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INVARIANT DISTANCES AND METRICS IN $\mathbb{C}^n$

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It is well known that Kobayashi distance and Caratheodory distance are important in complex analysis. It is because of that these distances have decreasing property under holomorphic mappings. For example the little Picard theorem which asserts that nonconstant entire function has at most one exceptional value is proved by the fact that Kobayashi distance has decreasing property and Kobayashi distance on $\mathbb{C}^n$ is identically zero and $\mathbb{C}^n$-$\{a,b\}$ is Kobayashi hyperbolic.

In this paper mainly according to Jarnicki-Pflug [4], we survey distances and metrics on a domain in $\mathbb{C}^n$. First we deal with general theory of pseudodistance and pseudometric. Let G be a domain in $\mathbb{C}^n$. Let $d_G : G\times G \longrightarrow [0,\infty)$ be a pseudodistance .

For $z', z''\in$G, *let* α:[0,1]$\longrightarrow G$ be a curve such that $\alpha(0) = z', \alpha(1) = z''$.*Put* $L_d(\alpha) = \sup\sum_{j=1}^{n} d(\alpha(t_{j-1}), \alpha(t_j))$,where supremum is taken over all partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of [0,1]. We define $d^i(z', z'') = \inf L_d(\alpha)$ for $z', z'' \in G$, where infimum is taken over all curve α *joining* z$'$ and z''. By definition we have $d(z', z'') \le d^i(z', z'')$ for $z', z'' \in G$.

Proposition 1. $(d^i)^i = d^i$.

Proof. $d^i \le (d^i)^i$ is trivial. It is sufficient to prove $(d^i)^i \ge d^i$. We have

$$d^i(\alpha(t_{j-1}), \alpha(t_j)) \le L_d(\alpha \mid [t_{j-1}, t_j]).$$

Therefore

$$\sum_{j=1}^{n} d^i(\alpha(t_{j-1}), \alpha(t_j)) \le \sum_{j=1}^{n} L_d(\alpha \mid [t_{j-1} - t_j]) = L_d(\alpha).$$

By taking supremum over all partition of [0,1], we have $L_{d^i}(\alpha) \le L_d(\alpha)$. Moreover taking infimum over all curve α *such that* $\alpha(0) = z'$ and $\alpha(1) = z''$, we have $(d^i)^i(z', z'') \le d^i(z', z'')$. $\square$

d is called inner if $d^i = d$ holds. By Proposition 1, d^i is inner.

Let $\delta : G\times\mathbb{C}^n \longrightarrow [0,\infty)$ be a pseudometric. Let $\alpha : [0,1] \to G$ be a piecewise C^1*curve. Put* $L_\delta(\alpha) = \int_0^1 \delta(\alpha(t); \alpha'(t))dt$.

Define

$$(\int \delta)(z', z'') = \inf L_\delta(\alpha),$$

where infimum is taken over all piecewise C^1*curve* α *joining* z$'$ and z''.

Proposition 2. $(\int \delta)^i = \int \delta$.

Proof. $(\int \delta) \leq (\int \delta)^i$ is trivial. It is sufficient to show $(\int \delta)^i \leq \int \delta$. By definition $(\int \delta)(\alpha(t_{j-1}), \alpha(t_j)) \leq L_\delta(\alpha \mid [t_{j-1}, t_j])$. Then

$$\sum_{j=1}^{n} (\int \delta)(\alpha(t_{j-1}), \alpha(t_j)) \leq \sum_{j=1}^{n} L_\delta(\alpha \mid [t_{j-1}, t_j]) = L_\delta(\alpha).$$

Taking supremum over all partition of [0,1], we have $L_{(\int \delta)}(\alpha) \leq L_\delta(\alpha)$. Moreover taking infimum over all piecewise C^1 curve α *joining* z' and z'', we have

$$(\int \delta)^i(z', z'') \leq (\int \delta)(z', z''). \quad \square$$

Let $d : G \times G \longrightarrow [0, \infty)$ be a pseudodistance. We define pseudometric Dd as follows:

$$(Dd)(a; X) = \lim_{\lambda \to 0} \sup_{z \to a} d(z, z + \lambda X).$$

for $a, z \in G$, $X \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. Dd is called the derivative of d.

Proposition 3. $d \leq \int (Dd)$.

Proof. Let α be a piecewise C^1 curve joining $z' = \alpha(0)$ and $z'' = \alpha(1)$. We show

$$\alpha(z', z'') \leq \int_0^1 (Dd)(\alpha(t), \alpha'(t))dt$$

for $z', z'' \in G$. Put $\varphi(t) = d(\alpha(0), \alpha(t))$, $\psi(t) = (Dd)(\alpha(t), \alpha'(t)), 0 \leq t \leq 1$. It is sufficient to show $\varphi(1) \leq \int_0^1 \psi(t)dt$. Since $\varphi(1) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)dt$, We show

$$\lim_{t', t'' \to t} \sup_{t' \neq t''} \left| \frac{\varphi(t') - \varphi(t'')}{t' - t''} \right| \leq \psi(t)$$

for $t \in [0, 1]$. Now we fix $t_0 \in [0, 1]$. Let $X(t', t'') = \frac{\alpha(t'') - \alpha(t')}{t'' - t'}$ *for* $t', t'' \in [0, 1]$, then

$$\lim_{t', t'' \to t_0} \sup_{t' \neq t''} X(t', t'') = \lim_{t', t'' \to t_0} \sup_{t' \neq t''} \frac{\alpha(t'') - \alpha(t')}{t'' - t'} = \alpha'(t_0).$$

We have

$$\begin{aligned}
\lim_{t', t'' \to t_0} \sup_{t' \neq t''} \left| \frac{\varphi(t') - \varphi(t'')}{t' - t''} \right| &= \lim_{t', t'' \to t_0} \sup_{t' \neq t''} \left| \frac{d(\alpha(0), \alpha(t')) - d(\alpha(0), \alpha(t''))}{t' - t''} \right| \\
&\leq \lim_{t', t'' \to t_0} \sup_{t' \neq t''} \frac{d(\alpha(t'), \alpha(t''))}{|t' - t''|} \\
&= \lim_{t', t'' \to t_0} \sup_{t' \neq t''} \frac{d(\alpha(t'), \alpha(t') + (t'' - t')X(t', t''))}{|t' - t''|} \\
&= (Dd)(\alpha(t_0); \alpha'(t_0)) \\
&= \psi(t_0). \quad \square
\end{aligned}$$

Proposition 4.

$$\int(Dd) \geq d^i.$$

Proof. From Proposition 3, we have $\int(Dd) \geq d$. So $(\int(Dd))^i \geq d^i$. From Proposition 2, we have $(\int(Dd))^i = \int(Dd)$. So $\int(Dd) \geq d^i$ holds. □

Proposition 5.

$$D(\int \delta) \leq \delta.$$

Proof. We have

$$\begin{aligned} D(\int \delta)(a;X) &= \lim_{\lambda\to 0}\sup_{z\to a} \frac{1}{|\lambda|}(\int \delta)(z, z+\lambda X) \\ &\leq \lim_{\lambda\to 0}\sup_{z\to a} \frac{1}{|\lambda|} L_\delta([z, z+\lambda X]) \\ &= \lim_{\lambda\to 0}\sup_{z\to a} \frac{1}{|\lambda|} \int_0^1 \delta(z+t\lambda X; \lambda X)dt \\ &= \lim_{\lambda\to 0}\sup_{z\to a} \int_0^1 \delta(z+t\lambda X; X)dt \\ &\leq \delta(a;X) \end{aligned}$$

,where $[z, z+\lambda X]$ denotes the segment joining z and z+$\lambda X, i.e.,\alpha(t) = (1-t)z + t(z+\lambda X) = z + tX, t \in [0,1]$. □

Proposition 6.

$$D(d^i) = Dd.$$

Proof. Since $d \leq d^i$, we have $Dd \leq Dd^i$. From Proposition 4 and 5, we have $Dd^i \leq D(\int Dd) \leq Dd$. □

Proposition 7. *Let G_1 and G_2 be domain in $\mathbb{C}^n$. Let $F : G_1 \longrightarrow G_2$ be a holomorphic mapping. If*

$$d_{G_2}(F(z'), F(z'')) \leq d_{G_1}(z', z''),$$

then

$$d^i_{G_2}(F(z'), F(z'')) \leq d^i_{G_1}(z', z'') \ for\ z', z'' \in G_1.$$

Proof. Let $\alpha : [0,1] \longrightarrow G_1$ be a curve joining z' and z''. From the assumption, we have

$$d_{G_2}(F\circ\alpha(t_{j-1}), F\circ\alpha(t_j)) \leq d_{G_1}(\alpha(t_{j-1}), \alpha(t_j)).$$

Taking summation from $j = 1$ to $j = n$ and taking supremum over all partition 0=t$_0$ $\leq t_1 \leq \cdots \leq t_n = 1$, we have $L_{d_{G_2}}(F\circ\alpha) \leq L_{d_{G_1}}(\alpha)$. Moreover taking infimum over all curve α *joining* z$'$ and z'', we have $d^i_{G_2}(F(z'), F(z'')) \leq d^i_{G_1}(z', z'')$. □

Proposition 8. *Let $F : G_1 \longrightarrow G_2$ be a holomorphic mapping. Let $\alpha : [0,1] \longrightarrow G_1$ be a piecewise C^1 curve joining z' and z''. If*

$$\delta_{G_2}(F(\alpha(t)); F'(\alpha(t))\alpha'(t)) \leq \delta_{G_1}(\alpha(t); \alpha'(t)),$$

then

$$(\int \delta_{G_2})(F(z'), F(z'')) \leq (\int \delta_{G_1})(z', z'').$$

Proof. From the assumption, we have

$$L_{\delta_{G_2}}(F \circ \alpha) \leq L_{\delta_{G_1}}(\alpha).$$

Taking infimum over all α, we conclude the proof. □

In Proposition 7 and 8, if $F : G_1 \longrightarrow G_2$ is a biholomorphic mapping, equation holds respectively.

Next, we recall the definition of Caratheodory pseudodistance, Caratheodory-Reiffen pseudometric, Kobayashi pseudodistance and Kobayashi-Royden pseudometric. Let Δ be a unit disc in $\mathbb{C}^n$. Let G be a domain in $\mathbb{C}^n$. $Hol(G, \Delta)$ and $Hol(\Delta, G)$ denote the set of holomorphic mappings from G to Δ and from Δ to G respectively.

Definition 1.

$$\begin{aligned} c_G(z', z'') &= \sup\{\varrho(f(z'), f(z'')) \mid f \in Hol(G, \Delta)\} \\ &= \sup\{\varrho(0, f(z'')) \mid f \in Hol(G, \Delta), f(z') = 0\} \end{aligned}$$

for z', $z'' \in G$, where ϱ is the Poincare distance in Δ.

$c_G(z', z'')$ is called Caratheodory pseudodistance.

Definition 2.

$$\begin{aligned} \gamma_G(z; X) &= \sup\{\gamma(f(z))|f'(z)X| \mid f \in Hol(G, \Delta)\} \\ &= \sup\{|f'(z)X| \mid f \in Hol(G, \Delta), f(z) = 0\} \end{aligned}$$

for $z \in G, X \in C^n$, where

$$\gamma(f(z)) = \frac{1}{1 - |f(z)|^2},$$

$$f'(z)X = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)X_j.$$

γ_G is called Caratheodory-Reiffen pseudometric.

Definition 3.

$$k_G^{\sim}(z, w) = \inf\{\varrho(\lambda, \mu)|\lambda, \mu \in \Delta, \exists\varphi \in Hol(\Delta, G), \varphi(\lambda) = z, \varphi(\mu) = w\}$$

for z,w $\in G$.

$k_G^{\sim}$ does not satisfy triangle inequality. So $k_G^{\sim}$ is not pseudodistance. $k_G^{\sim}$ is called Lempert function.

Definition 4.

$$k_G(z,w) = \inf\{\sum_{i=1}^{n} k_G^{\sim}(z_{i-1}, z_i) \mid z = z_0, z_1, \ldots, z_n = w\}.$$

$k_G(z,w)$ is called Kobayashi pseudodistance.

Definition 5.

$$\kappa_G(z;X) = \inf\{\gamma(\lambda)|c| | \exists\varphi \in Hol(\Delta, G), \exists\lambda \in \Delta; \varphi(\lambda) = z, c\varphi'(\lambda) = X\}$$
$$= \inf\{c > 0 | \exists\varphi \in Hol(\Delta, G); \varphi(0) = z, \alpha\varphi'(0) = X\}$$

for $z \in G, X \in C^n$

$\kappa_G : G \times \mathbb{C}^n \longrightarrow [0,\infty)$ is called Kobayashi-Royden pseudometric. We give some propositions without proof.

Proposition 9. *Let* G_1 *and* G_2 *be domain in* $\mathbb{C}^n$*. If* $f : G_1 \to G_2$ *is a holomorphic mapping, then*

$$k_{G_2}(f(z), f(w)) \leq k_{G_1}(z,w),$$
$$c_{G_2}(f(z), f(w))) \leq c_{G_1}(z,w)$$

for $z, w \in G_1$.

In particular,if f is biholomorphic, then equalities hold respectively.

Proposition 10.

$$k_{\mathbb{C}} \equiv 0$$
$$c_{\mathbb{C}} \equiv 0$$

Proposition 11 ([2]).

$$k_G^i = k_G$$
$$c_G^i \neq c_G$$

Proposition 12.

$$c_G \leq k_G$$
$$\gamma_G \leq \kappa_G$$

for any domain G *in* $\mathbb{C}^n$

Proposition 13. *Let* $f : G_1 \to G_2$ *be a holomorphic mapping. Then*

$$\kappa_{G_2}(f(z); f'(z)X) \leq \kappa_{G_1}(z;X)$$
$$\gamma_{G_2}(f(z); f'(z)X) \leq \gamma_{G_1}(z;X)$$

for $z \in G_1, X \in C^n$.

Proposition 14 ([1]). *Let* G *be a domain in* $\mathbb{C}^n$*. Topology on* G *induced by* k_G *coincides with the standard euclidean topology of* G.

Proposition 15 ([3]). *In the case of* $n \geq 3$*, there exists a domain* $G \subset C^n$ *whose standard euclidean topology does not coincide with the induced topology by* c_G.

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