatrix} \begin{bmatrix} i_s \\ i_l \\ i_\delta \\ i_j \end{bmatrix} \quad (5.5)$$

$$p \begin{bmatrix} v_c \\ v_s \\ i_l \\ i_f \end{bmatrix} = \begin{bmatrix} D_c^{-1} & & & \\ & D_s^{-1} & & 0 \\ & & D_l^{-1} & \\ & 0 & & D_f^{-1} \end{bmatrix} \begin{bmatrix} i_c \\ i_s \\ v_l \\ v_f \end{bmatrix} \quad (5.6)$$

$$\begin{bmatrix} v_\beta \\ i_\delta \end{bmatrix} = \begin{bmatrix} D_{\beta\alpha} & 0 \\ 0 & D_{\delta\gamma} \end{bmatrix} \begin{bmatrix} i_\alpha \\ v_\gamma \end{bmatrix} \quad (5.7)$$

where v . and i . are the voltage and current vectors, respectively, associated with the elements as noted by the subscripts. $B_{..}$ and $Q_{..}$ are the submatrices of the principal parts of the fundamental loop and cutset matrices, respectively, associated with the elements as noted by the subscripts. D_c , D_s , D_l , D_f , $D_{\beta\alpha}$ and $D_{\delta\gamma}$ are diagonal matrices of the element values as are respectively indicated by the subscripts. Since we are mainly concerned with the state equations, we are satisfied without immediately knowing the currents through the voltage sources or the voltages across the current sources. Therefore the equations for these variables are not shown.

We first eliminate i_α and v_γ from the equations in (5.7) and the first equations in (5.4) and (5.5). Then we get

$$M_{\alpha\beta} v_\beta = -Q_{\alpha s} i_s - Q_{\alpha l} i_l - Q_{\alpha j} i_j + Q_{\alpha\delta} D_{\delta\gamma} (B_{\gamma f} v_f + B_{\gamma c} v_c + B_{\gamma e} v_e) \quad (5.8)$$

$$M_{\gamma\delta} i_{\delta} = -B_{\gamma f} v_f - B_{\gamma c} v_c - B_{\gamma e} v_e + B_{\gamma\beta} D_{\beta\alpha} (Q_{\alpha s} i_s + Q_{\alpha l} i_l + Q_{\alpha j} i_j), \quad (5.9)$$

where

$$M_{\alpha\beta} = D_{\beta\alpha}^{-1} - Q_{\alpha\delta} D_{\delta\gamma} B_{\gamma\beta} \quad (5.10)$$

$$M_{\gamma\delta} = D_{\delta\gamma}^{-1} - B_{\gamma\beta} D_{\beta\alpha} Q_{\alpha\delta}. \quad (5.11)$$

We observe that $M_{\alpha\beta}$ or $M_{\gamma\delta}$ becomes singular if and only if the mutual resistances in $D_{\beta\alpha}$ and the mutual conductances in $D_{\delta\gamma}$ satisfy a certain relation specified by the network topology. We distinguish the cases where this occurs from the cases where the rank of a matrix decreases due to the network topology only. If the element values can be chosen arbitrary, $M_{\alpha\beta}$ and $M_{\gamma\delta}$ can be made nonsingular, and we can solve (5.8) and (5.9) for v_{β} and i_{δ} . Then from the third set of equations in (5.4) and (5.5) we obtain the relation between v_s , i_f , i_s and v_f , which can be written in the form;

$$\begin{bmatrix} v_s \\ i_f \end{bmatrix} = H_{sf} \begin{bmatrix} i_s \\ v_f \end{bmatrix} + H_{cl} \begin{bmatrix} v_c \\ i_l \end{bmatrix} + H_{ej} \begin{bmatrix} v_e \\ i_j \end{bmatrix} \quad (5.12)$$

where

$$H_{sf} = \begin{bmatrix} B_{s\beta} & 0 \\ 0 & Q_{f\delta} \end{bmatrix} \begin{bmatrix} M_{\beta\alpha} & 0 \\ 0 & M_{\delta\gamma} \end{bmatrix} \begin{bmatrix} U & -Q_{\alpha\delta} D_{\delta\gamma} \\ -B_{\gamma\beta} D_{\beta\alpha} & U \end{bmatrix} \begin{bmatrix} Q_{\alpha s} & 0 \\ 0 & B_{\gamma f} \end{bmatrix}. \quad (5.13)$$

$M_{\beta\alpha}$ and $M_{\delta\gamma}$ are the inverse matrices of $M_{\alpha\beta}$ and $M_{\gamma\delta}$ respectively.

If H_{sf} is nonsingular, then (5.12) can be solved for i_s and v_f . Eliminating these variables together with i_c and v_l using

(5.5), we obtain a set of state equations. The order of complexity for this case is the sum of the numbers of capacitors and inductors. If H_{sf} is singular and the nullity of the matrix is μ_{sf} , there must be μ_{sf} relations among v_s , i_f , v_c and i_l (Possibly some of these variables may become constant.). Thus an upper bound on the number of state variables is given by the sum of the numbers of the capacitors in T and inductors in $\bar{T}$ plus the rank of H_{sf} . Therefore we investigate the rank of this matrix in the following.

There are four matrices on the right-hand side of (5.13). The second and the third matrices are square and become singular only if there are certain relations among the mutual resistances in $D_{\beta\alpha}$ and the mutual conductances in $D_{\delta\gamma}$. The rank of the other two matrices can be determined from the network topology only. We define graph $G_{vs}[G_{is}]$ as the graph obtained from the voltage graph G_v [the current graph G_i] by contracting the capacitors in T and the inductors in T , and by deleting the inductors in $\bar{T}$, the voltage sensors [the controlled current sources] and the controlled voltage sources [the current sensors] leaving only the capacitors in $\bar{T}$. Then we have:

Lemma 5.1

The rank of $B_{s\beta}[Q_{\alpha s}]$ is equal to the rank of $G_{vs}[G_{is}]$.

Proof: Submatrix $B_{s\beta}[Q_{\alpha s}]$ corresponds to the graph, denoted by $G_{vs\beta}[G_{is\beta}]$, obtained from $G_v[G_i]$ by contracting all the tree-branches except the controlled voltage sources [current sensors] and by deleting all the chords except the capacitors in $\bar{T}$.

Then Lemma 5.1 follows immediately from Corollary 2.2 [Corollary

2.1].

Q.E.D.

We also define $G_{vf}[G_{if}]$ as the graph obtained from $G_v[G_i]$ by deleting the inductors in $\bar{T}$ and the capacitors in $\bar{T}$, and by contracting the capacitors in T , the controlled voltage sources[current sensors] and the voltage sensors[controlled current sources], leaving only the inductors in T . Then:

Lemma 5.2

The rank of $B_{\gamma f}[Q_{f\delta}]$ is equal to the nullity of $G_{vf}[G_{if}]$.

Corollary 5.2

Matrix $B_{s\beta}[Q_{\alpha s}]$ is square and nonsingular if and only if $G_{vs\beta}[G_{i\alpha s}]$ is a graph consisting of a pair of disjoint trees, that is, a tree of controlled voltage sources[current sensors] and a tree of capacitors.

The dual corollary to Corollary 5.2 can also be stated, but is omitted here.

Theorem 5.7

$$\text{rank of } H_{sf} \leq \min(\text{rank of } G_{vs} + \text{nullity of } G_{if}, \text{rank of } G_{is} + \text{nullity of } G_{vf}) . \quad (5.14)$$

Proof: Applying Sylvester's theorem for the rank of a matrix product, Lemma 5.1 and Lemma 5.2 to (5.13), we easily get the result.

Q.E.D.

Now in order to make the following discussions simpler we further replace all the inductors in the network by their equivalent circuits as shown in Fig.5.1 (c) and (d). Then the terms with subscripts l and f in (5.4), (5.5) and (5.6) disappear. The terms concerning inductors are also dropped from (5.8), (5.9) and (5.12). From the resultant equations we obtain

$$\begin{bmatrix} B_{s\beta} M_{\beta\alpha} Q_{\alpha s} & 0 \\ Q_{cs} + Q_{c\delta} M_{\delta\gamma} B_{\gamma\beta} D_{\beta\alpha} Q_{\alpha s} & U \end{bmatrix} \begin{bmatrix} i_s \\ i_c \end{bmatrix} = \begin{bmatrix} U & B_{sc} + B_{s\beta} M_{\beta\alpha} Q_{\alpha\delta} D_{\delta\gamma} B_{\gamma c} \\ 0 & Q_{c\delta} M_{\delta\gamma} B_{\gamma c} \end{bmatrix} \begin{bmatrix} v_s \\ v_c \end{bmatrix} + \begin{bmatrix} B_{se} + B_{s\beta} M_{\beta\alpha} Q_{\alpha\delta} D_{\delta\gamma} B_{\gamma e} & B_{s\beta} M_{\beta\alpha} Q_{\alpha j} \\ Q_{c\delta} M_{\delta\gamma} B_{\gamma e} & Q_{cj} + Q_{c\delta} M_{\delta\gamma} B_{\gamma\beta} D_{\beta\alpha} Q_{\alpha j} \end{bmatrix} \begin{bmatrix} v_e \\ i_j \end{bmatrix} \quad (5.15)$$

We are now going to derive a set of state equations from (5.15). Let us denote the rank of $B_{s\beta} M_{\beta\alpha} Q_{\alpha s}$ by ρ_s . The rank of the matrix in the left-hand side of (5.15) is the sum of ρ_s and the number of the capacitors in T . If there exists a $(\rho_s \times \rho_s)$ nonsingular principal minor matrix in $B_{s\beta} M_{\beta\alpha} Q_{\alpha s}$, we proceed as follows. First dividing the capacitors in T into two sets, indicated by s_1 and s_2 , such that the capacitors in the second set correspond to the nonsingular principal minor matrix, we can write (5.15) as

$$\begin{bmatrix} 0 & 0 & 0 \\ A_{2s_1} & A_{2s_2} & 0 \\ A_{3s_1} & A_{3s_2} & U \end{bmatrix} \begin{bmatrix} i_{s_1} \\ i_{s_2} \\ i_c \end{bmatrix} = \begin{bmatrix} K_{1s_1} & K_{1s_2} & K_{1c} \\ 0 & U & K_{2c} \\ 0 & 0 & K_{3c} \end{bmatrix} \begin{bmatrix} v_{s_1} \\ v_{s_2} \\ v_c \end{bmatrix} + \begin{bmatrix} F_{1e} & F_{1j} \\ F_{2e} & F_{2j} \\ F_{3e} & F_{3j} \end{bmatrix} \begin{bmatrix} v_e \\ i_j \end{bmatrix} \quad (5.16)$$

where A_{2s_2} and K_{1s_1} are nonsingular. Eliminating i_{s_1} , i_{s_2} and i_c by use of (5.6) and then v_{s_1} , we have

$$\begin{aligned}
& \begin{bmatrix} A_{2s_2} D_{s_2} - A_{2s_1} D_{s_1} K_{1s_1}^{-1} K_{1s_2} & -A_{2s_1} D_{s_1} K_{1s_1}^{-1} K_{1c} \\ A_{3s_2} D_{s_2} - A_{3s_1} D_{s_1} K_{1s_1}^{-1} K_{1s_2} & -A_{3s_1} D_{s_1} K_{1s_1}^{-1} K_{1c} + D_c \end{bmatrix} \begin{bmatrix} p v_{s_2} \\ p v_c \end{bmatrix} \\
& = \begin{bmatrix} U & K_{2c} \\ 0 & K_{3c} \end{bmatrix} \begin{bmatrix} v_{s_2} \\ v_c \end{bmatrix} + \begin{bmatrix} F_{2e} & F_{2j} \\ F_{3e} & F_{3j} \end{bmatrix} \begin{bmatrix} v_e \\ i_j \end{bmatrix} \\
& \quad - \begin{bmatrix} A_{2s_1} D_{s_1} K_{1s_1}^{-1} F_{1e} & A_{2s_1} D_{s_1} K_{1s_1}^{-1} F_{1j} \\ A_{3s_1} D_{s_1} K_{1s_1}^{-1} F_{1e} & A_{3s_1} D_{s_1} K_{1s_1}^{-1} F_{1j} \end{bmatrix} \begin{bmatrix} p v_e \\ p i_j \end{bmatrix} \quad (5.17)
\end{aligned}$$

The matrix on the left-hand side of (5.17) is nonsingular unless there exists a special relation among the element values. No further decrease in the order of complexity is caused by the network topology only. [5.5] A set of state equations can be obtained by multiplying the inverse of this matrix to both sides of the equation. Note that only the first derivative terms of the independent source voltages and/or currents may appear in the state equations. If the matrix in the left-hand side of (5.17) is singular, the process used to derive (5.17) from (5.16) is repeated, and each repetition of the process may yield higher order derivatives of the independent source voltages and/or currents. Summarizing, we have:

Theorem 5.8

If

- (i) T contains all the controlled voltage sources, but no controlled current sources,
- (ii) no special relation among the element values, or the

element values are arbitrary, and

- (iii) $B_{s\beta} M_{\beta\alpha} Q_{\alpha s}$ has a $(\rho_s \times \rho_s)$ nonsingular principal minor matrix,

then

- (i) $M_{\alpha\beta}$ and $M_{\gamma\delta}$ are nonsingular,
- (ii) the coefficient matrix in the left-hand side of (5.17) is nonsingular and v_c and v_{s_2} constitute the state variables, and
- (iii) at most the first derivatives of the independent source voltages and/or currents appear in the state equations.

Now we look into the networks, for which the conditions in Theorem 5.8 do not hold.

Case 1-1. Even if the rank of $B_{s\beta} M_{\beta\alpha} Q_{\alpha s}$ is ρ_s , there may not always exist a $(\rho_s \times \rho_s)$ nonsingular principal minor matrix. For example, if capacitor C_2 in the network shown in Fig.5.2 is included in an overnormal tree, $B_{s\beta} M_{\beta\alpha} Q_{\alpha s}$ is

$$\begin{bmatrix} 0 & r_2 & r_2+r_3 & r_2+r_3+r_4 \\ 0 & 0 & 0 & 0 \\ 0 & r_1 & r_1 & r_1 \\ 0 & r_1 & r_1+r_3 & r_1+r_3 \end{bmatrix}.$$

The rank of this matrix is easily seen to be 3, but there is no (3×3) nonsingular principal minor matrix. The largest nonsingular principal minor matrix is a (1×1) matrix. The order of complexity of this network is 2. There is no $(\rho_s \times \rho_s)$ nonsingular principal minor matrix for any overnormal tree of

the network. For this kind of network the derivation of the state equations needs more complex processes.

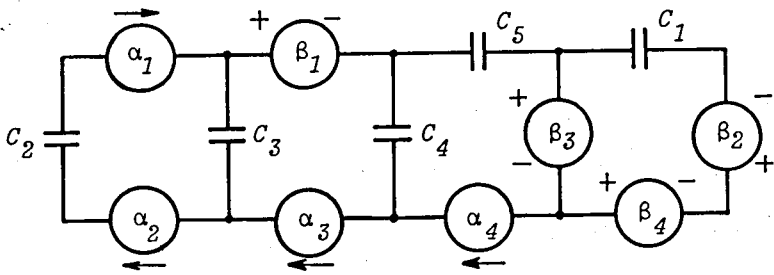


Fig.5.2 Example 5.1(by T.Nitta[5.5]).

Case 1-2. For some networks the existence of a $(\rho_g \times \rho_g)$ nonsingular principal minor matrix depends on the choice of the overnormal tree. If capacitor C_1 in the network shown in Fig.5.3 as Example 5.2 is included in T , the rank of $B_{s\beta} M_{\beta\alpha} Q_{\alpha s}$ is 2, but there is no (2×2) nonsingular principal minor matrix. Some other overnormal tree, however, say a tree with capacitor C_7 as a tree element, leads to the matrix with (2×2) nonsingular principal minor matrix. The order of complexity of this network is 3.

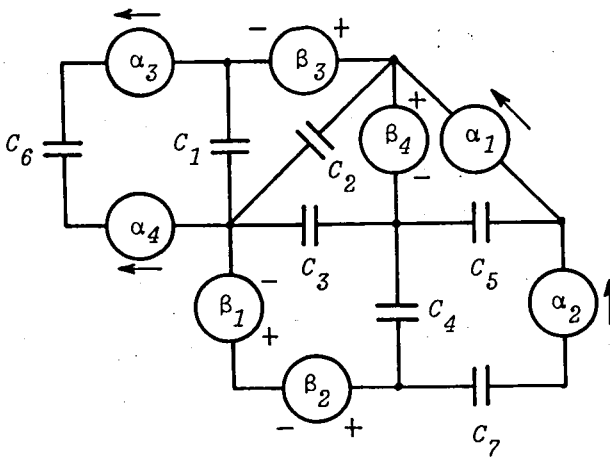


Fig.5.3 Example 5.2.

Case 1-3. For the network obtained from the network in Fig.5.2 by exchanging α_3 and β_3 , we get a $(\rho_g \times \rho_g)$ nonsingular principal minor matrix for any possible overnormal tree. The network is shown in Fig.5.4 as Example 5.3.

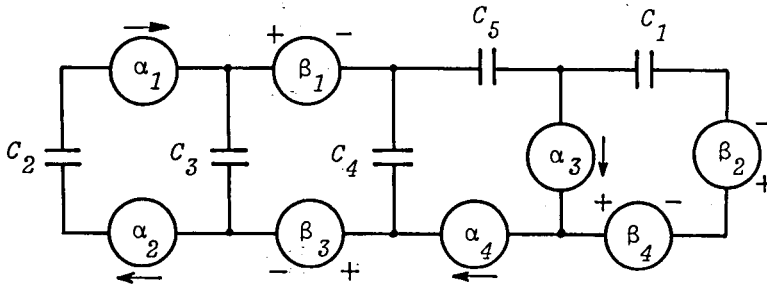


Fig.5.4 Example 5.3 (by T.Nitta[5.5]).

Let us now study the cases where there are some special relations among the element values, and $M_{\alpha\beta}$ or $M_{\gamma\delta}$ becomes singular. In these cases we go back to the original network equations. It may or may not be possible to get a set of state equations depending on the network topology and the element values, as are illustrated in the following examples.

Case 2-1. Even if $M_{\alpha\beta}$ is singular, a set of state equations can be obtained. An example is given in Fig.5.5 as Example 5.4. If $r_1 = r_3$, $M_{\alpha\beta}$ for this network is singular, but a state equation for v_{S_2} can be derived.

Case 2-2. $M_{\alpha\beta}$ is singular, and some of the variables in (5.4)-(5.7) are indeterminate. No set of state equations exists. For example, if $r_1 g_1 = 1$ in Example 5.5 in Fig. 5.6, $M_{\alpha\beta}$ is singular and v_{β_1} is indeterminate.

Case 2-3. $M_{\alpha\beta}$ is singular and no solution of the network

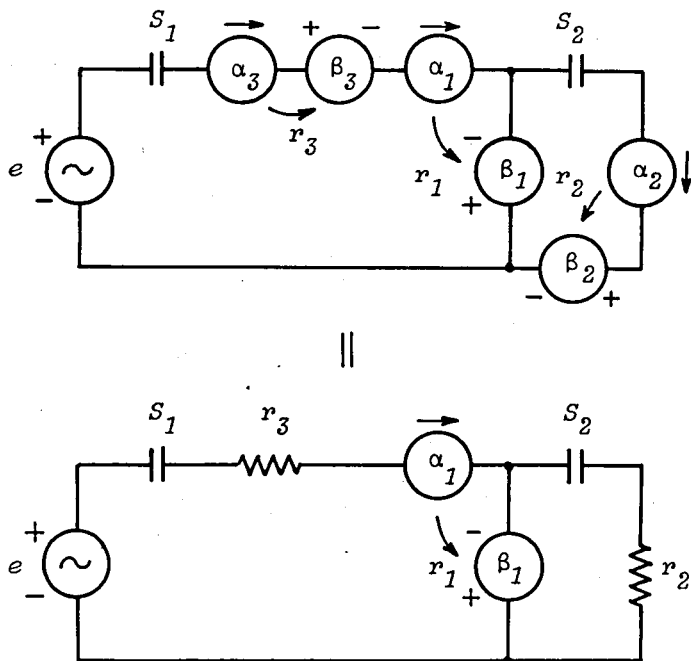


Fig.5.5 Example 5.4.

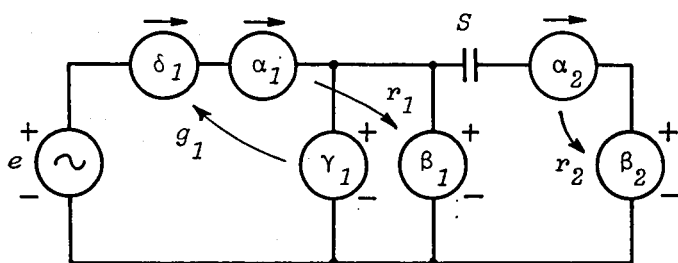


Fig.5.6 Example 5.5.

equation is possible. See Example 5.6 in Fig. 5.7. If $r_1 = R$, $M_{\alpha\beta}$ is singular. No solution of the network equations exists for $v_e \neq 0$.

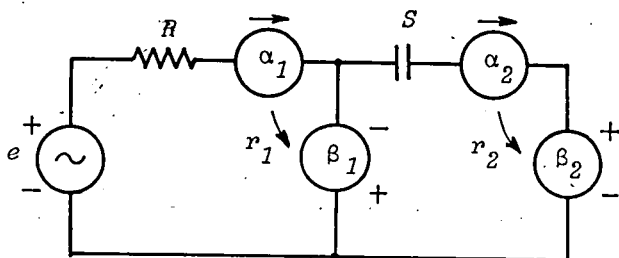


Fig.5.7 Example 5.6.

The order of complexity may or may not decrease compared with the network with the same network topology and element kind but with nonsingular $M_{\alpha\beta}$ or $M_{\gamma\delta}$. From Example 5.4 a disadvantage of using an overnormal tree can be pointed out. The priority of current-controlled voltage sources to be included in an overnormal tree is higher than that of capacitors, and the priority of resistors is lower than that of capacitors. There are cases, however, where a current-controlled voltage source can be regarded as only a resistor, rather than a mutually coupling twoport element. Then the capacitor which constitutes a loop with such a current-controlled voltage source can be removed from the cotree and included in the tree. In these cases, therefore, it can be said that the number of capacitors in the cotree becomes unnecessarily large due to the priority specified for the overnormal tree. Incidentally Example 5.4 serves as a counter example to Theorem 3 of reference [1.2], showing that the theorem is not always valid.

Finally we remark on the condition (i) in Theorem 5.8. It is exactly the sufficient condition in Theorem 4.2 for the existence of a common tree of the voltage and current graphs.

Thus it guarantees the existence of state equations if there is no special relations among the element values. Moreover it simplifies the form of the network equations as can be seen in (5.4) and (5.5), and makes it possible to treat an active network almost like an RLC network. It is not an essential condition, however, for the existence or the derivation of state equations.

5.4 CONCLUDING REMARKS

The state equations and the order of complexity of active networks are studied from the network topological point of view. With the effort to preserve the topological properties in the manipulations, we have succeeded in showing the results which are very much alike those of RLC networks. The order of complexity is stated with respect to a normal common tree, which is exactly a direct extension of a normal tree of an RLC network. The state equations which can be derived from (5.17) has only the first derivative terms of external forces like the passive case. A question might arise that the derivation of state equations using a normal common tree instead of an overnormal tree, would be more appropriate. Usually we have to go through a more complicated elimination process if we use a normal common tree.

ACKNOWLEDGEMENT

The author wishes to express his deep gratitude to Professor Akira Kishima and Professor Yoshio Hattori for their constant encouragement and guidance during the course of this study. Special thanks are due for their reading the manuscript and making precise and helpful suggestions to complete the dissertation. The author also wishes to thank Mr. Tanzo Nitta for his useful and stimulating discussions.

REFERENCES

Chapter 1.

- [1.1] Hattori, Y. and Ozawa, T., *Graph-riron Kaisetsu*
(*Introduction to Graph Theory*), Shokodo, Tokyo, 1974.
- [1.2] De Claris, N. and Saeks, R., "Theoretic foundations of
finite network and system analysis," in *Aspects of Network
and System Theory*, Kalman, R. E. and De Claris, N. edit.,
Holt, Rinehart and Winston, Inc., N.Y., 1971.
- [1.3] Mayeda, W., *Graph Theory*, Wiley-Interscience, N.Y., 1972.
- [1.4] Davies, A.C., "The significance of nullators, norators
and nullars in active-network theory," *Radio Elect. Eng.*
vol.34, pp.259-267, 1967.

Chapter 2.

- [2.1] Reza, F.M., "Order of complexity and minimal structure
in network analysis," *Proc. Symp. on Circuit Analysis*,
Univ. of Illinois, Urbana, Ill., pp.(7-1)-(7-33), 1955.
- [2.2] Hattori, Y., "Network analysis by using effective inde-
pendent variables," *CT Monograph*, No.69-3, Technical
Group on Circuit Theory, Inst. Elect. Commun. Eng. Japan,
1963.
- [2.3] Hattori, Y., Kobuchi, Y. and Uyama, C., "A state variable
approach to network analysis: Part I," *Tech. Report*,
Inst. Atom. Energy, Kyoto Univ, No.156, 1972.
- [2.4] Tutte, W.T., *Introduction to the Theory of Matroid*,
p.23, American Elsevier, N.Y., 1971.
- [2.5] Ozawa, T., "On the degree of interference and certain

trees of a multicolored-branch graph," Trans. Inst. Elect. Commun. Eng. Japan, vol.55-A, pp.642-643, 1972.

- [2.6] Ozawa, T., Hirao, S. and Hattori, Y., "Some properties of multicolored-branch graphs," Mem. Facul. Eng. Kyoto Univ., vol.35, pp.381-391, 1973.
- [2.7] Kishi, G. and Kajitani, Y., "Maximally distant trees and principal partition of a linear graph," IEEE Trans. Circuit Theory, vol.CT-16, pp.323-330, 1969; also, "Maximally distant trees in a linear graph," Trans. Inst. Elect. Commun. Eng. Japan, vol.51-A, pp.196-203, 1968.
- [2.8] Tutte, W.T., "Menger's theorem for matroid," Jour. Res. NBS, vol.69B, pp.49-53, 1965.

Chapter 3.

- [3.1] Tsuchiya, T., Ohtsuki, T., Ishizaki, Y. and Watanabe, H., "Topological degree of freedom of electrical networks," Proc. 5th Allerton Conf. Circuit Syst. Theory, pp.644-653, 1967; also Ohtsuki, T., Ishizaki, Y. and Watanabe, H., "Network analysis and topological degree of freedom," Trans. Inst. Elect. Commun. Eng. Japan, vol.51-A, pp.238-245, 1968.
- [3.2] Ozawa, T., "On the minimal graphs of a certain class of graphs," IEEE Trans. Circuit Theory, vol.CT-18, p.387, 1971.
- [3.3] Ozawa, T., "A procedure of graph-partitioning for the mixed analysis of electrical networks," Mem. Facul. Eng. Kyoto Univ. vol.33, pp.128-133, 1971.
- [3.4] Dulmage, A.L. and Mendelsohn, N.S., "Covering of bipart-

- ite graphs," *Canad. Jour. Math.*, vol.10, pp.517-534, 1958.
- [3.5] Dulmage, A.L. and Mendelsohn, N.S., "A structure theory of bipartite graphs of finite exterior dimension," *Trans. Roy. Soc. Canada, Sect.III*, vol.53, pp.1-13, 1959.
- [3.6] Iri, M., "Combinatorial canonical form of a matrix with applications to the principal partition of a graph," *Trans. Inst. Elect. Commun. Eng. Japan*, vol.54-A, pp.30-37, 1971.
- [3.7] Tomizawa, N., "Irreducible matroids and principal partition of a matroid into irreducible minors," *CST Monograph, CST74-8*, Technical Group on Circuit and System Theory, *Inst. Elect. Commun. Eng. Japan*, 1974.

Chapter 4.

- [4.1] Malic, N.R. and Hale, H.W., "Equations for active networks: Existence of unique solutions," *IEEE Trans. Circuit Theory*, vol.CT-14, pp.37-43, 1967.
- [4.2] Parker, S.R. and Barmes, V.T., "Existence of numerical solutions and the order of linear circuits with dependent sources," *IEEE Trans. Circuit Theory*, vol.CT-18, pp.368-374, 1971.
- [4.3] Purslow, E.J., "Solvability and analysis of linear active networks by use of the state equations," *IEEE Trans. Circuit Theory*, vol.CT-17, pp.469-475, 1970.
- [4.4] Chen, W.K. and Chen, F.N., "On the unique solvability of linear active networks," *IEEE Trans. Circuit and Systems*, vol.CAS-21, pp.26-35, 1974.

- [4.5] Ho, Y.S. and Roe, P.H., "Existence theorem in the time domain for linear active networks," IEEE Trans. Circuit and Systems, vol.CAS-21, pp.175-177, 1974.
- [4.6] Carlin, H.J., "Singular network elements," IEEE Trans. Circuit Theory, vol.CT-11, pp.67-72, 1964.
- [4.7] Ozawa, T., "On the existence of common trees in the two-graph method," Trans. Elect. Commun. Eng. Japan, vol.56-A, pp.371-372, 1973.
- [4.8] Tow, J., "Order of complexity of linear active networks," Proc. IEE, vol.115, pp.1259-1262, 1968.
- [4.9] Ozawa, T., "On the network topology of linear active networks," CT Monograph, CT72-32, Technical Group on Circuit and System Theory, Inst. Elect. Commun. Eng. Japan, 1972.
- [4.10] Minty, G., "A simple algorithm for listing all the trees of a graph," IEEE Trans. Circuit Theory, vol.CT-12, p.120, 1965.
- [4.11] Ozawa, T., "Common trees and partition of two-graphs," Trans. Elect. Commun. Eng. Japan, vol.57-A, pp.383-389, 1974.
- [4.12] Mirsky, L., *Transversal Theory*, Academic Press, N.Y., 1971.
- [4.13] Kuhn, H.W., "The Hungarian method for the assignment problem," Naval Res. Logist. Quart., vol.2, pp.83-97, 1955, or Ford, L.D. Jr. and Fulkerson, D.R., *Flows in Networks*, Princeton Univ. Press, Princeton, 1962.

Chapter 5.

- [5.1] Kuh, E.S. and Rohrer, R.A., "The state-variable approach to network analysis," Proc. IEEE, vol.53, pp.672-686, 1965.
- [5.2] Hirano, K., Nishi, F. and Tominaga, S., "Some considerations on state equations of linear active networks," Intern. Jour. Circuit Theory and Appl. vol.2, pp.39-50, 1974.
- [5.3] Ozawa, T., "Order of complexity of linear active networks and a common tree in the 2-graph method," Electron. Letters, vol.8, No.22, pp.542-543, 1972.
- [5.4] Tomizawa, N. and Iri, M., "An algorithm for solving the 'independent assignment problem' with application to the problem of determining the order of complexity of a network," Trans. Inst. Elect. Commun. Eng. Japan, vol.57-A, pp.627-629, 1974.
- [5.5] Ozawa, T. and Nitta, T., "Some considerations on the state equations of linear active networks and the network topology," Mem. Facul. Eng. Kyoto Univ. vol.34, pp.413-424, 1972.