

AN ESTIMATE OF THE ISOVARIANT BORSUK-ULAM CONSTANT

Ikumitsu NAGASAKI

Department of Mathematics
Kyoto Prefectural University of Medicine

ABSTRACT. We shall discuss the isovariant Borsuk-Ulam constant determined from the weak isovariant Borsuk-Ulam theorem. We first illustrate some properties of the Borsuk-Ulam constant and next provide an estimate of the isovariant Borsuk-Ulam constant for the special unitary group $SU(n)$.

1. BACKGROUND

Borsuk-Ulam type results for G -maps between (linear) G -spheres were studied by many researchers and various generalizations were shown. In particular, the following generalization is well known; see [3] for example.

Theorem 1.1. *Let G be $(C_p)^k$ a product of cyclic groups of prime order p or T^k a (k -dimensional) torus. Suppose that G acts smoothly and fixed-point-freely on spheres S_1 and S_2 . If there exists a (continuous) G -map $f : S_1 \rightarrow S_2$, then the inequality*

$$\dim S_1 \leq \dim S_2$$

holds.

On the other hand, T. Bartsch [1] proved that such a Borsuk-Ulam result does not hold for G not being a p -toral group. A compact Lie group G is called p -toral if there is an exact sequence $1 \rightarrow T \rightarrow G \rightarrow P \rightarrow 1$, where T is a torus and P is a finite p -group.

As a variation of the Borsuk-Ulam theorem, the isovariant Borsuk-Ulam theorem was first studied by A. G. Wasserman [9]. Let G be a compact Lie group. A G -map $f : X \rightarrow Y$ is called G -isovariant if f preserves the isotropy subgroups, i.e., $G_x = G_{f(x)}$ for any $x \in X$. In other words, it is a G -map such that $f|_{G(x)} : G(x) \rightarrow Y$ is injective on each orbit $G(x)$ of $x \in X$. From Wasserman's results, one sees the following.

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Theorem 1.2 (Isovariant Borsuk-Ulam theorem). *Let G be a solvable compact Lie group. If there exists a G -isovariant map $f : SV \rightarrow SW$ between linear G -spheres, then*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Wasserman conjectures that this theorem holds for all finite groups. This is unsolved at present; however, we showed a *weak version* of the isovariant Borsuk-Ulam theorem for an *arbitrary* compact Lie group.

Theorem 1.3 (Weak isovariant Borsuk-Ulam theorem ([5, 6])). *There exists a positive constant $c > 0$ such that*

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

for any pair of representations V and W with a G -isovariant map $f : SV \rightarrow SW$.

Definition. The *isovariant Borsuk-Ulam constant* c_G of G is defined to be the supremum of such a constant c . (If $G = 1$, then set $c_G = 1$ as convention.)

When $c_G = 1$, G is called a *Borsuk-Ulam group* (BUG for short); namely, a Borsuk-Ulam group G is a compact Lie group for which the isovariant Borsuk-Ulam theorem holds. In particular, a solvable compact Lie group is a Borsuk-Ulam group by Theorem 1.2, and several nonsolvable Borsuk-Ulam finite groups are also known; for the detail, see [7, 8, 9]. However, no one knows connected Borsuk-Ulam groups other than a torus. Therefore we would like to investigate c_G and provide some estimates at least. We illustrate general properties of c_G in section 2 and we provide an estimate c_G for $G = U(n)$ in section 3; in fact, we notice

$$c_{U(n)} \geq \frac{n}{n+1}$$

whose complete proof will be written elsewhere.

2. PROPERTIES OF c_G

The following result is a generalization of Wasserman's result and is proved by a similar argument as in [9].

Proposition 2.1. *If $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is an exact sequence of compact Lie groups, then*

$$\min\{c_K, c_Q\} \leq c_G \leq c_Q.$$

In particular, if K is a Borsuk-Ulam group, then $c_G = c_Q$.

Using this inductively, we have

Corollary 2.2. *If $1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G$, then*

$$\min_{1 \leq i \leq r} \{c_{H_i/H_{i-1}}\} \leq c_G.$$

As an example, one sees the following.

Example 2.3. *It follows that $c_{U(n)} = c_{SU(n)} = c_{PSU(n)}$. In particular, $c_{SU(2)} = c_{SO(3)}$ since $PSU(2) \cong SO(3)$.*

Proof. There is an exact sequence

$$1 \rightarrow C_n \rightarrow S^1 \times SU(n) \rightarrow U(n) \rightarrow 1.$$

Since C_n is a Borsuk-Ulam group, it follows from Proposition 2.1 that $c_{U(n)} = c_{S^1 \times SU(n)}$. Next, there is an exact sequence

$$1 \rightarrow S^1 \rightarrow S^1 \times SU(n) \rightarrow SU(n) \rightarrow 1.$$

Since S^1 is a Borsuk-Ulam group, it follows that $c_{S^1 \times SU(n)} = c_{SU(n)}$. Thus $c_{U(n)} = c_{SU(n)}$. Since the center of $SU(n)$ is isomorphic to C_n , it follows that $c_{PSU(n)} = c_{SU(n)}$. $\square$

3. ESTIMATION OF $c_{U(n)}$

Let T denote the maximal torus T of $U(n)$ given by diagonal matrices:

$$T = \left\{ \begin{pmatrix} t_1 & & O \\ & \ddots & \\ O & & t_n \end{pmatrix} \mid t_i \in S^1 (\subset \mathbb{C}) \right\}.$$

We set

$$d_{U(n)} = \sup \left\{ \frac{\dim U^T}{\dim U} \mid U : \text{nontrivial irreducible } U(n)\text{-representation} \right\}.$$

In order to estimate $c_{U(n)}$, we use the fact $c_{U(n)} \geq 1 - d_{U(n)}$ deduced from a result of [6].

Theorem 3.1. $d_{U(n)} = \frac{1}{n+1}$, and hence $c_{U(n)} \geq \frac{n}{n+1}$.

This is proved by representation theory. The irreducible complex representations of $U(n)$ are parametrized by λ in

$$\Lambda = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}.$$

Let V_λ denote the irreducible $U(n)$ -representation corresponding to $\lambda \in \Lambda$. (Then λ is the highest weight of V_λ .) Since $\text{Res}_T : R(U(n)) \rightarrow R(T)^{W_n}$ is isomorphic, where $W_n \cong S_n$ is

the Weyl group of $U(n)$, the character χ_λ of $\text{Res}_T V_\lambda$ is a homogenous symmetric Laurent polynomial in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with a form

$$\chi_\lambda(t) = \sum_{\mu \in \mathbb{Z}^n} m_\lambda(\mu) t^\mu = \sum_{\mu \in \mathbb{Z}^n} m_\lambda(\mu) t_1^{\mu_1} \cdots t_n^{\mu_n} \quad (t = \text{diag}(t_1, \dots, t_n) \in T).$$

The coefficient $m_\lambda(\mu)$ is the multiplicity of a weight μ , i.e., the dimension of the weight space corresponding to μ :

$$m_\lambda(\mu) = \dim\{v \in V_\lambda \mid t \cdot v = t^\mu v \text{ for all } t \in T\} \geq 0.$$

Let $M_\lambda := \{\mu \in \mathbb{Z}^n \mid |\mu| = |\lambda| \text{ and } \mu \preceq \lambda\}$, which is a finite set. Here $|\mu| = \sum_{i=1}^n \mu_i$, and $\preceq$ is the dominant order on $\mathbb{Z}^n$ defined by

$$\mu \preceq \lambda \iff \sum_{i=1}^k \mu_i \leq \sum_{i=1}^k \lambda_i \quad (1 \leq \forall k \leq n).$$

The following results can be found in [2, 4].

Proposition 3.2. *Let $\lambda \in \Lambda$ and $\mu \in \mathbb{Z}^n$.*

- (1) $m_\lambda(\mu) \neq 0 \iff \mu \in M_\lambda$.
- (2) $m_\lambda(\lambda) = 1$ for $\lambda \in \Lambda$.
- (3) $m_\lambda(w \cdot \mu) = m_\lambda(\mu)$ for any $w \in W_n$, where $w \cdot \mu = (\mu_{w^{-1}(1)}, \dots, \mu_{w^{-1}(n)})$.
- (4) W_n acts on M_λ by permutation as in (3) and for any $\mu \in M_\lambda$, $W_n(\mu) \cap \Lambda$ consists of one element. Therefore $M_\lambda \cap \Lambda$ is a complete system of representatives of M_λ/W_n .

Thus the character has a form

$$\chi_\lambda(t) = \sum_{\mu \in M_\lambda} m_\lambda(\mu) t^\mu = \sum_{\mu \in M_\lambda \cap \Lambda} m_\lambda(\mu) P_\mu(t),$$

where $P_\mu(t) = \sum_{\nu \in W_n(\mu)} t^\nu$.

Proposition 3.3. *Let $G = U(n)$ and $\lambda \in \Lambda$.*

- (1) $\dim V_\lambda = \chi_\lambda(1) = \sum_{\mu \in M_\lambda} m_\lambda(\mu)$.
- (2) $\dim V_\lambda^T = m_\lambda(0)$, the constant term of $\chi_\lambda(t)$.
- (3) $\dim V_\lambda^T > 0 \iff 0 \in M_\lambda \iff \lambda \in \Lambda_0 := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n, \sum_i \lambda_i = 0\}$.

Furthermore, the dimension of V_λ is described in terms of the highest weight $\lambda \in \Lambda$.

Proposition 3.4 (dimension formula for $U(n)$ ([2, 4])).

$$\dim V_\lambda = \frac{\prod_{i < j} (\lambda_i - \lambda_j + j - i)}{\prod_{i < j} (j - i)}.$$

On the other hand, computation of the multiplicity is not so easy (if λ is large); however several multiplicity formulas are known; for example, Freudenthal formula, Kostant formula, and combinatorially $m_\lambda(\mu)$ can be given as a Kostka number (= the number of certain semi-standard Young tableaux). We use Freudenthal's multiplicity formula; see [4] for example.

3.1. Outline of proof of Theorem 3.1. We may assume $\lambda \in \Lambda_0$ and $\lambda \neq 0$, since $\dim V_\lambda^T = 0$ if $\lambda \notin \Lambda_0$. Let $(-, -)$ denote the (standard) inner product on $\mathbb{R}^n$. Let $\alpha_{ij} = e_i - e_j$ for $i \neq j$, where e_i is the i -th fundamental unit vector. All α_{ij} form the root system of type A_{n-1} . Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ the set of positive roots and set

$$\rho := \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-3}{2}, -\frac{n-1}{2} \right).$$

Applying Freudenthal's multiplicity formula to $\mu = 0$, we have an inequality

$$(*) : m_\lambda(0)K_\lambda \leq 2n(n-1)d \sum_{k=1}^d m_\lambda(\mu_k),$$

where $K_\lambda := \|\lambda\|^2 + 2(\lambda, \rho)$ and $\mu_k := k\alpha_{1n} = (k, 0, \dots, 0, -k) \in \Lambda_0$. Since $\mu_k \in M_\lambda$ ($1 \leq k \leq d$), $\chi_\lambda(t)$ has a form

$$\chi_\lambda(t) = m_\lambda(0) + \sum_{k=1}^d m_\lambda(\mu_k)P_{\mu_k}(t) + \text{other terms},$$

where $P_{\mu_k}(t) = \sum_{i \neq j} t^{k\alpha_{ij}} = \sum_{i \neq j} t_i^k t_j^{-k}$, which has $n(n-1)$ terms. This shows

$$\dim V_\lambda = \chi_\lambda(1) \geq m_\lambda(0) + \sum_{k=1}^d m_\lambda(\mu_k)n(n-1).$$

Using the inequality (*), we obtain

$$\dim V_\lambda \geq \left(1 + \frac{K_\lambda}{2d}\right) m_\lambda(0).$$

Since $K_\lambda = \|\lambda\|^2 + 2(\lambda, \rho) \geq \lambda_1^2 + \lambda_n^2 + (n-1)(\lambda_1 - \lambda_n)$, it follows that

$$\frac{\dim V_\lambda^T}{\dim V_\lambda} \leq \frac{1}{n+1}.$$

On the other hand, applying the multiplicity formula to $\lambda = \mu_1$, one sees

$$\dim V_{\mu_1}^T = n-1,$$

and by the dimension formula, $\dim V_{\mu_1} = (n+1)(n-1)$. Hence it follows that

$$\frac{\dim V_{\mu_1}^T}{\dim V_{\mu_1}} = \frac{1}{n+1}.$$

Thus we have $d_{U(n)} = \frac{1}{n+1}$. □

Remark. In case of $n = 2$, the theorem provide an estimate $c_{U(2)} \geq 2/3$; however, this may be improved by a further argument; in fact, we show that $c_{U(2)} \geq 4/5$ in [6].

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DEPARTMENT OF MATHEMATICS, KYOTO PREFECTURAL UNIVERSITY OF MEDICINE, 1-5 SHIMOGAMO HANGI-CHO, SAKYO-KU, KYOTO 606-0823, JAPAN

E-mail address: nagasaki@koto.kpu-m.ac.jp