

Basic sequences of torsion free graded modules

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Let $R := k[x_1, \dots, x_r]$ denote a polynomial ring in r indeterminates over an infinite field k of arbitrary characteristic, $\mathfrak{m} := (x_1, \dots, x_r)$ its maximal ideal, and E a finitely generated graded R -module. All modules treated here are graded.

Let us recall first the main results of [7].

Theorem 1 ([7, Corollary 2.6, Theorem 2.11]). *There exist a finitely generated graded $k[x_i, \dots, x_r]$ -submodule $E^{[i]} \subset E$ and a finitely generated graded free $k[x_i, \dots, x_r]$ -submodule $E^{(i)} \subset E$ for each $i = 1, \dots, r+1$ such that*

$$(1.1) \quad E^{[1]} = E, \quad E^{[r+1]} = E^{(r+1)},$$

$$(1.2) \quad E^{[i]} = E^{(i)} \oplus E^{[i+1]} \text{ as } k[x_{i+1}, \dots, x_r]\text{-module, and}$$

$$(1.3) \quad x_i E^{[i+1]} \subset (x_{i+1}, \dots, x_r) E^{(i)} \oplus E^{[i+1]}$$

for all $i = 1, \dots, r$, if and only if

$$(1.4) \quad (x_r, \dots, x_{i+1})E :_E x_i \subset (x_r, \dots, x_{i+1})E :_E \langle \mathfrak{m} \rangle$$

for all $i = 1, \dots, r$, where

$$Z :_E \langle \mathfrak{m} \rangle = \{ e \in E \mid \mathfrak{m}^t e \subset Z \text{ for some } t \in \mathbf{N} \}.$$

If the submodules as above exist, denoting homogeneous free bases of $E^{(i)}$ by e_l^i ($1 \leq l \leq m_i$), we call $W := \{ e_l^i \mid 1 \leq i \leq r+1, 1 \leq l \leq m_i \}$ a *weak Weierstrass basis* of E with respect to $x_1, \dots, x_r$.

Remark 2. (1) If the condition (1.4) is satisfied, we say that $x_r, \dots, x_1$ form a filter-regular sequence with respect to E (see [19, Appendix:Definition 1]).

(2) For fixed $x_1, \dots, x_r$, the structures of $E^{(i)}$ and $E^{[i]}$ are uniquely determined up to isomorphism over $k[x_i, \dots, x_r]$ for each $i = 1, \dots, r+1$ by the conditions (1.1), (1.2) and (1.3).

(3) By (1.2),

$$(2.1) \quad E^{[i]} = E^{(i)} \oplus E^{(i+1)} \oplus \dots \oplus E^{[r+1]}$$

as k -vector space.

Remark 3. (1) If E satisfies $l_R(H_m^i(E)) < \infty$ for all $i < r$, then the sequence $x_1, \dots, x_r$ is always filter-regular with respect to E (see [19, Appendix: Proposition 16]).

(2) If the variables $x_1, \dots, x_r$ are chosen sufficiently generally regarding E , then $x_1, \dots, x_r$ form a filter-regular sequence with respect to E .

Example 4. Let $E = R/(x_1, \dots, x_{j-1})$ and let

$$\begin{aligned} E^{(1)} &= \dots = E^{(j-1)} = 0, & E^{(j+1)} &= \dots = E^{(r+1)} = 0, \\ E^{(j)} &= k[x_j, \dots, x_r]e_1^j \quad \text{with} \quad e_1^j = 1, \\ E^{[i]} &= 0 \quad (j+1 \leq i \leq r+1), & E^{[i]} &= E \quad (1 \leq i \leq j). \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. It is clear that $E = R \cdot 1 = k[x_j, \dots, x_r] \cdot 1$, since $x_i = 0$ in E for $i = 1, \dots, j-1$. Hence (1.1) and (1.2) hold. Verification of (1.3) is easy. $\square$

Example 5. Let $r := 3$ and $E := \text{Syz}_2^R(R/\mathfrak{m})$. Then E is the image of the matrix

$$\begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}.$$

Denote the first column by e_1^1 , the second by e_2^1 , and the third by e_1^2 . Set

$$\begin{aligned} E^{(1)} &= Re_1^1 \oplus Re_2^1, & E^{(2)} &= k[x_2, x_3]e_1^2, & E^{(3)} &= E^{(4)} = 0, \\ E^{[1]} &= E, & E^{[2]} &= E^{(2)}, & E^{[3]} &= E^{[4]} = 0. \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. The condition (1.2) follows from e.g. [7, Lemma 1.1]. Since $x_3e_1^1 - x_2e_2^1 + x_1e_1^2 = 0$, we find $x_1e_1^2 = -x_3e_1^1 + x_2e_2^1 \in (x_2, x_3)E^{(1)}$. Hence (1.3) holds. This implies on the other hand that $E = E^{[1]}$ by [7, Lemma 2.7]. Thus (1.1) – (1.3) are satisfied. $\square$

Example 6. Let $r := 2$ and $E := (x_1^2, x_2^2) \subset R = k[x_1, x_2]$. Let further

$$\begin{aligned} e_1^1 &= x_1^2, & e_1^2 &= x_2^2, & e_2^2 &= x_1x_2^2, \\ E^{(1)} &= Re_1^1, & E^{(2)} &= k[x_2]e_1^2 \oplus k[x_2]e_2^2, & E^{(3)} &= 0, \\ E^{[1]} &= E, & E^{[2]} &= E^{(2)}, & E^{[3]} &= 0. \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. It is easy to verify (1.1) and (1.2). Since $x_1e_1^2 = x_1x_2^2 = 1 \cdot e_2^2 \in E^{[2]}$ and $x_1e_2^2 = x_1^2x_2^2 = x_2^2e_1^1 \in (x_2)E^{(1)}$, the condition (1.3) holds, too. $\square$

Example 7. Let $r := 4$ and $E := (x_1^2, x_1x_2, x_2^2, x_1x_3 - x_2x_4) \subset R = k[x_1, x_2, x_3, x_4]$. Let further

$$\begin{aligned} e_1^1 &= x_1^2, \quad e_1^2 = x_1x_2, \quad e_2^2 = x_2^2, \quad e_1^3 = x_1x_3 - x_2x_4, \\ E^{(1)} &= Re_1^1, \quad E^{(2)} = k[x_2, x_3, x_4]e_1^2 \oplus k[x_2, x_3, x_4]e_2^2, \quad E^{(3)} = k[x_3, x_4]e_1^3, \\ E^{[1]} &= E, \quad E^{[2]} = E^{(2)} \oplus E^{(3)}, \quad E^{[3]} = E^{(3)}, \quad E^{(4)} = E^{[4]} = 0, \quad E^{(5)} = E^{[5]} = 0. \end{aligned}$$

Then these submodules satisfy (1.1) – (1.3).

Proof. It is easy to verify (1.1) and (1.2). Since $x_1e_1^2 = x_2e_1^1 \in (x_2)E^{(1)}$, $x_1e_2^2 = x_2e_1^2 \in E^{[2]}$, $x_1e_1^3 = x_3e_1^1 - x_4e_2^1 \in (x_3)E^{(1)} \oplus E^{[2]}$, and $x_2e_2^3 = x_3e_1^2 - x_4e_2^2 \in (x_3, x_4)E^{(2)}$ the condition (1.3) holds, too. $\square$

Definition 8. Choose a sufficiently general set of variables $x_1, \dots, x_r$. Let $\{e_i^l \mid 1 \leq i \leq r+1, 1 \leq l \leq m_i\}$ be a weak Weierstrass basis of E with respect to $x_1, \dots, x_r$. We define the *basic sequence* $B_R(E)$ of E to be the sequence $(\bar{n}^1; \bar{n}^2; \dots; \bar{n}^{r+1})$ made of the nondecreasing sequences of integers $\bar{n}^i$ ($1 \leq i \leq r+1$) such that $\bar{n}^i = (\deg(e_1^i), \dots, \deg(e_{m_i}^i))$ up to permutation. In case $E^{(i)} = 0$, then $m_i = 0$ and $\bar{n}^i = \emptyset$.

Remark 9. (1) $\text{depth}_m(E) = r + 1 - \max\{i \mid E^{(i)} \neq 0\}$.

(2) $\bar{n}^1 = (n_1^1, \dots, n_{m_1}^1)$ with $m_1 = \text{rank}_R(E)$

(3) If $E \subset R$, then $m_1 = 1$ and $n_1^1 = \min\{d \mid [E]_d \neq 0\}$. Further $n_1^1 = m_2$ if $\text{ht}(E) \geq 2$.

The notion of basic sequence was established first in [2] for the case of homogeneous ideals defining space curves and was applied to the study of arithmetically Buchsbaum curves in $\mathbf{P}^3$. It was extended to homogeneous ideals in polynomial rings of arbitrary number of variables in [6]. Later, for the purpose of generalizing a result on the structure of homogeneous ideals defining graded Buchsbaum rings obtained in [6], we further extended it to arbitrary finitely generated graded modules over polynomial rings.

One of the most difficult and important problems concerning basic sequence is to give a characterization of those of the homogeneous prime ideals. We have only few results so far in this direction, among which we think the following generalization of Gruson-Peskine's connectedness theorem (see [16]) a good one.

Theorem 10 (cf. [3, Corollary 1.2]). Let $I \subset R$ be a homogeneous prime ideal of height larger than or equal to two and let $(\bar{n}^1; \dots; \bar{n}^{r+1})$ its basic sequence. Then $n_l^2 \leq n_{l+1}^2 \leq n_l^2 + 1$ for all $l = 1, \dots, m_2 - 1$ (note that $m_2 = n_1^1$).

Proof. You will find a proof for the case $r = 4$ in [3, Section 1]. Reading it carefully, you will be convinced that the assertion is true for arbitrary k and $r \geq 2$. See [14] for another method. $\square$

There are a lot of other applications of basic sequences to the study homogeneous ideals defining curves in $\mathbf{P}^3$. For them, see [2] – [5].

We pass on to the next topic we are most interested in now. Let $p \geq 2$ and let M be a finitely generated torsion-free graded R -module with no free direct summand satisfying $\text{Ext}_R^i(M, R) = 0$ for $i = 1, \dots, p-1$. Let further $\mathfrak{I}(M, p)$ be the set of all homogeneous ideals I in R of height p fitting into exact sequences of the form

$$0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0.$$

where c is an integer and S_i ($0 \leq i \leq p-1$) are finitely generated graded free R -modules. By this sequence one obtains

$$H_{\mathfrak{m}}^{i-1}(R/I)(c) \cong H_{\mathfrak{m}}^i(M) \quad \text{for } i = 1, \dots, \dim(R/I) = r - p.$$

Since $H_{\mathfrak{m}}^i(M) = 0$ for $i = r - p + 1, \dots, r - 1$ and $i = 0$ by local duality, considering the local cohomologies of R/I is the same thing as considering those of M .

Problem 11. Describe $B_R(I)$ for all $I \in \mathfrak{I}(M, p)$.

Our first result is this.

Theorem 12 ([9, Theorem 3]). *For all M and p as above, the set $\mathfrak{I}(M, p)$ is not empty.*

Though it may not be explicit in [2], the argument of [2] and [6] on the structure of homogeneous ideals defining graded Buchsbaum rings was based wholly on the comparison of $I^{[p]}$ and $M^{[p]}$. This comparison theorem can be generalized in the following form.

Theorem 13 ([8, Theorem 2.3]). *Let $I \in \mathfrak{I}(M, p)$. If the variables $x_1, \dots, x_r$ satisfy (1.4) for both I and M then $I^{[p]} \cong C \oplus M^{[p]}(-c)$ as $k[x_p, \dots, x_r]$ -module with a finitely generated graded free $k[x_p, \dots, x_r]$ -module C .*

Corollary 14 ([8, Corollary 2.4]). *Let $I \in \mathfrak{I}(M, p)$, $B_R(I) = (\bar{n}^1; \bar{n}^2; \dots; \bar{n}^{r+1})$, and $B_R(M) = (\bar{\gamma}^1; \bar{\gamma}^2; \dots; \bar{\gamma}^{r+1})$. Then we have*

$$\begin{cases} \bar{n}^p = (\bar{w}', \bar{\gamma}^p + c) & \text{up to permutation and} \\ \bar{n}^i = \bar{\gamma}^i + c & \text{for } i = p+1, \dots, r+1 \end{cases}$$

with a suitable sequence of integers $\bar{w}'$, where $\bar{\nu} + c = (\nu_1 + c, \dots, \nu_l + c)$ for a sequence $\bar{\nu} = (\nu_1, \dots, \nu_l)$.

We have some results obtained as applications of the above theorem. They are formulated mainly in terms of C and $\bar{w}'$.

- An answer to Problem 11 for the case $p = 2$ (see [10] and [11]).

- An answer to Problem 11 for the case M is Buchsbaum (see [6, Sections 5 and 6]).
- Almost complete description of $B_R(I)$ for I defining graded Buchsbaum integral domains of codimension two (see [5] and [6, Section 7]).
- Some description of $B_R(I)$ for I defining graded integral domains of codimension two (see [17] and [11]).

Finally we explain the computational aspect of (weak) Weierstrass bases and basic sequences in the case where E is a submodule of a graded free module, namely, the case where E is torsion-free. If $\text{rank}_R(E) = 1$, then essentially E is a homogeneous ideal and weak Weierstrass bases can be obtained with the use of generalized Weierstrass preparation theorem due to Hironaka or Grauert (see [18] and [15]). In fact, for generic coordinates, Gröbner bases with respect to reverse lexicographic order form Weierstrass bases if $\text{char}(k) = 0$ and $\text{rank}_R(E) = 1$. We will make this point clear for the general torsion-free case below.

Let $\bar{a} = (a_1, \dots, a_s)$ be a sequence of integers. Suppose $E \subset R(-\bar{a}) := \bigoplus_{i=1}^s R(-a_i)$. Denote by v_i the free base $\overset{\circ}{(0, \dots, 0, 1, 0, \dots, 0)} \in R(-\bar{a})$ of degree a_i for each $i = 1, \dots, s$. Let $v = \sum_{i=1}^s f_i v_i$ be an element of $R(-\bar{a})$. We define the degree of v to be $\max\{\deg(f_i) + a_i \mid 1 \leq i \leq s\}$ and denote it by $\deg_{\bar{a}}(v)$. If every f_i is homogeneous and there is an integer b such that $\deg(f_i) + a_i = b$ for all i with $f_i \neq 0$, then we say that v is homogeneous of degree b . An element of $R(-\bar{a})$ of the form $f v_i$ ($1 \leq i \leq s$) with a monomial $f \in R$ in $x_1, \dots, x_r$ will be called a monomial of $R(-\bar{a})$ in $x_1, \dots, x_r$. An element of $R(-\bar{a})$ is homogeneous if and only if it is the linear combination of monomials of the same degree over k .

Let $<$ denote the reverse lexicographic order on the monomials of R in $x_1, \dots, x_r$. We denote by the same symbol $<$ the term order on the monomials of $R(-\bar{a})$ in $x_1, \dots, x_r$ such that

$$f v_i < g v_j \quad \text{if and only if} \quad \begin{cases} f < g & \text{or} \\ f = g & \text{and } i > j \end{cases}$$

(cf. [13, Definition 3.5.2]). Taking the gradation determined by $\deg_{\bar{a}}(\)$ into account, we further consider the term order $<_{\bar{a}}$ on the monomials of $R(-\bar{a})$ in $x_1, \dots, x_r$ such that

$$f v_i <_{\bar{a}} g v_j \quad \text{if and only if} \quad \begin{cases} \deg_{\bar{a}}(f v_i) < \deg_{\bar{a}}(g v_j) & \text{or} \\ \deg_{\bar{a}}(f v_i) = \deg_{\bar{a}}(g v_j) & \text{and } f v_i < g v_j. \end{cases}$$

The initial term of $v \in R(-\bar{a})$ with respect to $<_{\bar{a}}$ will be denoted by $\text{in}_{\bar{a}}(v)$.

As at the beginning the base field k is infinite of arbitrary characteristic.

Theorem 15 ([12]). Assume that $x_1, \dots, x_r$ are sufficiently general. Then there exists a set $W = \{ e_l^i \mid 1 \leq i \leq r, 1 \leq l \leq m_i \}$ of homogeneous generators of E such that $e_l^i \neq 0$ for all i, l which satisfies the following conditions.

$$(15.1) \quad E = \bigoplus_{i=1}^r E^{(i)} \quad \text{as } k\text{-module with } E^{(i)} := \bigoplus_{l=1}^{m_i} k[x_1, \dots, x_r] e_l^i,$$

$$(15.2) \quad x_{i'} e_l^j \in (x_{i'+1}, \dots, x_r) \left(\bigoplus_{l=1}^{m_{i'}} k[x_{i'+1}, \dots, x_r] e_l^{i'} \right) \oplus \left(\bigoplus_{i=i'+1}^r E^{(i)} \right)$$

for every triple i', j, l such that $1 \leq i' < j \leq r, 1 \leq l \leq m_{i'}$,

$$(15.3) \quad \text{the coefficient of } \text{in}_{\bar{a}}(e_l^i) \text{ is one for all } i, l,$$

$$(15.4) \quad \text{in}_{\bar{a}}(e_l^1) \in k[x_1]v_j \quad \text{for some } j = 1, \dots, s,$$

$$(15.5) \quad \text{in}_{\bar{a}}(e_l^i) \in k[x_1, \dots, x_i]x_i v_j \quad \text{for some } j = 1, \dots, s$$

for all $i = 2, \dots, r$,

$$(15.6) \quad \sum_{i=1}^r \sum_{l=1}^{m_i} g_l^i \text{in}_{\bar{a}}(e_l^i) = 0 \quad \text{with } g_l^i \in k[x_1, \dots, x_r] \quad (1 \leq i \leq r, 1 \leq l \leq m_i)$$

if and only if $g_l^i = 0$ for all i, l ,

$$(15.7) \quad e_l^i - \text{in}_{\bar{a}}(e_l^i) \notin \sum_{j=1}^r \sum_{l'=1}^{m_j} k[x_1, \dots, x_r] \text{in}_{\bar{a}}(e_{l'}^j) \quad \text{for all } i, l,$$

where $\bigoplus_{l=1}^m ()$ means that the sum $\sum_{l=1}^m ()$ is direct and we understand $\bigoplus_{l=1}^m () = 0$ if $m = 0$.

Corollary 16 ([12]). Let $E^{(i)}$ ($1 \leq i \leq r$) be as in the above theorem and $E^{(r+1)} := 0$. Let further $E^{[i]}$ be the subsets of E defined by the formula (2.1) for each $i = 1, \dots, r+1$. Then $E^{(i)}$ ($1 \leq i \leq r+1$) and $E^{[i]}$ ($1 \leq i \leq r+1$) satisfy the conditions (1.1) – (1.3).

Definition 17 ([12]). We call the system W of generators of E stated in Theorem 15 a perfect Weierstrass basis of E with respect to $x_1, \dots, x_r$.

Proposition 18 ([12]). Let W be a perfect Weierstrass basis of E with respect to $x_1, \dots, x_r$. Then the members of W form a Gröbner basis of E with respect to the term order $<_{\bar{a}}$. In particular, the basic sequence of E is a sequence consisting of the degrees of the members of a generic Gröbner basis with respect to the term order $<_{\bar{a}}$.

Remark 19. See [6, Section 3] for free resolutions starting with Weierstrass bases.

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