

## Several Questions on Singularities: Theories and Applications

Go-o ISHIKAWA; 石川 剛郎 (北海道大学大学院理学研究科)

**Question:** Why do you study SINGULARITIES?

**Answer 1:** Because it's there.

**Answer 2:** Singularities appear everywhere. We *can not avoid* singularities, for studying regular objects. So studying singularities is *indispensable* in mathematics and other area.

**Answer 3:** Any information on an object *concentrates on its singularities*. Thus studying singularities is one of fundamental methods in mathematics and other area. We must face with singularities *positively*.

**Question:** Are there any applications of singularity theory?

**Answer:** YES. I have collected below some of naïve questions that I have faced during the usual study of applications of singularity theory.

§1. Singularities of Bäcklund Transformations: Classical Theory and Problems.

§2. Frontal Surfaces: Genericity of Mappings to Singular Spaces.

§3. Plane-to-Plane Mappings: Global Configurations.

§4. Singularities in Projective Differential Geometry: Singular Surface Theory.

I would like to thank Satoshi Koike for his providing me the opportunity to talk at the symposium and to write this survey article.

## 1 Singularities of Bäcklund Transformations: Classical Theory and Problems

Bäcklund transformations are transformations of partial differential equations as well as their solutions. They are first introduced around surface theory. See [3]. There are many references on them, related to soliton theory [6]. Recently, Bäcklund transformations have been re-cast in the context of integrable systems in differential geometry [9][2].

In this note we recall the classical definition of Bäcklund transformations following [3], and pose problems related to singularity theory.

A smooth function  $z = f(x, y)$ , as is well-known, can be described by the surface  $\{(x, y, z) \mid z = f(x, y)\}$ , the graph of  $f$ , in the  $(x, y, z)$ -space, endowed with the projection  $(x, y, z) \mapsto (x, y)$ . If we forget the projection, namely, if we do not distinguish the variables  $(x, y)$  and the value  $z$ , then the study on functions of two variables is reduced to the study of surfaces in

the three space.

A tangent plane to a surface in  $(x, y, z)$ -space can be represented by additional two parameters  $p$  and  $q$ . When the surface is the graph of a function  $f(x, y)$ , we take  $p = f_x, q = f_y$ , the partial derivatives. Thus a graphical surface  $M = \{z = f(x, y)\}$  can be lifted naturally to a surface

$$\tilde{M} = \{(x, y, f(x, y), f_x(x, y), f_y(x, y))\}$$

in the five dimensional space  $\{(x, y, z, p, q)\}$ .

Consider the canonical one-form  $\alpha = dz - p dx - q dy$ . Then  $\alpha$  is a contact one-form on this  $\mathbf{R}^5$ . The canonical *contact structure* on  $\mathbf{R}^5$  is defined by the Pfaff equation  $\alpha = 0$ , namely by the distribution  $\{v \in T\mathbf{R}^5 \mid \langle \alpha, v \rangle = 0\} \subset T\mathbf{R}^5$ .

Then the lifting  $\tilde{M}$  is a *Legendre surface*, namely  $\alpha|_{\tilde{M}} = 0$  [1].

To treat non-graphical surfaces, it is natural to introduce the manifold of contact elements of  $\mathbf{R}^3$ . A contact element of  $\mathbf{R}^3$  is, by definition, a linear (hyper)plane of the tangent space to a point in  $\mathbf{R}^3$ . Since a contact element is defined by a non-zero cotangent vector up to non-zero scalar multiplication, the manifold consisting of all contact elements of  $\mathbf{R}^3$  is identified with the fiber-wise projectification  $PT^*\mathbf{R}^3$ .

Let  $\pi : PT^*\mathbf{R}^3 \rightarrow \mathbf{R}^3$  be the natural projection, mapping a contact element to its base point. Then each fiber is a projective plane  $\mathbf{R}P^2$ , which is a compactification of the  $(p, q)$ -plane: If we fix the decomposition  $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ , we have the natural embedding  $\mathbf{R}^5 \hookrightarrow PT^*\mathbf{R}^3$ , defined by  $(x, y, z, p, q) \mapsto (x, y, z, [p, q, 1])$ .

The canonical contact structure on  $\mathbf{R}^5$  naturally extends to a contact structure  $D \subset TPT^*\mathbf{R}^3$  on the manifold  $PT^*\mathbf{R}^3$  of contact elements: A tangent vector  $u \in T_c PT^*\mathbf{R}^3$  to  $PT^*\mathbf{R}^3$  at a contact element  $c$  belongs to  $D$  if and only if  $\pi_*(u) \subset c(\subset T_{\pi(c)}\mathbf{R}^3)$ . Here  $\pi_* : TPT^*\mathbf{R}^3 \rightarrow T\mathbf{R}^3$  is the linearization of  $\pi : PT^*\mathbf{R}^3 \rightarrow \mathbf{R}^3$ .

Any surface in  $\mathbf{R}^3$ , then, lifts naturally to a Legendre surface in  $PT^*\mathbf{R}^3$  with respect to the contact structure  $D$  defined above.

In what follows, we talk on  $PT^*\mathbf{R}^3$  for the theoretical naturality, but you may replace it by  $\mathbf{R}^5$  without loss of significance of the problem.

Now we consider a transformation of surfaces in  $\mathbf{R}^3$ . We regard the transformed surfaces lie in another  $\mathbf{R}^3$  which is a copy of  $\mathbf{R}^3$  with coordinates  $x', y', z'$ . Set  $M = PT^*\mathbf{R}^3$  and denote by  $M'$  the corresponding copy of  $M$ : This  $M'$  has the affine coordinate  $x', y', z', p', q'$  and the local contact form  $\alpha' = dz' - p'dx' - q'dy'$ .

Consider the product manifold  $M \times M'$  of dimension 10. Thus  $M \times M'$  has affine coordinates  $x, y, z, p, q, x', y', z', p', q'$ .

Denote by  $\text{pr} : M \times M' \rightarrow M$  and  $\text{pr}' : M \times M' \rightarrow M'$  the natural projections respectively. Then the contact structures on  $M$  and  $M'$  provide the distribution  $(\text{pr}_*)^{-1}D \cap (\text{pr}'_*)^{-1}D'$  of rank 8, which is locally defined by the Pfaff system

$$\alpha = dz - pdx - qdy = 0, \quad \alpha' = dz' - p'dx' - q'dy' = 0.$$

A *Bäcklund transformation* is a submanifold  $B$  of codimen-

sion 4 in  $M \times M'$  [3],[4].

**Example 1 ([8]).** Let  $N$  and  $N'$  be surfaces in  $\mathbf{R}^3$ , and  $\ell : N \rightarrow N'$  a diffeomorphism. Write  $P' = \ell(P)$ , for  $P \in N$ .  $\ell$  is called a Bäcklund transformation if the secant  $\overline{PP'}$  is tangent to  $N$  at  $P$  and  $N'$  at  $P'$ , and, the distance  $d(P, P') = r$  and the angle  $\text{angle}(\nu_P, \nu_{P'}) = \theta$  of normals  $\nu_P, \nu_{P'}$  is constant ( $P \in N$ ). If  $N : z = z(x, y)$ ,  $N' : z' = z'(x', y')$ , and  $P = (x, y, z)$ ,  $P' = (x', y', z')$ , then  $\ell$  is described by

$$\begin{aligned} F_1 : & p(x' - x) + q(y' - y) - (z' - z) = 0, \\ F_2 : & p'(x - x') + q'(y - y') - (z - z') = 0, \\ F_3 : & (x' - x)^2 + (y' - y)^2 = r^2, \\ F_4 : & \frac{pp' + qq' + 1}{\sqrt{p^2 + q^2 + 1}\sqrt{p'^2 + q'^2 + 1}} = \cos \theta, \end{aligned}$$

in the  $(x, y, z, p, q; x', y', z', p', q')$ -space.

Remark that a Bäcklund transformation  $B \subset M \times M'$  is endowed with a Pfaff system  $\alpha = 0, \alpha' = 0$  restricted to it. In the language of tangent vectors, the system defines

$$E = TB \cap (\text{pr}_*)^{-1}D \cap (\text{pr}'_*)^{-1}D' \subset TB,$$

which is a distribution over  $B$  with singularities in general.

We impose, in what follows, on a Bäcklund transformation  $B$  the condition that

*the projections  $\text{pr}|_B$  and  $\text{pr}'|_B$  are submersions.*

Then we see

**Proposition:** An integral manifolds of  $E$  are at most of dimension 2.

Here is an ad hoc proof of the proposition: Let  $S \subset B$  be an integral manifold of  $E$ . Since  $\text{pr}|_B : B^6 \rightarrow M^5$  is a submersion, the dimension of the kernel of the differential mapping  $(\text{pr}|_B)_*$  is equal to one. Moreover the rank of  $(\text{pr}|_S)_*$  must be at most two, since the image satisfies  $\alpha = 0$ . Therefore  $\dim S$  is at most three. Furthermore if  $\dim S = 3$ , then the image of  $(\text{pr}|_S)_*$  is of dimension two, and the inverse image of the image of  $(\text{pr}|_S)_*$  coincides with the tangent space to  $S$ . This leads to that the dimension of the kernel of  $(\text{pr}'|_S)_*$  is at least two, and to a contradiction.  $\square$

Now let  $I \subset B$  be an integral submanifold of dimension 2 of  $E$ :

$$\alpha|_I = 0, \quad \alpha'|_I = 0.$$

Then naturally posed questions are these:

**Question:** What are generic singularities of  $\text{pr}|_I : I \rightarrow M$  and  $\text{pr}'|_I : I \rightarrow M'$ ? What are generic singularities of  $\pi \circ \text{pr}|_I : I \rightarrow \mathbf{R}^3$  and  $\pi' \circ \text{pr}'|_I : I \rightarrow \mathbf{R}^3$ ?

Remark that  $\text{pr}|_I$  is an *integral mapping*, namely  $(\text{pr}|_I)^*\alpha = 0$ , and therefore the image  $\text{pr}(I) \subset M = PT^*\mathbf{R}^3$  is a Legendre variety, in other words, the regular part of  $\text{pr}(I)$  is an integral manifold (Legendre submanifold) of the contact structure  $\alpha = 0$ .

**Question:** Are there generating families for this singularity

problem, like in ordinary Legendre singularity theory?

Ideally we wish to find a function of type  $F(x, y, z; x', y', z')$ , for a given  $I \subset B$ , which is a generating family (with parameter  $x, y, z$ ) of  $\text{pr}(I)$  with respect to  $\pi$ , and at the same time, is a generating family (with parameter  $x', y', z'$ ) of  $\text{pr}'(I)$  with respect to  $\pi'$ . Since  $\text{pr}(I)$  and  $\text{pr}'(I)$  may have singularities, the generating family may define other extra components than  $\text{pr}(I)$  and  $\text{pr}'(I)$ .

Consider the case that the system of 4 equations defining a Bäcklund transformation  $B$  contains  $x = x', y = y'$ . Then we regard  $B$  as a submanifold in the  $(x, y, z, z', p, q, p', q')$ -space with equations

$$\alpha = dz - p dx - q dy = 0, \quad \alpha' = dz' - p' dx - q' dy = 0,$$

of codimension two, locally defined by two equations, say:

$$f(x, y, z, z', p, q, p', q') = 0, \quad g(x, y, z, z', p, q, p', q') = 0.$$

**Question:** Are there any local characterizations of the class of differential systems on  $(\mathbf{R}^6, 0)$  realized as Bäcklund transformations of above type.

If we eliminate  $z', p', q'$  using

$$dz' = p' dx + q' dy, \quad f = 0, \quad g = 0,$$

then we get a 2nd order differential equation of  $z = z(x, y)$ . If we eliminate  $z, p, q$  using

$$dz = p dx + q dy, \quad f = 0, \quad g = 0,$$

then we have a 2nd order differential equation of  $z' = z'(x, y)$ . Thus a Bäcklund transformation induces a transformation of 2nd order differential equations and solutions. (The graphs of solutions are  $\pi \circ \text{pr}(I)$  and  $\pi' \circ \text{pr}'(I)$ , in our notations.)

**Example(Sine-Gordon equation):** Let

$$\begin{aligned} f &= p' - p - 2 \sin \frac{z' + z}{2} \\ g &= q' + q - 2 \sin \frac{z' - z}{2}. \end{aligned}$$

Then we have

$$p'_y = p_y + \left( \cos \frac{z' + z}{2} \right) (q' + q) = p_y + \sin z' - \sin z,$$

and

$$q'_x = -q_x + \left( \cos \frac{z' - z}{2} \right) (p' - p) = -q_x + \sin z' + \sin z.$$

Thus we have

$$p'_y - \sin z' = p_y - \sin z, \quad q'_x - \sin z' = -q_x + \sin z,$$

and two differential equations:

$$z_{xy} = \sin z, \quad z'_{xy} = \sin z',$$

the same sine-Gordon equation. The transformation of solution, then, is closely related the transformation of surfaces with negative curvature.

I believe it is necessary to give the rigorous foundation to the elimination process:

**Question:** Are there any theory of elimination for partial differential equations, like in algebraic and analytic geometry.

I am very grateful to Toshizumi Fukui for his turning my attention to Bäcklund transformations and for the encouragement.

## References

- [1] V.I. Arnol'd, *Singularities of Caustics and Wave Fronts*, Kluwer Academic Publishers, Dordrecht, 1990.
- [2] E.V. Ferapontov, *Integrable systems in projective differential geometry*, math. DG/9903150 (25 Mar. 1999).
- [3] E. Goursat, *Le problème de Bäcklund*, Mémor. Sci. Math. **6**, Gauthier-Vilars, Paris, 1925.
- [4] M. Matsuda, *Theory of Exterior Differential Forms*, Iwanami Shoten Co., 1976, (in Japanese).
- [5] T. Morimoto *Monge-Ampère equations viewed from contact geometry*, in Banach Center Publications vol. **39**, "Symplectic Singularities and Geometry of Gauge Fields", 1998 pp.105–120.
- [6] R.M. Miura (ed.), *Bäcklund Transformations*, Lecture Notes in Math., **515**, Springer-Verlag, 1976.
- [7] S.S. Shen, *A Course on Nonlinear Waves*, Nonlinear topics in the mathematical sciences, **3**, Kluwer Academic Publishers, Dordrecht, 1993.
- [8] C.-L. Terng, R.S. Palais, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math., **1353**, Springer-Verlag, 1988.
- [9] C.-L. Terng, K. Uhlenbeck, *Poisson actions and scattering theory for integrable systems*, dg-ga/9707004 (7 Jul. 1997).

## 2 Frontal Surfaces: Genericity of Mappings to Singular Spaces.

A surface in  $\mathbf{R}^3$  or  $\mathbf{C}^3$  is called *frontal* if it has "smooth" Nash lifting in  $PT^*\mathbf{R}^3$ . Exactly, if we give the surface by a parametriza-

tion  $f : M \rightarrow \mathbf{R}^3$  from a  $C^\infty$  surface  $M$ , then  $f$  is called *frontal* if it has a unique frontal lifting  $\tilde{f} : M \rightarrow PT^*\mathbf{R}^3$ . If the surface is an analytic surface in  $\mathbf{C}^3$ , then, the surface is called *frontal* if the projection from the Nash lifting of the surface to the surface itself is finite to one.

Similarly we define the notion of *frontal hypersurfaces* in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  and more generally in  $C^\infty$  or complex manifolds.

Since the behavior of tangent spaces to a frontal surfaces is very restrictive, we expect we can apply the stratification theory to studying families of frontal surfaces.

I have applied the stratification theory to verifying the topological triviality of families of tangent developables [5]

**Question:** Is there any simple criteria for topological triviality of families of frontal (hyper)surfaces?

Remark that frontal surfaces have only non-isolated singularities “generically”. However there are examples of frontal surfaces having isolated singularities:  $z^2 = x^4 + y^4$ .

Also, the following question should be naturally posed:

**Question:** Are there any algebraic (ring theoretical) characterization of frontal (hyper)surfaces?

The study on frontal surfaces is closely related to the study on integral mappings.

**Givental’ conjecture [1]:** Generic singularities of integral mappings  $\mathbf{R}^2 \rightarrow \mathbf{R}^5$  are contact equivalent to the Nash lifting

of *folded umbrella*

$$(u, v) \mapsto (x, y, p, q, z) = (u, v^2/2, v^3/3, uv, uv^3/3).$$

The corank one case of Givental' conjecture is proved by Givental' [1][2]. The higher dimensional generalization of corank one case is solved by me [3].

**Question:** How do we describe the generic conditions for integral mappings of corank  $> 1$ .

Here, let us recall the notion of integral jet spaces [4]. In the ordinary jet space  $J^r(\mathbf{R}^2, \mathbf{R}^5)$ , consider

$$I^r := \{j^r h(x) \mid x \in \mathbf{R}^2, h : \mathbf{R}^2, x \rightarrow \mathbf{R}^5 \text{ integral}\}.$$

If  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^5$  is integral, then the jet extension  $j^r f$  is regarded as a mapping to  $I^r$ :  $j^r f : \mathbf{R}^2, 0 \rightarrow I^r$ , that we call *the integral jet extension*:  $(j^r f)(x) := j^r f(x)$ , the  $r$ -jet of  $f$  at  $x$ .

Then a difficulty arises from the fact that the isotropic jet space  $I^r$  has quadratic singularities

$$\text{Sing}(I^r) = \{j^r h(x) \mid h: \text{integral of corank} \geq 2\}.$$

Then the natural and important question is this:

**Question:** Do any transversality theorems exist, for mappings to singular spaces?

## References

- [1] A.B. Givental', *Lagrangian imbeddings of surfaces and unfolded Whitney umbrella*, Funkt. Anal. Prilozhen, **20-3** (1986), 35–41.

- [2] A.B. Givental', *Singular Lagrangian varieties and their Lagrangian mappings*, Itogi Nauki Tekh., Ser. Sovrem. Prob. Mat., (Contemporary Problems of Mathematics) **33**, VITINI, 1988, pp. 55–112.
- [3] G. Ishikawa *The local model of an isotropic map-germ arising from one dimensional symplectic reduction*, Math. Proc. Camb. Phil. Soc., **111**–1 (1992), 103–112.
- [4] G. Ishikawa, *Transversalities for Lagrange singularities of isotropic mappings of corank one*, in Singularities and Differential Equations, Banach Center Publications, **33** (1996), 93–104.
- [5] G. Ishikawa, Topological classification of tangent developables to space curves, to appear in Journal of London Math. Soc..

### 3 Plane-to-Plane Mappings: Global Configurations.

Let  $f : \mathbf{R}^2 \rightarrow PT^*\mathbf{R}^3$  be a proper generic integral mapping. Consider the projection  $\Pi : PT^*\mathbf{R}^3 \rightarrow \mathbf{R}^2$ ,  $(x, y, z, p, q) \mapsto (x, y)$  and the composition  $\Pi \circ f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ , which is called a Lagrange mapping. The critical value set of  $\Pi \circ f$  is called *the caustic*.

**Question:** (The Question on the Topology of Caustics.) Are there any differences on the topology of generic Lagrange mappings and the topology of generic mappings  $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ .

If we pose the condition that  $f$  is a Legendre immersion, then the question is classical:

**Question:** (The Classical Question on the Topology of Caustics.) Are there any differences on the topology of generic Lagrange mappings of *Legendre immersions* and the topology of generic mappings  $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ .

The topology of generic mappings  $\mathbf{R}^2 \rightarrow \mathbf{R}^2$  itself is also

interesting problem. See [1][2] for the characterization of the discriminant set. Even it seems to be not so clearly understood.

The problem should be treated again elsewhere.

I am grateful to Osamu Saeki for his informing me the related references. I would like to thank Kazuhiko Aomoto and Toru Ohmoto for the important questions and comment.

## References

- [1] G.K. Francis, S.F. Troyer, *Excellent maps with given folds and cusps*, Houston J. of Math., **3-2** (1977), 165–192.
- [2] I. Malta, N.C. Saldanha, C. Tomei, *Critical sets of proper Whitney functions in the plane*,

## 4 Singularities in Projective Differential Geometry: Singular Surface Theory.

Let  $f, f' : (\mathbf{R}^2, 0) \rightarrow \mathbf{R}P^3$  be map-germs to the projective three space.  $f$  and  $f'$  are called *projectively equivalent* if there exist a projective transformation  $\tau : \mathbf{R}P^3 \rightarrow \mathbf{R}P^3$  and a diffeomorphism-germ  $\sigma : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$  such that  $\tau \circ f = f' \circ \sigma$ .

Classical theory treats the projective classification of immersions: There exist relations of classical surface theory to the study on integrable systems, Bäcklund transformations and so on [1].

**Question:** Are there any generalization of classical theory of projective differential geometry to singular surfaces?

I believe that the projective differential geometry of singularities of ruled surfaces, developable surfaces, and frontal surfaces is a fruitful and promising area for studying; as the manifestation of the “contact nature” of projective geometry.

## References

- [1] E.V. Ferapontov, *Integrable systems in projective differential geometry*, math. DG/9903150 (25 Mar. 1999).
- [2] E.I. Wilczynski, *Projective-differential geometry of curves and ruled surfaces*, B.G. Teubners Sammling von Lehrbüchern, Mathematischen Wissenschaften, 18, Leipzig: B.G. Teubner, 1906.