

Infinite Dimensional Chaotic Synchronization for Randomly Coupling Systems and Dynamical Phase Transition

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We rigorously investigate the macroscopic dynamical phase transitions and chaotic synchronization in infinite-dimensional networks governed by generalized Boole transformations under symmetric structural quenched disorder. By leveraging the heavy-tailed statistics of the chaotic mappings, we establish a self-consistent macroscopic renormalization group framework and show that the background fields asymptotically converge to an invariant Cauchy distribution characterized by a stable scale parameter γ^* . Through a single-node deviation analysis in the thermodynamic limit, combined with exact complex residue calculus, we analytically derive the conditional Lyapunov exponent in a parsimonious closed form: $\lambda_c = 2 \ln[\sqrt{\alpha} + \sqrt{1 - \alpha - KE[|\epsilon_{ij}|]}]$. This enables us to establish the exact, universal synchronization critical coupling threshold: $K_c = 2\sqrt{\alpha}(1 - \sqrt{\alpha})/E[|\epsilon_{ij}|]$. Crucially, our pure theory demonstrates that the macroscopic synchronization threshold is governed solely by the first absolute moment of the coupling weights, revealing a profound universality class that is fundamentally resilient against microscopic structural fluctuations. Furthermore, we show that this transition marks a coherent macroscopic condensation of infinite chaotic degrees of freedom into a single effective one-dimensional orbit, in close analogy with Bose-Einstein condensation (BEC).

INTRODUCTION

Chaotic synchronization in coupled nonlinear systems serves as a fundamental paradigm for collective behaviors in nature, ranging from neural networks to laser arrays [1]. While synchronization criteria for low-dimensional systems have recently been derived rigorously [2, 17], analyzing coupled chaotic maps in the thermodynamic limit ($N \rightarrow \infty$) under structural disordered coupling remains a profound challenge in non-equilibrium statistical mechanics [3, 4]. In real-world systems, interactions are inherently random and competitive, reminiscent of frustrated spin glasses [8, 10]. However, while conventional spin-glass theories heavily rely on equilibrium frameworks [9], dynamical networks operate far from equilibrium, continuously driven by deterministic chaos at each node. Understanding how macroscopic order emerges from such a “dynamical spin glass” requires a rigorous description of dynamical phase transitions under structural quenched disorder. Furthermore, deciphering how large-scale dynamical networks achieve optimal information processing with low energy consumption implies a robust mechanism that compresses macroscopic entropy while preserving microscopic chaotic complexity.

In this Letter, we provide an exact analytical solution for the dynamical phase transition in an infinite-dimensional randomly coupled map (GCM). By employing generalized Boole transformations [18, 19], $F(x) = \alpha(x - 1/x)$ for $0 < \alpha < 1$, as the nodal dynamics, we bypass the mathematical barriers of infinite-dimensional Perron-Frobenius operators. Utilizing the invariant properties of Cauchy stable distributions, we derive a closed-form self-consistent equation for the macroscopic scale parameter under arbitrary α [19, 21]. Crucially, our framework establishes a grand unification of coupled chaotic fields: the conventional uniformly coupled systems are naturally embedded as an idealized subset where the variance of structural disorder vanishes. Furthermore, at the maximum-chaos limit $\alpha = 1/2$, our general formulation seamlessly yields an analytical threshold for synchronization. This exact solution bridges microscopic random chaos and macroscopic entropy compression, offering a novel physical principle for the energy-efficient computing observed in high-dimensional chaotic fields, conceptually resonating with the macroscopic reduction theories of synchronization ensembles [5–7].

II. MODEL FRAMEWORK

We consider an ensemble of N chaotic elements whose local nodes are governed by the generalized Boole transformations [18, 19]. To investigate the exact mechanism of the dynamical phase transition under structural disorder, we consider the globally and randomly coupled map (GCM) framework first introduced by Umeno [20]. The governing

equation with random linear interactions is defined as follows:

$$x_i(t+1) = F(x_i(t)) + \frac{K}{N-1} \sum_{j \neq i}^N \epsilon_{ij} x_j(t), \quad (1)$$

where $x_i(t) \in \mathbb{R}$ is the state variable of the i -th element ($i = 1, \dots, N$) at discrete time t , and K represents the global coupling strength. The uncoupled local chaotic dynamics $F(x)$ is given by the generalized Boole transformation:

$$F(x) = \alpha \left(x - \frac{1}{x} \right), \quad (0 < \alpha < 1). \quad (2)$$

Here, ϵ_{ij} denotes the random coupling coefficient from the j -th to the i -th element. We assume that ϵ_{ij} are independent and identically distributed (i.i.d.) random variables symmetrically distributed with a finite expectation value of the absolute value, $E[|\epsilon_{ij}|] < \infty$, and a finite variance. This condition naturally models the quenched spatial disorder of the interactions—serving as a dynamical version of spin glasses [8–10]—while crucially allowing the system to smoothly reduce to a uniformly coupled symmetric network in the zero-variance limit.

In the infinite-dimensional limit ($N \rightarrow \infty$), the coupling term represents a randomized linear summation of infinite chaotic degrees of freedom. Due to the heavy-tailed property inherently induced by the local generalized Boole map $F(x)$, whose invariant measure is given by a Cauchy distribution, the distribution of $x_j(t)$ also asymptotically follows a Cauchy distribution by the stable property of the Cauchy distribution [11, 13]. By virtue of the generalized central limit theorem for chaotic processes [12, 19], the collective random inputs via $\epsilon_{ij}x_j(t)$ preserve the stable profile and robustly converge to a Cauchy limiting distribution. This profound algebraic property allows us to bypass the intractable complexity of infinite-dimensional Perron-Frobenius equations. Instead, the macroscopic behavior of the entire network can be completely reduced to a low-dimensional deterministic map governed solely by the scale parameter γ of the Cauchy limiting distribution as the fixed point of the renormalization group self-consistent equation. This provides a rigorous and exact analytical window into the macroscopic dynamical phase transition from chaotic desynchronization to global synchronization.

III. RENORMALIZATION GROUP SELF-CONSISTENT EQUATION

To elucidate the exact mechanism of the infinite-dimensional chaotic synchronization, we track the macroscopic dynamics of the network through the time evolution of the probability density function $\rho_i(x, t)$ for each element $x_i(t)$. Given that the local node dynamics are governed by the generalized Boole transformations, the distribution of each state variable asymptotically follows a Cauchy distribution by the generalized central limit theorem (GCLT) [19]. Let $\gamma_i(t)$ be the scale parameter of the Cauchy distribution for the i -th element at time t . By applying the stable properties of the Cauchy distribution to the randomized linear GCM equation, the addition of these heavy-tailed variables directly translates into a coupled dynamical system of the scale parameters. Consequently, we obtain a set of N -coupled deterministic equations for the individual scale parameters γ_i :

$$\gamma_i(t+1) = \alpha \left(\gamma_i(t) + \frac{1}{\gamma_i(t)} \right) + \frac{K}{N-1} \sum_{j \neq i}^N |\epsilon_{ij}| \gamma_j(t), \quad (3)$$

where the first term represents the transformation of the scale parameter $G(\gamma) = \alpha \left(\gamma + \frac{1}{\gamma} \right)$ under the uncoupled generalized Boole transformation $F(x)$ [21], and the second term accounts for the linear aggregation of the randomized interactions. Now, we take the infinite-dimensional limit ($N \rightarrow \infty$) to uncover the macroscopic phase. Under the fundamental assumption that the coupling coefficients ϵ_{ij} are independent and identically distributed (i.i.d.) random variables, we invoke the law of large numbers and the standard central limit theorem (CLT) for the interaction term. In this thermodynamic limit, the spatial disorder is self-averaging, meaning that the sum over the randomized network structure effectively flattens out the sample-dependent fluctuations of the individual nodes. Specifically, as $N \rightarrow \infty$, the summation term becomes structurally identical for all elements, as the quenched randomness converges to its expected value:

$$\frac{1}{N-1} \sum_{j \neq i}^N |\epsilon_{ij}| \gamma_j(t) \longrightarrow E[|\epsilon_{ij}|] \cdot \bar{\gamma}(t), \quad (4)$$

where $\bar{\gamma}(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \gamma_j(t)$ is the mean scale parameter of the entire ensemble. This statistical unification implies that the interaction field exerts a uniform macroscopic pressure on every single node, regardless of its specific local connection topology. Therefore, all individual scale parameters asymptotically collapse into a single, universal scale parameter, $\gamma_i(t) \rightarrow \gamma(t)$, for all i . As a glorious consequence of this statistical reduction, the N -coupled intractable equations collapse beautifully into a single, closed, one-dimensional deterministic map for the universal scale parameter $\gamma(t)$:

$$\gamma(t+1) = \alpha \left(\gamma(t) + \frac{1}{\gamma(t)} \right) + KE[|\epsilon_{ij}|] \gamma(t). \quad (5)$$

This is the exact renormalization group self-consistent equation that dictates the macroscopic dynamical phase transition of the system. To explicitly determine the steady-state macroscopic profile of this infinite-dimensional disordered network, we solve the algebraic equation for the fixed point by setting $\gamma(t+1) = \gamma(t) = \gamma^*$ in Eq. (3). This directly leads to the following self-consistent algebraic equation:

$$\gamma^* = \alpha \left(\gamma^* + \frac{1}{\gamma^*} \right) + KE[|\epsilon_{ij}|] \gamma^*. \quad (6)$$

By dividing both sides by γ^* (under the physical requirement that the scale parameter must be strictly positive, $\gamma^* > 0$), we isolate the term originating from the local generalized Boole transformation:

$$1 = \alpha \left(1 + \frac{1}{(\gamma^*)^2} \right) + KE[|\epsilon_{ij}|]. \quad (7)$$

Rearranging the terms to group the macroscopic parameters (α, K) together yields:

$$1 - \alpha - KE[|\epsilon_{ij}|] = \frac{\alpha}{(\gamma^*)^2}. \quad (8)$$

Thus, we obtain the explicit closed-form expression for the square of the universal scale parameter:

$$(\gamma^*)^2 = \frac{\alpha}{1 - \alpha - KE[|\epsilon_{ij}|]}. \quad (9)$$

Since γ^* is fundamentally a scale parameter of a probability density function, it must be a real and positive value ($\gamma^* \in \mathbb{R}^+$). Therefore, a non-trivial, physically stable macroscopic phase exists if and only if the denominator is strictly positive, $1 - \alpha - KE[|\epsilon_{ij}|] > 0$.

Taking the positive root, the universal scale parameter γ^* is explicitly and uniquely determined as:

$$\gamma^* = \sqrt{\frac{\alpha}{1 - \alpha - KE[|\epsilon_{ij}|]}}. \quad (10)$$

Equation (8) is one of the central analytical breakthroughs of this Letter. It establishes that the infinite-dimensional macroscopic state under quenched spatial disorder is governed by a simple algebraic balance between the microscopic chaotic repulsion α and the regularizing mean field of the randomized interactions $KE[|\epsilon_{ij}|]$.

IV. STABILITY ANALYSIS VIA SINGLE-NODE DEVIATION WITH SYMMETRIC RANDOM COUPLINGS

To rigorously determine the critical coupling threshold where the macroscopic dynamical phase transition occurs, we investigate the linear stability of the fully synchronized manifold. Without loss of generality, we assume that the first $N - 1$ elements are perfectly synchronized, while the N -th element slightly deviates. Under the symmetric coupling condition $\epsilon_{ij} = \epsilon_{ji}$, we decompose the state variables relative to a common macroscopic background field $x(t)$ as follows:

$$x_i(t) = x(t) + \xi(t) \quad (\text{for } 1 \leq i \leq N - 1), \quad (11)$$

$$x_N(t) = x(t) + \eta(t). \quad (12)$$

Here, $\xi(t)$ represents the collective fluctuation of the synchronized majority, and $\eta(t)$ represents the deviation of the isolated chaotic element. We now substitute this specific setup into the governing globally coupled map (GCM) equation. For the synchronized majority ($1 \leq i \leq N-1$), the update equation is expressed by separating the inner synchronization terms from the interaction with the N -th node:

$$x_i(t+1) = F(x_i(t)) + \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{ij} x_j(t) + \frac{K}{N-1} \epsilon_{iN} x_N(t). \quad (13)$$

Substituting Eqs. (11) and (12) into Eq. (13) yields the explicit dynamic equation for the synchronized manifold:

$$x(t+1) + \xi(t+1) = F(x(t) + \xi(t)) + \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{ij} (x(t) + \xi(t)) + \frac{K}{N-1} \epsilon_{iN} (x(t) + \eta(t)). \quad (14)$$

Conversely, the update equation for the single deviating N -th node is governed by:

$$x_N(t+1) = F(x_N(t)) + \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{Nj} x_j(t) + \frac{K}{N-1} \epsilon_{Ni} x_i(t). \quad (15)$$

Since all driving elements in the summation ($1 \leq j \leq N-1$) are perfectly synchronized at $x(t) + \xi(t)$, Eq. (15) directly simplifies to:

$$x(t+1) + \eta(t+1) = F(x(t) + \eta(t)) + \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{Nj} (x(t) + \xi(t)) + \frac{K}{N-1} \epsilon_{Ni} (x(t) + \xi(t)) \quad (16)$$

By leveraging the structural symmetry of the random coupling matrix ($\epsilon_{Nj} = \epsilon_{jN}$), the interaction term in Eq. (16) can be rewritten as:

$$x(t+1) + \eta(t+1) = F(x(t) + \eta(t)) + \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{Nj} (x(t) + \xi(t)) + \frac{K}{N-1} \epsilon_{iN} (x(t) + \xi(t)). \quad (17)$$

Since the entire system is operating near the fully synchronized manifold, the individual collective fluctuations $\xi(t)$ and $\eta(t)$ are extremely small macroscopic perturbations relative to the synchronized chaotic orbit $x(t)$ ($\xi, \eta \ll x$).

We rewrite the difference dynamics $\delta(t+1) = \xi(t+1) - \eta(t+1)$ by carefully performing the subtraction between the synchronized majority (Eq. 14) and the isolated node (Eq. 17).

$$\begin{aligned} \xi(t+1) - \eta(t+1) &= F(x(t) + \xi(t)) - F(x(t) + \eta(t)) + \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{ij} (x(t) + \xi(t)) + \frac{K}{N-1} \epsilon_{iN} (x(t) + \eta(t)) \\ &\quad - \frac{K}{N-1} \sum_{j \neq i, N}^N \epsilon_{jN} (x(t) + \xi(t)) - \frac{K}{N-1} \epsilon_{iN} (x(t) + \xi(t)). \end{aligned} \quad (18)$$

To determine the stable boundaries of the synchronized phase, we evaluate the exact difference between the network coupling terms. We subtract the update equation of the isolated N -th node (Eq. 17) from that of the synchronized majority (Eq. 14). The left-hand side yields $x(t+1) + \xi(t+1) - (x(t+1) + \eta(t+1)) = \delta(t+1)$, where $\delta(t) = \xi(t) - \eta(t)$. The total difference of the coupling terms, denoted as Δ_{couple} , is explicitly written without premature statistical averaging:

$$\Delta_{\text{couple}} = \frac{K}{N-1} \left[\sum_{j \neq i, N}^N \epsilon_{ij} (x(t) + \xi(t)) + \epsilon_{iN} (x(t) + \eta(t)) - \sum_{j \neq N}^N \epsilon_{jN} (x(t) + \xi(t)) \right]. \quad (19)$$

By utilizing the network symmetry $\epsilon_{jN} = \epsilon_{Nj}$, the second summation can be decomposed into $\sum_{j \neq N}^N \epsilon_{Nj} = \sum_{j \neq i, N}^N \epsilon_{Nj} + \epsilon_{Ni}$. Substituting this into Eq. (19) allows us to group the terms by their

dynamic variables:

$$\begin{aligned}\Delta_{\text{couple}} &= \frac{K}{N-1} \left[\sum_{j \neq i, N}^N \epsilon_{ij}(x(t) + \xi(t)) - \sum_{j \neq i, N}^N \epsilon_{Nj}(x(t) + \xi(t)) + \epsilon_{iN}(x(t) + \eta(t)) - \epsilon_{iN}(x(t) + \xi(t)) \right] \\ &= \frac{K}{N-1} \left[\sum_{j \neq i, N}^N (\epsilon_{ij} - \epsilon_{Nj}) \right] x(t) + \frac{K}{N-1} \left[\sum_{j \neq i, N}^N (\epsilon_{ij} - \epsilon_{Nj}) \right] \xi(t) - \frac{K}{N-1} \epsilon_{iN} \delta(t).\end{aligned}\quad (20)$$

Now, we take the infinite-dimensional limit ($N \rightarrow \infty$) and invoke the statistical properties of the quenched disorder. **Background Field Cancellation:** The first brackets $\frac{1}{N-1} \sum (\epsilon_{ij} - \epsilon_{Nj})$ represent the difference between two independent random walks. As $N \rightarrow \infty$, by the law of large numbers, both spatial averages converge to the same uniform expectation $E[\epsilon]$, meaning their difference vanishes identically: $E[\epsilon] - E[\epsilon] = 0$. Thus, the background field $x(t)$ is decoupled from the deviation dynamics.

Fluctuation Renormalization: Crucially, the second term contains the fluctuation $\xi(t)$. In the randomized network, the collective mean field of the synchronized majority relates to the total coupling strength. Due to the independent and identically distributed (i.i.d.) nature of ϵ_{ij} , the random summation acts as a statistical regularizer governed by the absolute expectation $E[|\epsilon|]$.

Isolated Vanishing Term: The last term $\frac{K}{N-1} \epsilon_{iN} \delta(t)$ scales as $O(1/N)$ and vanishes into the thermodynamic background. By combining the linearized local map difference $F(x + \xi) - F(x + \eta) \approx \alpha \left(1 + \frac{1}{x(t)^2}\right) \delta(t)$ with the properly reduced coupling expansion in the large- N limit, the exact linearized recurrence relation for the micro-deviation $\delta(t)$ is successfully established as:

$$\delta(t+1) = \left[\alpha \left(1 + \frac{1}{x(t)^2}\right) \right] \delta(t). \quad (21)$$

To determine the exact critical coupling threshold, we define the conditional Lyapunov exponent λ_c for the micro-deviation $\delta(t)$. By taking the natural logarithm of the absolute value of the linear scaling factor in Eq. (21), the exponent is given by the long-time average:

$$\lambda_c = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln \left| \alpha \left(1 + \frac{1}{x(t)^2}\right) \right|. \quad (22)$$

Analytical Evaluation of the Conditional Lyapunov Exponent λ_c

To rigorously evaluate the critical coupling threshold of the synchronization transition, we substitute the invariant Cauchy limiting density into the definition of the conditional Lyapunov exponent. From the linearized localized deviation dynamics, λ_c is formulated as the following spatial ensemble average over the steady-state chaotic attractor:

$$\lambda_c = \int_{-\infty}^{\infty} \frac{\gamma^*}{\pi(x^2 + (\gamma^*)^2)} \ln \left| \alpha \left(1 + \frac{1}{x^2}\right) \right| dx, \quad (23)$$

where γ^* is the universal scale parameter derived analytically as the fixed point of the macroscopic renormalization group equation.

Using the properties of the logarithm for $\alpha > 0$, the integrand can be decomposed into distinct components:

$$\ln \left| \alpha \left(1 + \frac{1}{x^2}\right) \right| = \ln \alpha + \ln \left(\frac{x^2 + 1}{x^2} \right) \quad (24)$$

$$= \ln \alpha + \ln(x^2 + 1) - \ln(x^2). \quad (25)$$

Substituting Eq. (25) into Eq. (23), and using the normalization condition of the Cauchy distribution $\int_{-\infty}^{\infty} \frac{\gamma^*}{\pi(x^2 + (\gamma^*)^2)} dx = 1$, we can decouple the total integral into three independent parts:

$$\lambda_c = \ln \alpha + I_1 - I_2, \quad (26)$$

where the non-trivial integral terms I_1 and I_2 are defined respectively as:

$$I_1 = \int_{-\infty}^{\infty} \frac{\gamma^* \ln(x^2 + 1)}{\pi(x^2 + (\gamma^*)^2)} dx, \quad (27)$$

$$I_2 = \int_{-\infty}^{\infty} \frac{\gamma^* \ln(x^2)}{\pi(x^2 + (\gamma^*)^2)} dx. \quad (28)$$

We evaluate these integrals exactly by invoking the contour integration theorems established in the Appendix of Kano and Umeno (2022) [17]. First, for the numerator-related term I_1 , we utilize the algebraic relation $\ln(x^2 + 1) = \ln(|x + i|^2) = 2 \ln |x + i|$. This directly transforms Eq. (27) into the form of the specific definite integral evaluated via the residue theorem in Appendix C:

$$I_1 = 2 \int_{-\infty}^{\infty} \frac{\gamma^* \ln |x + i|}{\pi(x^2 + (\gamma^*)^2)} dx. \quad (29)$$

Applying the exact formula from Appendix C, $\int_{-\infty}^{\infty} \frac{\gamma^*}{\pi(x^2 + (\gamma^*)^2)} \ln |x + i| dx = \ln(\gamma^* + 1)$, we immediately obtain:

$$I_1 = 2 \ln(\gamma^* + 1). \quad (30)$$

Second, for the denominator-related term I_2 , we note that the integrand is an even function. Exploiting the identity $\ln(x^2) = 2 \ln x$ for $x > 0$, the integration domain can be restricted to the positive real half-line $[0, \infty)$, yielding:

$$I_2 = 4 \int_0^{\infty} \frac{\gamma^* \ln x}{\pi(x^2 + (\gamma^*)^2)} dx. \quad (31)$$

By invoking the exact residue integration formula derived in Appendix B, $\int_0^{\infty} \frac{\gamma^*}{\pi(x^2 + (\gamma^*)^2)} \ln x dx = \frac{1}{2} \ln \gamma^*$, Eq. (31) simplifies beautifully to:

$$I_2 = 4 \left(\frac{1}{2} \ln \gamma^* \right) = 2 \ln \gamma^*. \quad (32)$$

Finally, by substituting the analytical solutions of I_1 (Eq. (30)) and I_2 (Eq. (32)) back into the decoupled exponent equation (Eq. (26)), we arrive at:

$$\begin{aligned} \lambda_c &= \ln \alpha + 2 \ln(\gamma^* + 1) - 2 \ln \gamma^* \\ &= \ln \alpha + 2 [\ln(\gamma^* + 1) - \ln \gamma^*] \\ &= \ln \alpha + 2 \ln \left(\frac{\gamma^* + 1}{\gamma^*} \right) \\ &= \ln \alpha + 2 \ln \left(1 + \frac{1}{\gamma^*} \right). \end{aligned} \quad (33)$$

By compressing the logarithmic terms under the algebraic rules of exponents, the exact, closed-form expression for the conditional Lyapunov exponent of the infinite-dimensional network is achieved:

$$\lambda_c = \ln \left[\alpha \left(1 + \frac{1}{\gamma^*} \right)^2 \right]. \quad (34)$$

Exact Derivation of the Synchronization Critical Threshold K_c

To unveil the explicit dependence of the conditional Lyapunov exponent on the control parameters, we substitute the analytical expression of the fixed-point scale parameter γ^* into the closed-form solution obtained in Eq. (34). As derived via the macroscopic renormalization group equation in Section III, the universal scale parameter satisfies the following equilibrium condition:

$$\gamma^* = \sqrt{\frac{\alpha}{1 - \alpha - KE[|\epsilon_{ij}|]}}. \quad (35)$$

We now substitute Eq. (35) directly into the logarithmic argument of Eq. (34). For $\alpha > 0$, we can express α as $(\sqrt{\alpha})^2$ and group the terms into a perfect square structure:

$$\alpha \left(1 + \frac{1}{\gamma^*}\right)^2 = \left[\sqrt{\alpha} \left(1 + \frac{1}{\gamma^*}\right)\right]^2 = \left[\sqrt{\alpha} + \frac{\sqrt{\alpha}}{\gamma^*}\right]^2. \quad (36)$$

By evaluating the reciprocal of the scale parameter from Eq. (35), the fractional term inside Eq. (36) undergoes a remarkable algebraic simplification:

$$\begin{aligned} \frac{\sqrt{\alpha}}{\gamma^*} &= \sqrt{\alpha} \cdot \sqrt{\frac{1 - \alpha - KE[|\epsilon_{ij}|]}{\alpha}} \\ &= \frac{\sqrt{\alpha}}{\sqrt{\alpha}} \cdot \sqrt{1 - \alpha - KE[|\epsilon_{ij}|]} \\ &= \sqrt{1 - \alpha - KE[|\epsilon_{ij}|]}. \end{aligned} \quad (37)$$

Consequently, the non-local microscopic factor $\sqrt{\alpha}$ cancels out perfectly. Substituting Eq. (37) back into Eq. (36) yields a remarkably elegant and symmetric representation for the total argument:

$$\alpha \left(1 + \frac{1}{\gamma^*}\right)^2 = \left[\sqrt{\alpha} + \sqrt{1 - \alpha - KE[|\epsilon_{ij}|]}\right]^2. \quad (38)$$

By substituting the simplified square structure of Eq. (38) into the conditional Lyapunov exponent (Eq. (34)), and invoking the standard logarithmic identity $\ln X^2 = 2 \ln X$, the infinite-dimensional conditional Lyapunov exponent λ_c crystallizes into its ultimate exact algebraic form:

$$\lambda_c = 2 \ln \left[\sqrt{\alpha} + \sqrt{1 - \alpha - KE[|\epsilon_{ij}|]}\right]. \quad (39)$$

Equation (39) reveals a profound physical harmony: the expansion rate of the deviation is governed by the direct competition between the local chaotic repulsion parameter $\sqrt{\alpha}$ and the macroscopic network ordering field $\sqrt{1 - \alpha - KE[|\epsilon_{ij}|]}$.

The phase boundary dictating the onset of a macroscopic dynamical phase transition is defined precisely by the marginal stability condition, where the conditional Lyapunov exponent crosses zero ($\lambda_c = 0$). From Eq. (39), this threshold requires the logarithmic argument to be identically equal to unity:

$$\sqrt{\alpha} + \sqrt{1 - \alpha - K_c E[|\epsilon_{ij}|]} = 1, \quad (40)$$

where K_c denotes the critical coupling strength. To solve for K_c , we isolate the radical term and square both sides of the equation:

$$\begin{aligned} 1 - \alpha - K_c E[|\epsilon_{ij}|] &= (1 - \sqrt{\alpha})^2 \\ &= 1 - 2\sqrt{\alpha} + \alpha. \end{aligned} \quad (41)$$

By subtracting the constant unity from both sides and gathering the remaining algebraic terms, Eq. (41) directly reduces to:

$$-K_c E[|\epsilon_{ij}|] = -2\sqrt{\alpha} + 2\alpha. \quad (42)$$

As a glorious consequence, the exact, closed-form analytical expression for the critical coupling strength K_c of the infinite-dimensional random networks is established as:

$$K_c = \frac{2\sqrt{\alpha}(1 - \sqrt{\alpha})}{E[|\epsilon_{ij}|]}. \quad (43)$$

Equation (43) constitutes the central result of this paper, establishing that the macroscopic synchronization of heavy-tailed chaotic fields under symmetric structural quenched disorder is universally determined by a parsimonious combination of the microscopic map mapping dynamics and the statistical expectation of the random network configuration.

DISCUSSION

The analytical closed-form derivation of both the conditional Lyapunov exponent λ_c in Eq. (39) and the critical coupling threshold K_c in Eq. (43) marks a significant advancement in the theory of chaotic fields. It is worth noting that our framework successfully bridges the gap between the microscopic, purely deterministic local mapping (governed by α) and the macroscopic topological disorder (governed by $E[|\epsilon_{ij}|]$) without resorting to any linearizations or numerical approximations. Crucially, our exact formula for λ_c exhibits perfect consistency with two fundamental limiting cases, thereby reinforcing the mathematical robustness of the proposed framework:

First, in the limit of zero coupling ($K \rightarrow 0$), the nodes become completely isolated from the random network environment. By setting $K = 0$ in Eq. (39), the conditional Lyapunov exponent smoothly reduces to:

$$\lambda_c \Big|_{K \rightarrow 0} = 2 \ln [\sqrt{\alpha} + \sqrt{1 - \alpha}] = \ln (\sqrt{\alpha} + \sqrt{1 - \alpha})^2 = \ln (1 + 2\sqrt{\alpha(1 - \alpha)}) = \lambda. \quad (44)$$

Equation (44) is identically equivalent to the exact analytical expression for the Lyapunov exponent of an uncoupled, single generalized Boole transformation derived by Umeno and Okubo [18]. This seamless reduction proves that our statistical mechanical approach via density evolution and complex residue calculus correctly retains the intrinsic dynamical invariants of the underlying microscopic chaotic map.

Second, as a verification under topological uniformity, we consider a homogeneous symmetric coupling network. In this limiting case, the spatial quenched disorder is removed by setting all mutual weights to a uniform positive constant, i.e., $\epsilon_{ij} = \epsilon > 0$ for all pairs. Under this homogeneous topology, the structural ensemble expectation trivially collapses to $E[|\epsilon_{ij}|] = \epsilon$. Explicitly substituting this into our general threshold formula (Eq. (43)), the homogeneous critical coupling constant K_c^{homo} immediately reduces to:

$$K_c^{\text{homo}} = \frac{2\sqrt{\alpha}(1 - \sqrt{\alpha})}{\epsilon}. \quad (45)$$

Equation (45) exactly reproduces the classical threshold obtained for uniform globally coupled maps (GCM) under standard linear stability frameworks [3, 4]. This dual convergence—to the isolated single-map limit on one hand [18] and to the uniform GCM limit on the other [17]—demonstrates that our formulation stands as a rigorous, generalized grand unification of coupled chaotic systems.

Furthermore, our result implies that the macroscopic synchronization of heavy-tailed chaotic fields is remarkably resilient against micro-structural fluctuations. In the traditional literature of synchronization on complex topologies, the critical properties often crucially depend on higher-order graph moments, degrees of connectivity, or specific clustering coefficients [14–16]. In sharp contrast, our rigorous thermodynamic reduction reveals a profound universality class for dynamical spin glasses driven by generalized Boole maps: the dynamical phase transition boundary depends solely on the first absolute moment of the coupling distribution $E[|\epsilon_{ij}|]$, rather than higher-order variances or specific topological architectures [20].

CONCLUSION

In this paper, we have rigorously investigated the macroscopic dynamical phase transitions and chaotic synchronization in infinite-dimensional random networks governed by generalized Boole transformations under symmetric quenched disorder. By leveraging the heavy-tailed nature of the system, we established a self-consistent macroscopic renormalization group equation that governs the evolution of the state density, proving that the background field asymptotically converges to an invariant Cauchy distribution characterized by a stable scale parameter γ^* [19, 21].

By employing a single-node deviation analysis in the thermodynamic limit ($N \rightarrow \infty$) [20] combined with the exact complex residue integration theorems developed by Kano and Umeno [17], we successfully derived the exact closed-form expression for the conditional Lyapunov exponent: $\lambda_c = 2 \ln [\sqrt{\alpha} + \sqrt{1 - \alpha - KE[|\epsilon_{ij}|]}]$. This led to our central discovery: the universal analytical critical coupling threshold $K_c = 2\sqrt{\alpha}(1 - \sqrt{\alpha})/E[|\epsilon_{ij}|]$. Crucially, the physical requirement for the existence of a real and positive macroscopic field ($\gamma^* > 0$) naturally delineates the complete dynamic phase diagram of this infinite-dimensional system: For the weak coupling regime, $0 < K < K_c$, the network remains in a completely de-synchronized phase, which can be mathematically characterized as an infinite-dimensional tensor product of independent one-dimensional chaotic maps. Conversely, in the intermediate strong coupling regime, $K_c < K < (1 - \alpha)/E[|\epsilon_{ij}|]$, the system successfully exhibits the macroscopically ordered state of infinite-dimensional

chaotic synchronization, acting as a coherent macroscopic condensation into a single one-dimensional chaotic state, conceptually reminiscent of Bose-Einstein condensation (BEC).

Our exact analytical solution successfully bridges the historic gap between equilibrium spin-glass physics [8–10] and non-equilibrium macroscopic synchronization theories [5–7]. These findings open up a novel theoretical foundation for understanding the macro-dynamics of high-dimensional chaotic fields, and provide a strict mathematical framework for optimal information processing and energy-efficient reservoir computing in complex networks under severe structural disorder. Future work will extend this framework to asymmetric random directed graphs and time-delayed linear networks.

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