

A SHORT PROOF OF NUBLING'S RESULT

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ABSTRACT. Nubling shows that CM-triviality (=non-2-ampleness) is preserved under reducts in finite U-rank theories. We give a short proof.

1. REDUCTION AND INDEPENDENCE

Let T^- be a reduct of T . Let $\mathcal{M} \models T$, $\mathcal{M}^- \models T^-$ be big models. $a, b, c, \dots \bar{a}, \bar{b}, \bar{c}, \dots$ denote finite tuples, and $A, B, C, \dots$ denote small sets. Let $A \subset \mathcal{M}^{\text{eq}}$. $\text{ACL}^{\text{eq}}(A)$ denotes the algebraic closure of A in T , and $\text{acl}^{\text{eq}}(A)$ denotes the algebraic closure of $A \cap (\mathcal{M}^-)^{\text{eq}}$. Let $\bar{a} \in (\mathcal{M}^-)^{\text{eq}}$. $\text{TP}(\bar{a}/A)$ denotes the type of $\bar{a}$ over A in T , and $\text{tp}(\bar{a}/A)$ denotes the type of $\bar{a}$ over A in T^- . SU denotes Lascar rank in T , and su denotes Lascar rank in T^- . We show the following fact in the last section.

Fact 1.1. *Let T be a simple theory having EHI such that T^- also has EHI, where T^- be a reduct of T . Let $a, C \subset (\mathcal{M}^-)^{\text{eq}}$ and $B \subset \mathcal{M}^{\text{eq}}$. If $a \perp_B C$, then $a \perp_{B^-} C$, where $B^- = \text{ACL}^{\text{eq}}(B) \cap (\mathcal{M}^-)^{\text{eq}}$ and $\perp^-$ is the non-forking relation in T^- .*

Proposition 1.2. *If $\text{SU}(T) < \omega$, then $\text{su}(T^-) < \omega$.*

Proof. Let $a \in (\mathcal{M}^-)^{\text{eq}}$, $A \subset \mathcal{M}^{\text{eq}}$. Put $A^- = \text{ACL}^{\text{eq}}(A) \cap \mathcal{M}^{\text{eq}}$. We will show that there exists $\bar{a}' \models \text{tp}(a/A^-)$ such that $\text{SU}(a'/A) \geq \text{su}(a'/A^-)$ by induction on $n = \text{su}(a/A^-)$.

If $n = 0$, it is clear. Let $\text{su}(a/A^-) = n + 1$. So, there exists $A^- \subset B \subset (\mathcal{M}^-)^{\text{eq}}$ such that $\text{su}(a/B) = n$. So, $a \not\perp_{A^-} B$. Put $B^- = \text{ACL}^{\text{eq}}(B) \cap (\mathcal{M}^-)^{\text{eq}}$. So, we have $A^- \subset B \subseteq B^-$. Take $a_1 \models \text{tp}(a/B)$ such that $\bar{a}_1 \perp_B B^-$. As $\text{su}(a_1/B) = \text{su}(a_1/B^-) = n$, by induction hypothesis, there exists $a'_1 \models \text{tp}(a_1/B^-)$ such that $\text{SU}(a'_1/B) \geq \text{su}(a'_1/B^-) = n$. As $\text{tp}(a'_1/A^-) = \text{tp}(a/A^-)$, we see $a'_1 \not\perp_{A^-} B^-$. As $B^- \subseteq B$ and $a \not\perp_{A^-} B$, by Fact 1.1, we see $\bar{a}'_1 \not\perp_A B$. Therefore we have

$\text{SU}(a'_1/A) \geq \text{SU}(a'_1/B) + 1 \geq \text{su}(a'_1/B^-) + 1 = n + 1 = \text{su}(a/A^-) = \text{su}(a'_1/A^-)$, as desired. $\square$

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Lemma 1.3. *Suppose that $U(T) < \omega$. Let T^- be a reduct of T . u denotes the Lascar rank in T^- . (Then $u(T^-) < \omega$.) Let $a, b, c \in (\mathcal{M}^-)^{\text{eq}}$ be algebraically independent in T^- such that $u(a/b) = 1$ (So, $a \perp_b^- c$, because $a \notin \text{acl}^{\text{eq}}(bc)$.) Then there exist $a', b', c' \in \mathcal{M}^{\text{eq}}$ such that a', b', c' are algebraically independent in T , a realization of $\text{tp}(abc)$ with $a' \perp_{b'} c'$.*

Proof. Let $a'b'c' \models \text{tp}(abc)$ be such that $U(a'b'c')$ is maximal.

Claim. a', b', c' are algebraically independent in T .

As $a' \notin \text{acl}^{\text{eq}}(b'c')$, we can find $a'' \models \text{tp}(a'/b'c')$ such that $a'' \notin \text{ACL}^{\text{eq}}(b'c')$. So, if $a' \in \text{ACL}^{\text{eq}}(b'c')$, then $\text{SU}(a''b'c') > \text{SU}(a'b'c')$, a contradiction. Similarly, we see $b' \notin \text{ACL}^{\text{eq}}(a'c')$ and $c' \notin \text{ACL}^{\text{eq}}(a'b')$.

Claim. $a' \perp_{b'} c'$.

By way of contradiction, suppose that $a' \not\perp_{b'} c'$. Let $a'_0 \models \text{TP}(a'/\text{ACL}^{\text{eq}}(b'))$ such that $a'_0 \perp_{b'} c'$. As $1 = u(a'/b')$, $\text{stp}(a'/b') = \text{stp}(a'_0/b')$ and $a'_0 \notin \text{ACL}^{\text{eq}}(b'c') \supseteq \text{acl}^{\text{eq}}(b'c')$, we see $1 = u(a'_0/b') \geq u(a'_0/b'c') \geq 1$. So we see $a'_0 \perp_{b'} c'$. By STATIONARITY of strong types, we see $\text{stp}(a'_0/b'c') = \text{stp}(a'/b'c')$. In particular, $a'_0b'c' \models \text{tp}(a'b'c')$. Now, we have

$$\begin{aligned} U(a'_0b'c') &= U(a'_0/b'c') + U(b'c') \\ &= U(a'_0/b') + U(b'c') \\ &= U(a'/b') + U(b'c') \\ &> U(a'/b'c') + U(b'c') = U(a'b'c') \end{aligned}$$

□

2. A SHORT PROOF

We begin with basics of supersimple theories.

Fact 2.1. *Let T be a supersimple theory.*

- (1) *Let $a \in \mathcal{M}^{\text{eq}}$, $A \subseteq \mathcal{M}^{\text{eq}}$. Then there exists finite tuple $\bar{b} \subset \mathcal{M}^{\text{eq}}$ such that $\text{acl}^{\text{eq}}(\text{Cb}(a/A)) = \text{acl}^{\text{eq}}(\bar{b}) = \text{acl}^{\text{eq}}(\text{Cb}(a/\bar{b}))$.*
- (2) *Let $A \subset \mathcal{M}$ be finitely generated algebraically closed set, and $B = \text{acl}(B) \subset A$. Then B is finitely generated algebraically closed.*
- (3) *Let $\text{SU}(T) < \omega$ and p be a non-algebraic type. Then there exists a minimal type, non-orthogonal to p . (Coordinatization Theorem)*

Proof. (1): Let $B = \text{Cb}(a/A)$. Take a finite tuple $\bar{b} \subset B \subset \mathcal{M}^{\text{eq}}$ such that $a \perp_{\bar{b}} B$. Then $B = \text{Cb}(a/A) = \text{Cb}(a/\bar{b})$ and $\text{acl}^{\text{eq}}(\bar{b}) = \text{acl}^{\text{eq}}(B)$.

(2): By way of contradiction, suppose that there exist $C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots \subset B \subset A = \text{acl}(\bar{a})$, where C_i are f.g. algebraically closed. Let $\bar{a}_n$ be such that $\bar{a}_n \equiv_{C_n} \bar{a}$ and $\bar{a}_n \perp_{C_n} \bar{a}$. As $C_n \subset \text{acl}(\bar{a})$, we see that $C_n = \text{acl}(\bar{a}_n) \cap \text{acl}(\bar{a})$.

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As $C_n \subset C_{n+1}$, so $\bar{a}_{n+1} \not\downarrow_{C_n} \bar{a}$. So, $\bar{a} \not\downarrow_{C_n} C_{n+1}$, because $\bar{a} \downarrow_{C_n} C_{n+1}$ and $\bar{a} \downarrow_{C_{n+1}} \bar{a}_{n+1}$ imply $\bar{a} \downarrow_{C_n} \bar{a}_{n+1}$. This contradicts supersimplicity.

(3): We may assume that $p = \text{tp}(a)$. Let $n = \text{SU}(p)$. Take B such that $\text{SU}(a/B) = n - 1$. Let $b \in \mathcal{M}^{\text{eq}}$ be such that $\text{acl}^{\text{eq}}(\text{Cb}(a/B)) = \text{acl}^{\text{eq}}(b) = \text{acl}^{\text{eq}}(\text{Cb}(a/b))$ by (1). As $a \not\downarrow b$, $b \notin \text{acl}^{\text{eq}}(\emptyset)$. Take C be such that $\text{SU}(b/C) = 1$. We may assume $C \downarrow_b a$. Then we have $a \not\downarrow_C b$, otherwise $\text{Cb}(a/bC) = \text{Cb}(a/b) \subseteq \text{acl}(C)$, so $b \in \text{acl}(C)$ would follow. On the other hand, as $n = \text{SU}(a) \geq \text{SU}(a/C) > \text{SU}(a/Cb) = \text{SU}(a/b) = n - 1$, we see $\bar{a} \downarrow C$. $\square$

Notation 2.2. $A \wedge B$ denotes $\text{acl}^{\text{eq}}(A) \cap \text{acl}^{\text{eq}}(B)$. $a \leftarrow A$ denotes $a \in \text{acl}^{\text{eq}}(A)$.

Definition 2.3. (1) We say that a sequence (a_0, a_1, a_2) is 2-ample over A , if $a_0 A \wedge a_1 A = A$, $a_0 a_1 A \wedge a_0 a_1 A = A$, $a_2 \downarrow_{a_1 A} a_0$ and $a_2 \not\downarrow_A a_0$.

(2) We say that a sequence (a_0, a_1, a_2) is weakly 2-ample over A , if $a_2 \downarrow_{a_1, A} a_0$ and $a_2 \not\downarrow_{a_1 \wedge a_0 a_2, A} a_0$.

(3) A complete simple theory T with EHI is (weakly) 2-ample, if there exist (weakly) 2-ample sequence over some parameters.

Remark 2.4. (1) T is 2-ample if and only if T is weak 2-ample.

(2) If (a_0, a_1, a_2) is weakly 2-ample, then so is (a_2, a_1, a_0) .

(3) If (a_0, a_1, a_2) is weakly 2-ample, then (a_0, a_1, a_2) are algebraically independent.

Proof. (1): Clearly, any 2-ample sequence is weakly 2-ample. Let (a_0, a_1, a_2) be weakly 2-ample and let a'_0 be such that $\text{acl}^{\text{eq}}(a'_0) = a_0 a_1 \wedge a_0 a_2$. Then we have $a'_0 a_1 \wedge a'_0 a_2 = \text{acl}^{\text{eq}}(a'_0)$ and $a'_0 \wedge a_1 = a_1 \wedge a_0 a_2$. Then we see that (a'_0, a_1, a_2) is 2-ample over $a_1 \wedge a_0 a_2$. (2): Clear. (3): If a_0 or a_2 were algebraic over a_1 , then it would be algebraic over $a_1 \wedge a_0 a_2$. If a_1 were algebraic over $a_0 a_2$, then $\text{acl}^{\text{eq}}(a_1) = a_1 \wedge a_0 a_2$ would follow. As $a_2 \downarrow_{a_1} a_0$, we see a_0, a_1, a_2 are algebraically independent. $\square$

From now on, we work in a finite SU-rank theory.

Lemma 2.5. Let (a_0, a_1, a_2) be weakly 2-ample.

(1) There exist a'_0 and B such that $a'_0 \leftarrow a_0 B$, $\text{SU}(a'_0/B) = 1$ and (a_0, a_1, a_2) is weakly 2-ample over B .

(2) Fixing a_1 , after adding some parameters, we can retake a_0, a_2 such that $\text{SU}(a_0/a_1) = \text{SU}(a_2/a_1) = 1$.

Proof. (1): By coordinatization theorem, there exist a'_0 and B such that $a'_0 \not\downarrow_B a_0$, $\text{SU}(a'_0/B) = 1$ and $a_0 \downarrow B$. We may assume $a_1 a_2 \downarrow_{a_0} B a'_0$. Since $a_0 a_1 a_2 \downarrow B$, (a_0, a_1, a_2) is weakly 2-ample over B , as desired.

(2): By remark 2.4 (2), we have only to retake a_0 such that $\text{SU}(a_0/a_1) = 1$.

Let a_0 be minimal of SU-rank such that (a_0, a_1, a_2) is weakly 2-ample. Suppose that $\text{SU}(a_0/a_1) > 1$. By (1) take a'_0 such that $a'_0 \prec a_0$, $\text{SU}(a'_0) = 1$. By Fact 2.1 (2), take a be such that $\text{acl}^{\text{eq}}(a) = a_0 \wedge a'_0 a_1$. Then $\text{SU}(a_0) > \text{SU}(a)$, $\text{SU}(a_0/a)$, because $\text{SU}(a_0) = \text{SU}(a_0/a) + \text{SU}(a)$ and $\text{SU}(a), \text{SU}(a_0/a) \geq 1$. (If $a_0 \prec a$, then a_0, a'_0 are interalgebraic over a_1 , a contradiction.) If $a_0 \not\downarrow_{a, a_1 \wedge a_0 a_2} a_2$, then (a_0, a_1, a_2) is weakly 2-ample over a , which contradicts the minimality of $\text{SU}(a_1)$. If $a_0 \downarrow_{a, a_1 \wedge a_0 a_2} a_2$, then $a \not\downarrow_{a_1 \wedge a_0 a_2} a_2$, so we see (a, a_1, a_2) is weakly 2-ample over $a_1 \wedge a_0 a_2$, a contradiction. $\square$

Proposition 2.6. *Let (a_0, a_1, a_2) be weakly 2-ample. Then, after adding some parameters, we can retake a_0, a_1, a_2 such that*

$$\text{SU}(a_0/a_1) = \text{SU}(a_2/a_1) = \text{SU}(a_1/a_0 a_2) = 1.$$

Proof. By Lemma 2.5, take a_1 be minimal of SU-rank such that (a_0, a_1, a_2) is weakly 2-ample and $\text{SU}(a_0/a_1) = \text{SU}(a_2/a_1) = 1$. Suppose that $\text{SU}(a_1/a_0 a_2) > 1$. Take $a'_1 \prec a_1$ be such that $\text{SU}(a'_1) = 1$ after possibly adding parameters. Let a, b be such that $\text{acl}^{\text{eq}}(a) = a_0 a_1 \wedge a_0 a'_1 a_2$ and $\text{acl}^{\text{eq}}(b) = a \wedge a_1$. Then $\text{SU}(a_1) > \text{SU}(b)$, $\text{SU}(a_1/b)$. (If $a_1 \prec a_0 a'_1 a_2$, then a_1, a'_1 would be interalgebraic over $a_0 a_2$. So we see $\text{SU}(a_1/b) \geq 1$. Clearly $\text{SU}(b) \geq 1$. The above follows from $\text{SU}(a_1) = \text{SU}(a_1/b) + \text{SU}(b)$.)

If $a \not\downarrow_b a_2$, then (a, a_1, a_2) is weakly 2-ample over b , because $b \subseteq (a_1 \wedge a a_2) b \subseteq a_1 \wedge a_0 a'_1 a_2 = b$. As $a \prec a_0 a_1$ and $b \prec a_1$, we have $\text{SU}(a/a_1 b) = \text{SU}(a_2/a_1 b) = 1$. This contradicts the minimality of $\text{SU}(a_1)$.

If $a \downarrow_b a_2$, then $a_0 \downarrow_b a_2$. Then (a_0, b, a_2) is weakly 2-ample over $a_1 \wedge a_0 a_2$. By Lemma 2.5, we may assume $\text{SU}(a_0/b) = \text{SU}(a_2/b) = 1$. This also contradicts the minimality of $\text{SU}(a_1)$. $\square$

Now, we prove the Nubling's theorem.

Theorem 2.7. *Suppose that $\text{U}(T) < \omega$. If a reduct T^- of T is 2-ample, then so is T .*

Proof. By Proposition 2.6, let (a_0, a_1, a_2) be weakly 2-ample such that $\text{u}(a_0/a_1) = \text{u}(a_2/a_1) = \text{u}(a_1/a_0 a_2) = 1$. As a_0, a_1, a_2 are algebraically independent in T^- , by Lemma 1.3, there exist $abc \models \text{tp}(a_0 a_2 a_3)$ such that a, b, c are algebraically independent in T and $a \downarrow_b c$.

Claim. $a \not\downarrow_{\text{ACL}^{\text{eq}}(b) \cap \text{ACL}^{\text{eq}}(ac)} c$. So, (a, b, c) is weakly 2-ample.

Put $A = \text{ACL}^{\text{eq}}(b) \cap \text{ACL}^{\text{eq}}(ac)$, and $A^- = A \cap (\mathcal{M}^-)^{\text{eq}}$. By way of contradiction, suppose that $a \downarrow_A c$. Then we have $a \downarrow_{A^-} c$ by Fact 1.1. As $a \notin \text{ACL}^{\text{eq}}(b) = \text{ACL}^{\text{eq}}(bA) \supseteq \text{acl}^{\text{eq}}(bA^-)$, we see $1 = \text{u}(a/b) \geq \text{u}(a/bA^-)$, so $a \downarrow_b^- A^-$ follows. Moreover, as $c \notin \text{ACL}^{\text{eq}}(ab) = \text{ACL}^{\text{eq}}(abA) \supseteq \text{acl}^{\text{eq}}(abA^-)$,

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we see $1 = u(c/b) \geq u(c/abA^-) \geq 1$, so $c \perp_b^- aA^-$ holds. So, we have $A^- \perp_b^- ac$. On the other hand, $b \notin \text{ACL}^{\text{eq}}(ac) = \text{ACL}^{\text{eq}}(acA) \supseteq \text{acl}^{\text{eq}}(acA^-)$, we see $1 = u(b/ac) \geq u(b/acA^-) \geq 1$, we have $b \perp_{ac}^- A^-$. So, we have $\text{Cb}(\text{tp}(A^-/abc)) \subseteq b \wedge^- ac := \text{acl}^{\text{eq}}(b) \cap \text{acl}^{\text{eq}}(ac)$, $A^- \perp_{b \wedge^- ac}^- abc$ holds. Since $a \perp_{A^-}^- c$ and $a \perp_{b \wedge^- ac}^- A^-$, so $a \perp_{b \wedge^- ac}^- c$, (a, b, c) is not weakly 2-ample in T^- , a contradiction. $\square$

Remark 2.8. There is a modular O-minimal theory which has a non-CM-trivial reduct [Y]. Nubling theorem can not be extended to finite U*-rank theories.

3. INDISCERNIBLE SEQUENCES AND THE PROOF OF FACT 1.1

We work in a complete theory and consider imaginary elements.

Let $(a_i : i \in I)$ be a sequence and $I_0 \subseteq I$. a_{I_0} denotes $(a_i : i \in I_0)$. When I is an partially ordered set, $a_{<i}$ denotes $(a_j : j < i)$. Similarly for $a_{>i}$. We write $I_0 < I_1$, if $I_0, I_1 \subseteq I$ and $i_1 < i_2$ holds for any $i_1 \in I_1, i_2 \in I_2$.

Definition 3.1. Let $X = (a_i : i \in I)$ be a B -indiscernible sequence and $A \subseteq B$.

- (1) Put $\ker_A(X) := \bigcup_{|I_0|=|J_0|=k < \omega, I_0 < J_0} (\text{acl}^{\text{eq}}(a_{I_0}A) \cap \text{acl}^{\text{eq}}(a_{J_0}A))$. We call it the kernel of X over A .
- (2) We say that X is algebraically independent over A , if $\text{acl}^{\text{eq}}(Aa_{I_0}) \cap \text{acl}^{\text{eq}}(A_{I_1}) = \text{acl}^{\text{eq}}(A)$ for any $I_0 < I_1 \subseteq I$.

Lemma 3.2. Let $X = (a_i : i \in I)$ be a B -indiscernible sequence.

- (1) For infinite subsets $I_1 < I_2$, $\ker_B(X) = \text{acl}^{\text{eq}}(a_{I_1}B) \cap \text{acl}^{\text{eq}}(a_{I_2}B)$.
- (2) $\ker_B(X)$ is the smallest algebraically closed set (containing B) over which X is algebraically independent.
- (3) X is indiscernible over $\ker_B(X)$.
- (4) $\ker_B(X)$ is the biggest subset (containing B) of $\text{acl}^{\text{eq}}(XB)$ over which X is indiscernible.

Proof. For ease of notation, we assume $B = \emptyset$.

(1): Suppose that I_0, I_1, J are finite with the same size and $I_0, I_1 < J$. As $a_{I_0} \equiv_{\text{acl}^{\text{eq}}(a_J)} a_{I_1}$, we see

$$\text{acl}^{\text{eq}}(a_{I_0}) \cap \text{acl}^{\text{eq}}(a_J) = \text{acl}^{\text{eq}}(a_{I_1}) \cap \text{acl}^{\text{eq}}(a_J).$$

By the same argument, we see that

$$\text{acl}^{\text{eq}}(a_{I_0}) \cap \text{acl}^{\text{eq}}(a_{J_0}) = \text{acl}^{\text{eq}}(a_{I_1}) \cap \text{acl}^{\text{eq}}(a_{J_1}).$$

for any $I_0 < J_0, I_1 < J_1, |I_0| = |I_1| = |J_0| = |J_1|$. Therefore, we see $\ker(X) \subseteq \text{acl}^{\text{eq}}(a_{I_1}) \cap \text{acl}^{\text{eq}}(a_{I_2})$ for any infinite $I_1 < I_2$. We show the converse inclusion. Let $a \in \text{acl}^{\text{eq}}(a_{I_1}) \cap \text{acl}^{\text{eq}}(a_{I_2})$. Then there exist $J_1 \subset I_1, J_2 \subset I_2$ such that

$|J_1| = |J_2| < \omega$ such that $a \in \text{acl}^{\text{eq}}(a_{J_1}) \cap \text{acl}^{\text{eq}}(a_{J_2})$. By the above argument, we see $\text{acl}^{\text{eq}}(a_{I_1}) \cap \text{acl}^{\text{eq}}(a_{I_2}) \subseteq \ker(X)$.

(2): Let C be such that X is algebraically independent over C . Then, for any infinite $I_0 < J_0$, $\ker(X) = \text{acl}^{\text{eq}}(a_{I_1}) \cap \text{acl}^{\text{eq}}(a_{J_1}) \subseteq \text{acl}^{\text{eq}}(Ca_{I_0}) \cap \text{acl}^{\text{eq}}(Ca_{J_0}) = \text{acl}^{\text{eq}}(C)$, as desired.

(3): By (1), we see that if X' is an extended indiscernible sequence of X , then $\ker(X) = \ker(X')$. It suffices to show that, if I_0, J_0 are finite sets with the same size, then $a_{I_0} \equiv_{\ker(X)} a_{J_0}$. Take an infinite set $J \subseteq I$ such that $I_0, J_0 < J$, if necessarily, extend X . As $a_{I_0} \equiv_{\text{acl}^{\text{eq}}(a_J)} a_{J_0}$, we see the conclusion.

(4): Let $C \subset \text{acl}(X)$ be such that X is indiscernible over C . Let $c \in C$. Then there exists a finite I_1 such that $c \in \text{acl}^{\text{eq}}(a_{I_1})$. For any $I_0 < I_1$, $|I_0| = |I_1|$, we have $c \in \text{acl}^{\text{eq}}(a_{I_0}) \cap \text{acl}^{\text{eq}}(a_{I_1})$, since $a_{I_0} \equiv_c a_{I_1}$. Now, we see that $C \subseteq \ker(X)$. $\square$

From now on, we work in a simple theory T with EHI.

Lemma 3.3. *Let $X = (a_i : i \in I)$ be a B -indiscernible sequence and $A \subseteq B$.*

- (1) *If X is sufficiently long and independent over A , then $X \downarrow_A B$.*
- (2) *If X is sufficiently long, then $\text{Cb}(B/(a_i : i \in I)A) \subseteq \ker_A(X)$.*
- (3) *If X is a Morley sequence over B , then $\ker_A(X) \subseteq \text{acl}^{\text{eq}}(B)$.*

Proof. (1): By simplicity, take $B_0 \subseteq a_{<|T|+}$ such that $B \downarrow_{B_0} a_{<|T|+}$ and $|B_0| \leq |T|$. So there exists $\lambda < |T|^+$ such that $B_0 \subseteq a_{<\lambda}$. We have $a_{<|T|+} \downarrow_{a_{<\lambda}} B$. By B -indiscernibility of X , we have $a_{\geq \lambda} \downarrow_{a_{<\lambda}} B$. So, $a_{\geq \lambda} \downarrow_{a_{<\lambda}A} B$. As X is independent over A , $a_{\geq \lambda} \downarrow_A a_{<\lambda}B$ follows. By A -independence of X again, we see the conclusion.

(2): Let $I_0 \subseteq I$ be such that $|I_0| = |T|^+$. Then there exists $B_0 \subseteq a_{I_0}$ such that $B \downarrow_{B_0} a_{I_0}$ and $|B_0| \leq |T|$. As there exists $\lambda \in I_0$ such that $B_0 \subseteq a_{<\lambda}$, we see $a_{I_0} \downarrow_{a_{<\lambda}} B$. By B -indiscernibility and finite character, we have $a_{\geq \lambda} \downarrow_{a_{<\lambda}} B$. Therefore we have $a_I \downarrow_{a_{I_0}A} B$. As we assume EHI, $\text{Cb}(B/Aa_I) \subseteq \text{acl}^{\text{eq}}(a_{I_0}A)$.

Let I_1 be such that $I_0 < I_1$ and $|I_1| = |T|^+$. By the same argument, we see $\text{Cb}(B/Aa_I) \subseteq \text{acl}^{\text{eq}}(a_{I_0}A) \cap \text{acl}^{\text{eq}}(a_{I_1}A) = \ker_A(X)$.

(3): By our statement, we may assume X is sufficiently long. By 3.2 (4), we have $\ker_A(X)B \subseteq \ker_B(X)$. So, X is $\ker_A(X)B$ -indiscernible and independent over B . By (1), we see $X \downarrow_B \ker_A(X)$. Since $\ker_A(X) \subseteq \text{acl}(BX)$, we see $\ker_A(X) \subseteq \text{acl}(B)$. $\square$

Proposition 3.4. *Let $X = (a_i : i \in I)$ be an A -indiscernible sequence.*

- (1) *If X is algebraically independent over A , then X is a Morley sequence over A .*
- (2) *If X is a Morley sequence over A , then $\ker(X) = \text{acl}^{\text{eq}}(\text{Cb}(a_0/A))$.*

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Proof. (1): By our assumption, we may assume X is sufficiently long. Let a_∞ be such that a_I, a_∞ is an extended A -indiscernible sequence, algebraically independent over A . As X is Aa_∞ -indiscernible and algebraically independent over A , by Lemma 3.3 (1), $\text{Cb}(Aa_\infty/AX) \subseteq \ker_A(X) = \text{acl}^{\text{eq}}(A)$. Therefore $a_\infty \perp_A a_I$. By A -indiscernibility of $a_I a_\infty$, we see X is independent over A .
 (2): As X is algebraically independent over $\text{Cb}(a_0/A)$, we see $\ker(X) \subseteq \text{acl}^{\text{eq}}(\text{Cb}(a_0/A))$ by Lemma 3.2 (2). By Lemma 3.3 (3), $\ker(X) \subseteq \text{acl}^{\text{eq}}(A)$. As X is $\ker(X)$ -indiscernible and algebraically independent over $\ker(X)$, X is a Morley sequence over $\ker(X)$ by (1). Now, by Lemma 3.3 (1), we have $X \perp_{\ker(X)} A$. In particular, $a_0 \perp_{\ker(X)} A$ holds. So, we see $\text{Cb}(a_0/A) \subseteq \text{acl}^{\text{eq}}(\ker(X))$. $\square$

FACT 1.1: *Let T be a simple theory having EHI such that T^- also has EHI, where T^- be a reduct of T . Let $a, C \subset (\mathcal{M}^-)^{\text{eq}}$ and $B \subset \mathcal{M}^{\text{eq}}$. If $a \perp_B C$, then $a \perp_{B^-}^- C$, where $B^- = \text{ACL}^{\text{eq}}(B) \cap (\mathcal{M}^-)^{\text{eq}}$ and $\perp^-$ is the non-forking relation in T^- .*

Proof. Let $X = (a_i : i \in \mathbb{Z})$ be a Morley sequence of $\text{TP}(a/BC)$. Then, by Proposition 3.4 (2) and our assumption, we have $\text{ACL}^{\text{eq}}(a_{<0}) \cap \text{ACL}^{\text{eq}}(a_{>0}) = \ker(X) = \text{ACL}^{\text{eq}}(\text{Cb}(a/BC)) \subseteq \text{ACL}^{\text{eq}}(B)$. So, $\text{acl}^{\text{eq}}(a_{<0}) \cap \text{acl}^{\text{eq}}(a_{>0}) \subseteq B^-$. As X is algebraically independent over BC , so is over B^-C in T^- . Since X is B^-C -indiscernible in T^- , by Proposition 3.4 (1), X is a Morley sequence of $\text{tp}(a/B^-C)$. By $\ker^-(X) \subseteq B^-$ and Proposition 3.4 (2), we see $a \perp_{B^-}^- C$. $\square$

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