

Iwahori-Hecke type algebras associated with the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$

By

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Abstract

In this paper we give Iwahori-Hecke type algebras $H_q(\mathbf{g})$ associated with the Lie superalgebras $\mathbf{g} = A(m, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$. We classify the irreducible representations of $H_q(\mathbf{g})$ for generic q .

Introduction

Recently, motivated by a question posed by V. Serganova [S] and study of the Weyl groupoids [H1][H2] associated with Nichols algebras [AS1][AS2] including generalizations of quantum groups, I. Heckenberger and the author [HY] introduced a notion of ‘Coxeter groupoids’ (in fact they can be defined as semigroups), and showed that a Matsumoto-type theorem holds for the groupoids, so they have the solvable word problem. We mention that the Coxeter groupoid associated with the affine Lie superalgebra $D^{(1)}(2, 1; x)$ was used in the study [HSTY], where Drinfeld second realizations of $U_q(D^{(1)}(2, 1; x))$ was analyzed by physical motivation in recent study of AdS/CFT correspondence.

It would be able to be said that one of the main purposes at present of the representation theory is to study the Kazhdan-Lusztig polynomials (cf. [Hu, 7.9]) and their versions. The polynomials are defined by using the standard and canonical bases of the Iwahori-Hecke algebras. The existence of those bases is closely related to the Matsumoto theorem of the Coxeter groups. So it would be natural to ask what to be

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the Iwahori-Hecke algebras of the Coxeter groupoids. In this paper, we give a tentative answer to this question for the Coxeter groupoids W associated with the Lie superalgebras $\mathfrak{g} = A(m, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$. We introduce the Iwahori-Hecke type algebra $H_q(\mathfrak{g})$ (in the text, it is also denoted by $H_q(W)$) as q -analogue of the semigroup algebra $\mathbb{C}W/\mathbb{C}0$, where 0 is the zero element of W . We also show that if q is nonzero and not any root of unity, $H_q(\mathfrak{g})$ is semisimple and there exists a natural one-to-one correspondence between the equivalence classes of the irreducible representations of $H_q(\mathfrak{g})$ and those of the Iwahori-Hecke algebra $H_q(W_0)$ associated with the Weyl group W_0 of the Lie algebra $\mathfrak{g}(0)$ obtained as the even part of $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$.

Until now, no relation has been achieved between the groupoids treated in [SV] and this paper.

This paper is composed of the two sections. Main results and their proofs are given in Section 2. Results of [HY] used in Section 2 are introduced in Section 1.

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§ 1. Preliminary—Matsumoto-type theorem of Coxeter groupoids

This section is preliminary. Here we collect the results which have already been given in [HY] and will be used in the next section.

§ 1.1. Semigroups and Monoids

Let K be a non-empty set. Assume that K has a product map $K \times K \rightarrow K$, $(x, y) \mapsto xy$. We call K a *semigroup* if $(xy)z = x(yz)$ for $\forall x, y, z \in K$. We call K a *monoid* if K is a semigroup and there exists a unit $1 \in K$, that is, $1x = x1 = x$ for all $x \in K$.

§ 1.2. Free semigroup $F_{-1}(N)$ and Free monoid $F_0(N)$

Let N be a non-empty set. Let $F_{-1}(N)$ be the set of all the finite sequences of elements of N , that is

$$F_{-1}(N) := \prod_{n=1}^{\infty} N^n = \{(h_1, \dots, h_n) | n \in \mathbb{N}, h_i \in N\}.$$

We regard $F_{-1}(N)$ as the semigroup by

$$(h_1, \dots, h_m)(h_{m+1}, \dots, h_{m+n}) = (h_1, \dots, h_m, h_{m+1}, \dots, h_{m+n}).$$

Then we call $F_{-1}(N)$ a *free semigroup*. Let $F_0(N)$ be the semigroup obtained by adding the unit 1 , that is, $F_0(N) := \{1\} \cup F_{-1}(N)$, $1 \notin F_{-1}(N)$, and $1x = x1 = x$ for all $x \in F_0(N)$.

§ 1.3. Semigroup generated by the generators and defined by the relations

Let $Q = \{(x_j, y_j) | j \in J\}$ be a subset of $F_{-1}(N) \times F_{-1}(N)$, where J is an index set. For $g_1, g_2 \in F_{-1}(N)$, we write $g_1 \sim_1 g_2$ if there exist $(x, y) \in Q$ and $(f_1, f_2) \in F_0(N) \times F_0(N)$ such that either of the following (i), (ii), (iii) holds.

- (i) $g_1 = f_1 x f_2 \neq g_2 = f_1 y f_2$.
- (ii) $g_1 = f_1 y f_2 \neq g_2 = f_1 x f_2$.
- (iii) $g_1 = g_2 = f_1 x f_2 = f_1 y f_2$.

For $g, g' \in F_{-1}(N)$, we write $g \sim g'$ if $g = g'$ or there exists $r \in \mathbb{N}$ and $g_1, \dots, g_r \in F_{-1}(N)$ such that $g_1 = g$, $g_r = g'$, and $g_i \sim_1 g_{i+1}$ for $1 \leq i \leq r-1$. Then $F_{-1}(N)/\sim$ can be regarded as a semigroup by the product $[g][g'] = [gg']$, where for $g \in F_{-1}(N)$, we denote $[g] := \{g' | g' \sim g\} \in F_{-1}(N)/\sim$. We call $F_{-1}(N)/\sim$ the semigroup generated by N and defined by the relations $x_j = y_j$ ($j \in J$). When there is no fear of misunderstanding, we also denote $[g]$ by its representative g by abuse of notation.

§ 1.4. Free group $F_1(N)$ and Involutive free group $F_2(N)$

Let N be a set. Let N^{-1} be a copy of N so that the bijective map $N \rightarrow N^{-1}$, $x \mapsto x^{-1}$, is given. Let $F_1(N)$ be the semigroup generated by

$$\{e\} \cup N \cup N^{-1} \quad (\text{disjoint union})$$

and defined by the relations

$$ee = e, \quad ex = xe = x, \quad ex^{-1} = x^{-1}e = x^{-1}, \quad xx^{-1} = x^{-1}x = e \quad \text{for } \forall x \in N.$$

We call $F_1(N)$ the *free group* over N .

Let $F_2(N)$ be the semigroup generated by

$$\{e\} \cup N \quad (\text{disjoint union})$$

and defined by the relations

$$ee = e, \quad ex = xe = x, \quad x^2 = e \quad \text{for } \forall x \in N.$$

We call $F_2(N)$ the *involutive free group* over N . Note that $F_2(N)$ can be identified with the quotient group of $F_1(N)$ in the natural sense:

$$F_2(N) = F_1(N) / \{g_1 y_1^2 g_1^{-1} \cdots g_r y_r^2 g_r^{-1} | r \in \mathbb{N}, y_i \in N \cup N^{-1}, g_i \in F_1(N)\}.$$

§ 1.5. Action $\triangleright$ of $F_2(N)$ on A

Let N and A be non-empty sets. An action $\triangleright$ of $F_2(N)$ on A is a map

$$\triangleright : F_2(N) \times A \rightarrow A$$

such that

$$e \triangleright a = a, \quad g \triangleright (h \triangleright a) = (gh) \triangleright a \quad \text{for } \forall g, \forall h \in F_2(N), \forall a \in A.$$

Note that $n \triangleright (n \triangleright a) = a$ for all $n \in N, a \in A$.

For $n, n' \in N$ and $a \in A$, define

$$\Theta(n, n'; a) := \{(nn')^m \triangleright a, (n'n)^m \triangleright a \mid m \in \mathbb{N} \cup \{0\}\}.$$

Let

$$\theta(n, n'; a) := |\Theta(n, n'; a)|.$$

This is the cardinality of $\Theta(n, n'; a)$, which is either in $\mathbb{N}$ or is ∞ . One obviously has $\Theta(n, n'; a) = \Theta(n', n; a)$ and $\Theta(n, n'; n \triangleright a) = n \triangleright \Theta(n', n; a)$.

Let $a_0 := a, b_0 := a$, and define recursively $a_{m+1} := n \triangleright b_m, b_{m+1} := n' \triangleright a_m$ for all $m \in \mathbb{N} \cup \{0\}$. That is:

$$\begin{aligned} b_0 &:= a, & a_1 &:= n \triangleright a, & b_2 &:= n' \triangleright n \triangleright a, & a_3 &:= n \triangleright n' \triangleright n \triangleright a, \dots \\ a_0 &:= a, & b_1 &:= n' \triangleright a, & a_2 &:= n \triangleright n' \triangleright a, & b_3 &:= n' \triangleright n \triangleright n' \triangleright a, \dots \end{aligned}$$

Then we have

$$\theta(n, n'; a) = \begin{cases} \infty & \text{if } a_m \neq b_m \text{ for all } m \in \mathbb{N}, \\ \min\{m \in \mathbb{N} \mid a_m = b_m\} & \text{otherwise.} \end{cases}$$

§ 1.6. Coxeter groupoids

Definition 1.1. [HY, Definition 1] Let N and A be non-empty sets. Let $\triangleright$ be a transitive action of $F_2(N)$ on A . For each $a \in A$ and $i, j \in N$ with $i \neq j$ let

$$m_{i,j;a} = m_{j,i;a} \in (\mathbb{N} + 1) \cup \{\infty\}$$

be such that $\theta(i, j; a) \in \mathbb{N} \implies \frac{m_{i,j;a}}{\theta(i,j;a)} \in \mathbb{N} \cup \{\infty\}$ or $\theta(i, j; a) = \infty \implies m_{i,j;a} = \infty$. Set

$$\mathbf{m} := (m_{i,j;a} \mid i, j \in N, i \neq j, a \in A).$$

Let

$$(1.1) \quad W = (W, N, A, \triangleright, \mathbf{m})$$

be the semigroup generated by the set

$$\{0, e_a, s_{i,a} \mid a \in A, i \in N\}$$

and defined by the relations

$$(1.2) \quad 00 = e_a 0 = 0 e_a = s_{i,a} 0 = 0 s_{i,a} = 0,$$

$$(1.3) \quad \begin{aligned} e_a^2 &= e_a, & e_a e_b &= 0 \text{ for } a \neq b, \\ e_{i \triangleright a} s_{i,a} &= s_{i,a} e_a = s_{i,a}, & s_{i,i \triangleright a} s_{i,a} &= e_a, \end{aligned}$$

$$(1.4) \quad \begin{aligned} s_i s_j \cdots s_j s_{i,a} &= s_j s_i \cdots s_i s_{j,a} \text{ } (m_{i,j;a} \text{ factors}) & \text{if } m_{i,j;a} \text{ is finite and odd,} \\ s_j s_i \cdots s_j s_{i,a} &= s_i s_j \cdots s_i s_{j,a} \text{ } (m_{i,j;a} \text{ factors}) & \text{if } m_{i,j;a} \text{ is finite and even,} \end{aligned}$$

where we use the convention:

$$(1.5) \quad s_j s_{i,a} := s_{j,i \triangleright a} s_{i,a}, \quad s_i s_j s_{i,a} := s_{i,j \triangleright i \triangleright a} s_j s_{i,a}, \dots$$

See also (1.7) below.

§ 1.7. Sign representation

Let $\mathbb{Z}A$ be the free $\mathbb{Z}$ -module generated by A , that is,

$$\mathbb{Z}A = \bigoplus_{a \in A} \mathbb{Z}a.$$

Then there exists a unique semigroup homomorphism

$$\widetilde{\text{sgn}} : W \rightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}A)$$

such that

$$(1.6) \quad \widetilde{\text{sgn}}(0)(b) = 0, \quad \widetilde{\text{sgn}}(e_a)(b) = \delta_{ab} b, \quad \widetilde{\text{sgn}}(s_{i,a})(b) = (-1) \delta_{ab} i \triangleright a$$

for $a, b \in A$ and $i \in N$, where δ means Kronecker's symbol. Hence for $w \in W$ one has

$$w \neq 0$$

if and only if $w = e_a$ for some $a \in A$ or there exist $m \in \mathbb{N}$ and $i_j \in N$, $b_j \in A$ with $1 \leq j \leq m$ such that $b_j = i_{j+1} \triangleright b_{j+1}$ and $w = s_{i_1, b_1} \cdots s_{i_{m-1}, b_{m-1}} s_{i_m, b_m}$. If this is the case, we use the convention

$$(1.7) \quad s_{i_1} \cdots s_{i_{m-1}} s_{i_m, b_m} := w,$$

and, if $m = 0$, $s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a}$ means e_a . We note again

Lemma 1.2. (1) $s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} \neq 0$ for all $a \in A$ and $m \in \mathbb{N} \cup \{0\}$.
 (2) If $s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} = s_{j_1} \cdots s_{j_{r-1}} s_{j_r, b}$, then $a = b$, $i_1 \cdots i_m \triangleright a = j_1 \cdots j_r \triangleright b$ and $(-1)^m = (-1)^r$.

§ 1.8. Generalization of Root systems

Definition 1.3. [HY, Definition 2] We call a quadruple $(R, N, A, \triangleright)$ a *multi-domains root system* if the following conditions hold.

1. N and A are non-empty sets and $\triangleright$ is a transitive action of $F_2(N)$ on A .
2. Let V_0 be the $|N|$ -dimensional $\mathbb{R}$ -linear space. Then

$$R = \{(R_a, \pi_a, S_a) \mid a \in A\},$$

where $\pi_a = \{\alpha_{n, a} \mid n \in N\} \subset R_a \subset V_0$, and π_a is a basis of V_0 for all $a \in A$.

3. $R_a = R_a^+ \cup -R_a^+$ for all $a \in A$, where $R_a^+ = (\mathbb{N} \cup \{0\})\pi_a \cap R_a$.
4. For any $i \in N$ and $a \in A$ one has $\mathbb{R}\alpha_{i, a} \cap R_a = \{\alpha_{i, a}, -\alpha_{i, a}\}$.
5. $S_a = \{\sigma_{i, a} \mid i \in N\}$, and for each $a \in A$ and $i \in N$ one has $\sigma_{i, a} \in \text{GL}(V_0)$,

$$\sigma_{i, a}(R_a) = R_{i \triangleright a}, \quad \sigma_{i, a}(\alpha_{i, a}) = -\alpha_{i, i \triangleright a}, \quad \sigma_{i, a}(\alpha_{j, a}) \in \alpha_{j, i \triangleright a} + (\mathbb{N} \cup \{0\})\alpha_{i, i \triangleright a}$$
for all $j \in N \setminus \{i\}$.
6. $\sigma_{i, i \triangleright a} \sigma_{i, a} = \text{id}$ for $a \in A$ and $i \in N$.
7. Let $a \in A$, $i, j \in N$, $i \neq j$, and $d = |((\mathbb{N} \cup \{0\})\alpha_{i, a} + (\mathbb{N} \cup \{0\})\alpha_{j, a}) \cap R_a|$. If d is finite then $\theta(i, j; a)$ is finite and it divides d .

Convention. We write

$$(R, N, A, \triangleright) \in \mathcal{R}$$

if $(R, N, A, \triangleright)$ is a multi-domains root system, that is, $\mathcal{R} = \{(R, N, A, \triangleright)\}$ denotes the family of all the multi-domains root systems.

Definition 1.4. [HY, Definition 4] Let $(R, N, A, \triangleright) \in \mathcal{R}$. Let $\mathbf{m} := (m_{i, j; a} \mid i, j \in N, i \neq j, a \in A)$ be such that $m_{i, j; a} := |((\mathbb{N} \cup \{0\})\alpha_{i, a} + (\mathbb{N} \cup \{0\})\alpha_{j, a}) \cap R_a|$. Then we call $(W, N, A, \triangleright, \mathbf{m})$ the *Coxeter groupoid associated with $(R, N, A, \triangleright)$* .

Theorem 1.5. [HY, Theorem 1] Let $(R, N, A, \triangleright) \in \mathcal{R}$. Set $V = \bigoplus_{a \in A} V_a$, where $V_a = V_0$. Let $P_a : V \rightarrow V_a$ and $\iota_a : V_a \rightarrow V$ be the canonical projection and the canonical inclusion map respectively. Then the assignment $\rho : 0 \mapsto 0 \cdot \text{id}_V$, $e_a \mapsto \iota_a P_a$, $s_{i, a} \mapsto \iota_{i \triangleright a} \sigma_{i, a} P_a$, gives a faithful representation (ρ, V) of the Coxeter groupoid $(W, N, A, \triangleright, \mathbf{m})$ associated with $(R, N, A, \triangleright)$.

§ 1.9. Matsumoto-type theorem

Define $\ell : W \rightarrow \mathbb{N} \cup \{0\} \cup \{-\infty\}$ to be the map such that $\ell(0) = -\infty$, $\ell(e_a) = 0$ for all $a \in A$, and

$$\ell(w) = \min\{m \in \mathbb{N} \mid w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} \text{ for some } i_1, \dots, i_m \in N, a \in A\}$$

for all $w \in W \setminus (\{0\} \cup \{e_a \mid a \in A\})$; we also refer to Lemma 1.2 (1) for this definition of ℓ . One has

$$(1.8) \quad \ell(w) = \ell(w^{-1})$$

for $w \in W \setminus \{0\}$, and

$$(1.9) \quad \ell(w w') \leq \ell(w) + \ell(w')$$

for $w, w' \in W$. We say that a product $w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} \in W$ is *reduced* if $m = \ell(w)$.

Definition 1.6. [HY, Definition 5] Let $W = (W, N, A, \triangleright, \mathbf{m})$ be a Coxeter groupoid. Let $\widetilde{W} = (\widetilde{W}, N, A, \triangleright, \mathbf{m})$ denote the semigroup generated by the set

$$\{0, \tilde{e}_a, \tilde{s}_{i,a} \mid a \in A, i \in N\}$$

and defined by the relations

$$(1.10) \quad 00 = 0, \quad 0\tilde{e}_a = \tilde{e}_a 0 = 0\tilde{s}_{i,a} = \tilde{s}_{i,a} 0 = 0,$$

$$(1.11) \quad \tilde{e}_a^2 = \tilde{e}_a, \quad \tilde{e}_a \tilde{e}_b = 0 \text{ for } a \neq b, \quad \tilde{e}_{i \triangleright a} \tilde{s}_{i,a} = \tilde{s}_{i,a} \tilde{e}_a = \tilde{s}_{i,a},$$

$$(1.12) \quad \begin{aligned} \tilde{s}_i \tilde{s}_j \cdots \tilde{s}_j \tilde{s}_{i,a} &= \tilde{s}_j \tilde{s}_i \cdots \tilde{s}_i \tilde{s}_{j,a} \text{ (} m_{i,j;a} \text{ factors)} && \text{if } m_{i,j;a} \text{ is finite and odd,} \\ \tilde{s}_j \tilde{s}_i \cdots \tilde{s}_j \tilde{s}_{i,a} &= \tilde{s}_i \tilde{s}_j \cdots \tilde{s}_i \tilde{s}_{j,a} \text{ (} m_{i,j;a} \text{ factors)} && \text{if } m_{i,j;a} \text{ is finite and even.} \end{aligned}$$

Theorem 1.7. [HY, Theorem 5] (Matsumoto-type theorem of the Coxeter groupoids) *Let*

$$W = (W, N, A, \triangleright, \mathbf{m})$$

be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ (see Definition 1.4). Suppose that $m \in \mathbb{N} \cup \{0\}$, $a \in A$, and $(i_1, \dots, i_m), (j_1, \dots, j_m) \in N^m$ such that

$$\ell(s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a}) = m$$

and equation

$$s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} = s_{j_1} \cdots s_{j_{m-1}} s_{j_m, a}$$

holds in W . Then in the semigroup $(\widetilde{W}, N, A, \triangleright, \mathbf{m})$ one has

$$\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m, a} = \tilde{s}_{j_1} \cdots \tilde{s}_{j_{m-1}} \tilde{s}_{j_m, a}.$$

Corollary 1.8. [HY, Corollary 6] *Let*

$$W = (W, N, A, \triangleright, \mathbf{m})$$

be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ (see Definition 1.4). Suppose that $m \in \mathbb{N} \cup \{0\}$, $a \in A$, and $(i_1, \dots, i_m) \in N^m$ such that $\ell(s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a}) < m$ holds in $(W, N, A, \triangleright, \mathbf{m})$. Then there exist $j_1, \dots, j_m \in N$ and $t \in \{1, \dots, m-1\}$ such that $j_t = j_{t+1}$ and in the semigroup $(\widetilde{W}, N, A, \triangleright, \mathbf{m})$ one has the equation

$$\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m, a} = \tilde{s}_{j_1} \cdots \tilde{s}_{j_t} \tilde{s}_{j_{t+1}} \cdots \tilde{s}_{j_{m-1}} \tilde{s}_{j_m, a}.$$

In the next section, we also need

Proposition 1.9. [HY, Corollary 3] *Let $m \in \mathbb{N}$, $(i_1, \dots, i_m, j) \in N^{m+1}$, and $a \in A$, and suppose that $\ell(s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a}) = m$. Then:*

- (1) $m = |\sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m, a}(R_a^+) \cap -R_{i_1 \dots i_m \triangleright a}^+|$.
- (2) $\ell(s_{i_1} \cdots s_{i_m} s_{j, \triangleright a}) = m - 1 \iff \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m, a}(\alpha_{j, a}) \in -R_{i_1 \dots i_m \triangleright a}^+.$
- (3) $\ell(s_{i_1} \cdots s_{i_m} s_{j, \triangleright a}) = m + 1 \iff \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m, a}(\alpha_{j, a}) \in R_{i_1 \dots i_m \triangleright a}^+.$

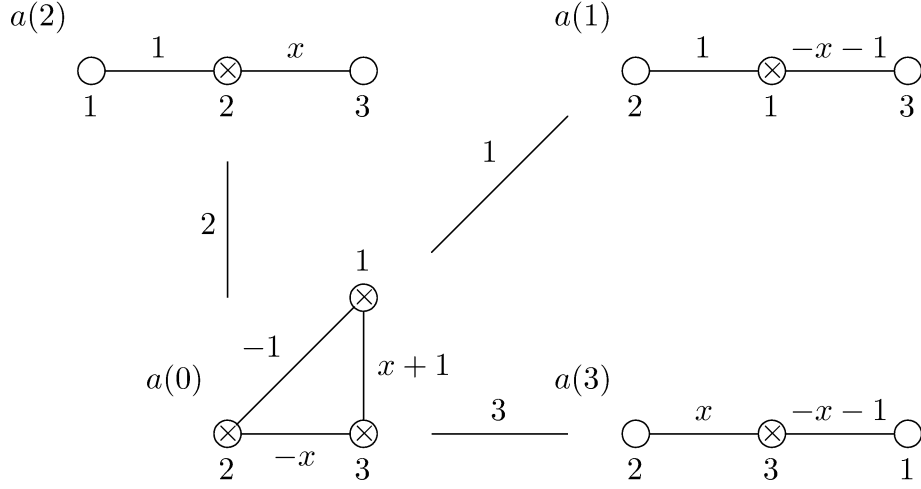
Example 1.10. Here we treat the finite dimensional simple Lie superalgebra $D(2, 1; x)$, where $x \notin \{0, -1\}$. Note that it has 14 (positive and negative) roots. One has $m_{i, j; a(k)} = 2 + \delta_{k0} + (1 - \delta_{k0})(\delta_{ik} + \delta_{jk})$ and $\sigma_{i, a(k)}(\alpha_{j, a(k)}) = \alpha_{j, i \triangleright a(k)} + \delta_{3, m_{i, j; a(k)}} \alpha_{i, i \triangleright a(k)}$ for $i \neq j$. Moreover

$$\begin{aligned} R_{a(k)}^+ &= \pi_{a(k)} \cup \{\alpha_{i, a(k)} + \alpha_{j, a(k)} | m_{i, j; a(k)} = 3\} \\ &\quad \cup \{\alpha_{1, a(k)} + \alpha_{2, a(k)} + \alpha_{3, a(k)}\} \\ &\quad \cup \{\alpha_{i, a(k)} + 2\alpha_{k, a(k)} + \alpha_{j, a(k)} | m_{i, j; a(k)} = 2\}. \end{aligned}$$

Note that $D(2, 1; 1) = D(2, 1) = \mathfrak{osp}(4|2)$ (see also Section 2.2).

Let $w_{a(2)} := s_{3, a(2)} s_{2, a(0)} s_{3, a(3)} s_{1, a(3)} s_{3, a(0)} s_{2, a(2)} s_{1, a(2)}$. Then $\rho(w_{a(2)}) = -\text{id}_{V_{a(2)}}$. Indeed:

$$\begin{aligned} \alpha_{1, a(2)} &\mapsto -\alpha_{1, a(2)} \mapsto -\alpha_{1, a(0)} - \alpha_{2, a(0)} \mapsto -\alpha_{1, a(3)} - \alpha_{2, a(3)} - 2\alpha_{3, a(3)} \\ &\mapsto -\alpha_{1, a(3)} - \alpha_{2, a(3)} - 2\alpha_{3, a(3)} \mapsto -\alpha_{1, a(0)} - \alpha_{2, a(0)} \mapsto -\alpha_{1, a(2)} \mapsto -\alpha_{1, a(2)}, \\ \alpha_{2, a(2)} &\mapsto \alpha_{1, a(2)} + \alpha_{2, a(2)} \mapsto \alpha_{1, a(0)} \mapsto \alpha_{1, a(3)} + \alpha_{3, a(3)} \mapsto \alpha_{3, a(3)} \mapsto -\alpha_{3, a(0)} \\ &\mapsto -\alpha_{2, a(2)} - \alpha_{3, a(2)} \mapsto -\alpha_{2, a(2)}, \\ \alpha_{3, a(2)} &\mapsto \alpha_{3, a(2)} \mapsto \alpha_{2, a(0)} + \alpha_{3, a(0)} \mapsto \alpha_{2, a(3)} \mapsto \alpha_{2, a(3)} \mapsto \alpha_{2, a(0)} + \alpha_{3, a(0)} \\ &\mapsto \alpha_{3, a(2)} \mapsto -\alpha_{3, a(2)}. \end{aligned}$$

Figure 1. Dynkin diagrams of the Lie superalgebra $D(2, 1; x)$

By Proposition 1.9(1), we have $\ell(w_{a(2)}) = |R_{a(2)}^+| = 7$, $w_{a(2)}$ is the longest word. Let $w' := (s_{3,a(2)}s_{2,a(0)})^{-1}w_{a(2)}$. Then $\ell(w') = 5$. By Theorem 1.7, w' has the following four reduced expressions:

$$\begin{aligned} w' &= \underline{s_{3,a(3)}s_{1,a(3)}s_{3,a(0)}}s_{2,a(2)}s_{1,a(2)} = s_{1,a(1)}s_{3,a(1)}\underline{s_{1,a(0)}s_{2,a(2)}s_{1,a(2)}} \\ &= s_{1,a(1)}\underline{s_{3,a(1)}s_{2,a(1)}}s_{1,a(0)}s_{2,a(2)} = s_{1,a(1)}s_{2,a(1)}s_{3,a(1)}s_{1,a(0)}s_{2,a(2)}. \end{aligned}$$

§ 2. Main theorems—Irreducible representations of the Iwahori-Hecke type algebras $H_q(A(m, n))$, $H_q(B(m, n))$, $H_q(C(n))$ and $H_q(D(m, n))$ associated with the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$

§ 2.1. Definition of Lie superalgebras

As for the terminology concerning Lie superalgebras, we refer to [K].

Let $\mathfrak{v} = \mathfrak{v}(0) \oplus \mathfrak{v}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}$ -linear space. If $i \in \{0, 1\}$ and $j \in \mathbb{Z}$ such that $j - i \in 2\mathbb{Z}$ then let $\mathfrak{v}(j) = \mathfrak{v}(i)$. If $X \in \mathfrak{v}(0)$ (resp. $X \in \mathfrak{v}(1)$) then we write

$$(2.1) \quad \deg(X) = 0 \text{ (resp. } \deg(X) = 1)$$

and we say that X is an *even* (resp. *odd*) element. If $X \in \mathfrak{v}(0) \cup \mathfrak{v}(1)$, then we say that X is a *homogeneous* element and that $\deg(X)$ is the *parity* (or *degree*) of X . If $\mathfrak{w} \subset \mathfrak{v}$ is a subspace and $\mathfrak{w} = (\mathfrak{w} \cap \mathfrak{v}(0)) \oplus (\mathfrak{w} \cap \mathfrak{v}(1))$ (resp. $\mathfrak{w} \subset \mathfrak{v}(0)$, resp. $\mathfrak{w} \subset \mathfrak{v}(1)$), then we say that $\mathfrak{w}$ is a *graded* (resp. *even*, resp. *odd*) subspace.

Let $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}$ -linear space equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ ($i, j \in \mathbb{Z}$); we recall from the above paragraph that

$$(2.2) \quad \mathfrak{g}(i) = \{X \in \mathfrak{g} \mid \deg(X) = i\}.$$

We say that $\mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot])$ is a $(\mathbb{C})$ -Lie superalgebra if for all homogeneous elements X, Y, Z of $\mathfrak{g}$ the following equations hold.

$$[Y, X] = -(-1)^{\deg(X)\deg(Y)}[X, Y], \quad (\text{skew-symmetry})$$

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\deg(X)\deg(Y)}[Y, [X, Z]]. \quad (\text{Jacobi identity})$$

We call the Lie algebra $\mathfrak{g}(0)$ the *even part* of $\mathfrak{g}$.

§ 2.2. Lie superalgebras $\mathfrak{gl}(m+1|n+1)$ and $\mathfrak{osp}(m|n)$

Let $m, n \in \mathbb{N} \cup \{0\}$. Let:

$$\mathcal{D}_{m+1|n+1} := \{(p_1, \dots, p_{m+n+2}) \in \mathbb{Z}^{m+n+2} \mid p_i \in \{0, 1\}, \sum_{i=1}^{m+n+2} p_i = n+1\}.$$

For $i, j \in \{1, \dots, m+n+2\}$, let $\mathbf{E}_{i,j}$ denote the $(m+n+2) \times (m+n+2)$ matrix having 1 in (i, j) position and 0 otherwise, that is, the (i, j) -matrix unit. Let $\mathbf{E}_{m+n+2}$ denote the $(m+n+2) \times (m+n+2)$ unit matrix, that is, $\mathbf{E}_{m+n+2} = \sum_{i=1}^{m+n+2} \mathbf{E}_{i,i}$. Denote by $M_{m+n+2}(\mathbb{C})$ the $\mathbb{C}$ -linear space of the $(m+n+2) \times (m+n+2)$ -matrices, i.e., $M_{m+n+2}(\mathbb{C}) = \oplus_{i,j=1}^{m+n+2} \mathbb{C}\mathbf{E}_{i,j}$.

Let $d = (p_1, \dots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$. The Lie superalgebra $\mathfrak{gl}(m+1|n+1) = \mathfrak{gl}(d)$ is defined by $\mathfrak{gl}(d) = M_{m+n+2}(\mathbb{C})$ (as a $\mathbb{C}$ -linear space),

$$(2.3) \quad \mathfrak{gl}(d)(0) = \oplus_{1 \leq p_i = p_j \leq m+n+2} \mathbb{C}\mathbf{E}_{i,j}, \quad \mathfrak{gl}(d)(1) = \oplus_{1 \leq p_i \neq p_j \leq m+n+2} \mathbb{C}\mathbf{E}_{i,j},$$

and $[X, Y] = XY - (-1)^{r_1 r_2} YX$ for $X \in \mathfrak{gl}(d)(r_1)$ and $Y \in \mathfrak{gl}(d)(r_2)$,

where XY and YX mean the matrix product, that is, $\mathbf{E}_{i,j}\mathbf{E}_{k,l} = \delta_{j,k}\mathbf{E}_{i,l}$. Define the $\mathbb{C}$ -linear map $\text{str} : \mathfrak{gl}(d) \rightarrow \mathbb{C}$ by $\text{str}(\mathbf{E}_{i,j}) = \delta_{i,j}(-1)^{p_i}$. The Lie subsuperalgebra $\{X \in \mathfrak{gl}(d) \mid \text{str}(X) = 0\}$ of $\mathfrak{gl}(d)$ is denoted as $\mathfrak{sl}(m+1|n+1) = \mathfrak{sl}(d)$. The finite dimensional simple Lie superalgebra $A(m, n)$ is defined as follows, where $m+n \geq 1$. Let $\mathfrak{z}$ be the one dimensional ideal $\mathbb{C}\mathbf{E}_{m+n+2}$ of $\mathfrak{gl}(d)$. If $m \neq n$, then $A(m, n)$ means $\mathfrak{sl}(d)$. On the other hand, $A(n, n)$ means $\mathfrak{sl}(d)/\mathfrak{z}$, and is also denoted as $\mathfrak{psl}(n+1|n+1)$.

Let $d = (p_1, \dots, p_{m+2n}) \in \mathcal{D}_{m|2n}$. Define the map $\theta : \{1, \dots, m+2n\} \rightarrow \{1, \dots, m+2n\}$ by $\theta(i) = m+2n+1-i$. Assume that $p_{\theta(i)} = p_i$. Let $g_i \in \{1, -1\}$ be such that $g_i = -1$ if $p_i = 1$ and $i < \theta(i)$ and $g_i = 1$ otherwise. We have an automorphism Ω of

$\mathfrak{gl}(d)$ defined by $\Omega(\mathbf{E}_{i,j}) = -(-1)^{p_i p_j + p_j} g_i g_j \mathbf{E}_{\theta(j), \theta(i)}$. The Lie superalgebra $\mathfrak{osp}(m|2n)$ means $\{X \in \mathfrak{gl}(d) | \Omega(X) = X\}$. We also denote $\mathfrak{osp}(m|2n)$ as follows:

$$\begin{aligned} B(m-1, n) &= \mathfrak{osp}(2m-1|2n) && \text{if } m, n \in \mathbb{N}, \\ D(m+1, n) &= \mathfrak{osp}(2m+2|2n) && \text{if } m, n \in \mathbb{N}, \\ C(n+1) &= \mathfrak{osp}(2|2n) && \text{if } n \in \mathbb{N}. \end{aligned}$$

We also note that $\mathfrak{osp}(2m+1|0)$, $\mathfrak{osp}(0|2n)$, and $\mathfrak{osp}(2m|0)$ are isomorphic to the simple Lie algebras of type B_m (if $m \geq 2$), C_n (if $n \geq 3$) and D_m (if $m \geq 4$) respectively, so $\mathfrak{osp}(2m+1|0) \cong \mathfrak{o}_{2m+1}$, $\mathfrak{osp}(0|2n) \cong \mathfrak{sp}_{2n}$ and $\mathfrak{osp}(2m|0) \cong \mathfrak{o}_{2m}$. As for the even part $\mathfrak{osp}(m|2n)(0)$ of $\mathfrak{osp}(m|2n)$, we have

$$(2.4) \quad \mathfrak{osp}(m|2n)(0) \cong \mathfrak{osp}(m|0) \oplus \mathfrak{osp}(0|2n).$$

§ 2.3. Definition of Iwahori-Hecke type algebras

Definition 2.1. Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the groupoid introduced in (1.1). Assume that A is finite. Let $q \in \mathbb{C}$. Let $H_q(W)$ be the $\mathbb{C}$ -algebra (with 1) generated by

$$(2.5) \quad \{E_a, T_{i,a} | a \in A, i \in N\}$$

and defined by the relations

$$(2.6) \quad E_a^2 = E_a,$$

$$(2.7) \quad E_{i \triangleright a} T_{i,a} E_a = T_{i,a},$$

$$(2.8) \quad \sum_{a \in A} E_a = 1$$

$$(2.9) \quad E_a E_b = 0 \quad \text{if } a \neq b,$$

$$(2.10) \quad (T_{i,a} - qE_a)(T_{i,a} + E_a) = 0 \quad \text{if } i \triangleright a = a,$$

$$(2.11) \quad T_{i, i \triangleright a} T_{i,a} = E_a \quad \text{if } i \triangleright a \neq a,$$

$$(2.12) \quad T_i T_j \cdots T_j T_{i,a} = T_j T_i \cdots T_i T_{j,a} \text{ (} m_{i,j;a} \text{ factors)} \quad \text{if } m_{i,j;a} \text{ is finite and odd,}$$

$$(2.13) \quad T_j T_i \cdots T_j T_{i,a} = T_i T_j \cdots T_i T_{j,a} \text{ (} m_{i,j;a} \text{ factors)} \quad \text{if } m_{i,j;a} \text{ is finite and even,}$$

where, in (2.12)-(2.13), we use the same convention as that of (1.5) with $s_{i,a}$ in place of $T_{i,a}$.

Lemma 2.2. Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the Coxeter groupoid associated with an element $(R, N, A, \triangleright)$ of $\mathcal{R}$ (see Definition 1.4). Assume that A is finite. Then there exists a map $f : W \rightarrow H_q(W)$ such that

$$(2.14) \quad f(0) = 0, \quad f(e_a) = E_a,$$

$$(2.15) \quad f(s_{i,a} w) = T_{i,a} f(w) \quad \text{if } w \in W \setminus \{0\} \text{ and } \ell(s_{i,a} w) = 1 + \ell(w).$$

Further, as a $\mathbb{C}$ -linear space, $H_q(W)$ is spanned by $f(W \setminus \{0\})$. In particular, if W is finite, then

$$(2.16) \quad \dim H_q(W) \leq |W| - 1.$$

Proof. Let $\widetilde{W}$ be the semigroup introduced in Definition 1.6 for W . It is easy to show that there exists a unique semigroup homomorphism $\widetilde{f} : \widetilde{W} \rightarrow H_q(W)$ such that $\widetilde{f}(0) = 0$, $\widetilde{f}(\tilde{e}_a) = E_a$ and $\widetilde{f}(\tilde{s}_{i,a}) = T_{i,a}$. By Theorem 1.7, there exists a unique map $f : W \rightarrow H_q(W)$ such that $f(0) = 0$ and $f(w) = \widetilde{f}(\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m,a})$ if $w \in W \setminus \{0\}$, $\ell(w) = m$ and $w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a}$. Then f satisfies (2.14)-(2.15), as desired.

We show

$$(2.17) \quad \forall w \in W, \forall i \in N, \forall a \in A, T_{i,a}f(w) \in \mathbb{C}f(s_{i,a}w) + \mathbb{C}f(w).$$

If $s_{i,a}w = 0$, then clearly $T_{i,a}f(w) = 0$ holds. If $w \neq 0$, $s_{i,a}w \neq 0$ and $\ell(s_{i,a}w) = 1 + \ell(w)$, then (2.17) follows from (2.15). Assume that $w \neq 0$, $s_{i,a}w \neq 0$ and $\ell(s_{i,a}w) \neq 1 + \ell(w)$. Then by (1.8) and Proposition 1.9, we have $\ell(s_{i,a}w) = \ell(w) - 1$, so $f(w) = T_{i,i \triangleright a}f(s_{i,a}w)$. Since $T_{i,a}f(w) = T_{i,a}T_{i,i \triangleright a}f(s_{i,a}w)$, we have $T_{i,a}f(w) = f(s_{i,a}w)$ if $i \triangleright a \neq a$, and $T_{i,a}f(w) = (q-1)f(w) + qf(s_{i,a}w)$ otherwise. Hence we have (2.17), as desired.

It is clear from (2.17) that the rest of the statement follows. $\square$

Notation. Let $r \in \mathbb{N}$. Let $V_0^{(r)}$ be the r -dimensional $\mathbb{R}$ -linear space with a basis $\{\varepsilon_i | 1 \leq i \leq r\}$. Let $V_0^{(r),\prime}$ be the subspace of $V_0^{(r)}$ formed by the elements $\sum_{i=1}^r x_i \varepsilon_i$ with $x_i \in \mathbb{R}$ and $\sum_{i=1}^r x_i = 0$, so $\dim V_0^{(r),\prime} = r - 1$. For a non-zero element $x = \sum_{i=1}^{[N]} x_i \varepsilon_i$ of $V_0^{(r)}$ with $x_i \in \mathbb{R}$, define $\tilde{\sigma}_x \in \text{GL}(V_0^{(r)})$ by $\tilde{\sigma}_x(\varepsilon_j) = \varepsilon_j - 2x_j(\sum_{i=1}^{[N]} x_i^2)^{-1}x$, that is, $\tilde{\sigma}_x$ is the reflection of $V_0^{(r)}$ with respect to the hyperplane of $V_0^{(r)}$ orthogonal to x . Note that if $x \in V_0^{(r),\prime}$, then $\tilde{\sigma}_x(V_0^{(r),\prime}) = V_0^{(r),\prime}$.

§ 2.4. Basic of Iwahori-Hecke algebras

For the basic facts about the Iwahori-Hecke algebras, we refer to [GU]. Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the groupoid introduced in (1.1). In this subsection we always assume that

$$(2.18) \quad |A| = 1 \text{ and } N \text{ and } W \text{ are finite.}$$

Let $a \in A$, so $A = \{a\}$. Then $W \setminus \{0\}$ is nothing but the Coxeter group associated with the Coxeter system $(W \setminus \{0\}, \{s_{i,a} | i \in N\})$. In this case, we also denote $H_q(W)$ and $T_{i,a}$ by $H_q(W \setminus \{0\})$ and T_i respectively. That is, $H_q(W \setminus \{0\})$ is the $\mathbb{C}$ -algebra (with 1) generated by T_i ($i \in N$) and defined by the relations

$$(2.19) \quad (T_i - q)(T_i + 1) = 0,$$

$$(2.20) \quad \begin{aligned} T_i T_j \cdots T_j T_i &= T_j T_i \cdots T_i T_j \text{ (} m_{i,j;a} \text{ factors)} & \text{if } m_{i,j;a} \text{ is odd,} \\ T_j T_i \cdots T_j T_i &= T_i T_j \cdots T_i T_j \text{ (} m_{i,j;a} \text{ factors)} & \text{if } m_{i,j;a} \text{ is even.} \end{aligned}$$

It is well-known that $\dim H_q(W \setminus \{0\}) = |W \setminus \{0\}|$. In this paper we fix a complete set of non-equivalent irreducible representations of $H_q(W \setminus \{0\})$ by

$$(2.21) \quad \{\rho_{H_q(W \setminus \{0\}), \lambda} : H_q(W \setminus \{0\}) \rightarrow \text{End}_{\mathbb{C}}(V_{H_q(W \setminus \{0\}), \lambda}) \mid \lambda \in \Lambda_{H_q(W \setminus \{0\})}\},$$

where $\Lambda_{H_q(W \setminus \{0\})}$ is an index set. Define the polynomial $P_{W \setminus \{0\}}(q)$ in q by

$$(2.22) \quad P_{W \setminus \{0\}}(q) := \sum_{w \in W \setminus \{0\}} q^{\ell(w)}.$$

This is called the *Poincaré polynomial* of $W \setminus \{0\}$.

It is well-known [GU] (see also [CR, (25.22) and (27.4)]) that for $q \in \mathbb{C} \setminus \{0\}$, the following three conditions are equivalent.

- (i) $P_{W \setminus \{0\}}(q) \neq 0$ holds.
- (ii) $H_q(W \setminus \{0\})$ is a semisimple algebra.
- (iii) The map

$$(2.23) \quad \bigoplus_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} \rho_{H_q(W \setminus \{0\}), \lambda} : H_q(W \setminus \{0\}) \rightarrow \bigoplus_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} \text{End}_{\mathbb{C}}(V_{H_q(W \setminus \{0\}), \lambda})$$

defined by $X \mapsto \bigoplus_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} \rho_{H_q(W \setminus \{0\}), \lambda}(X)$ is a $\mathbb{C}$ -algebra isomorphism.

In particular,

$$(2.24) \quad q \cdot P_{W \setminus \{0\}}(q) \neq 0 \implies \dim H_q(W \setminus \{0\}) = \sum_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} (\dim V_{H_q(W \setminus \{0\}), \lambda})^2.$$

Assume that $N = \{1, 2, \dots, n\}$ and $m_{i, i+1; a} = 3$ and $m_{i, j; a} = 2$ ($|j - i| \geq 2$). Then W is the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $V_0 = V_0^{(n+1)'}$, $R_a^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$, $\alpha_{i, a} = \varepsilon_i - \varepsilon_{i+1}$ and $\sigma_{i, a} = \tilde{\sigma}_{\alpha_{i, a}}$. As a group, $W \setminus \{0\}$ is isomorphic to the symmetric group S_{n+1} , so we also denote $W \setminus \{0\}$ by S_{n+1} by abuse of notation. Note that $\dim H_q(S_{n+1}) = (n+1)!$.

Assume that $N = \{1, 2, \dots, n\}$ and $m_{i, i+1; a} = 3$ ($1 \leq i \leq n-3$), $m_{n-1, n; a} = 4$ and $m_{i, j; a} = 2$ ($|j - i| \geq 2$). Then W is the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $V_0 = V_0^{(n)}$, $R_a^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i \mid 1 \leq i \leq n\}$, $\alpha_{i, a} = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq n-1$), $\alpha_{n, a} = \varepsilon_n$ and $\sigma_{i, a} = \tilde{\sigma}_{\alpha_{i, a}}$. We also denote $W \setminus \{0\}$ by $W(B_n)$ and $W(C_n)$. Note that $\dim H_q(W(B_n)) = 2^n n!$.

Assume that $N = \{1, 2, \dots, n\}$ and $m_{i, i+1; a} = 3$ ($1 \leq i \leq n-2$), $m_{n-1, n; a} = 2$, $m_{n-2, n; a} = 3$ and $m_{i, j; a} = 2$ ($|j - i| \geq 2$ and $1 \leq i \leq n-3$). We also denote $W \setminus \{0\}$ by $W(D_n)$. Note that $\dim H_q(W(D_n)) = 2^{n-1} n!$. Then W is the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $V_0 = V_0^{(n)}$, $R_a^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n\}$, $\alpha_{i, a} = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq n-1$), $\alpha_{n, a} = \varepsilon_{n-1} + \varepsilon_n$ and $\sigma_{i, a} = \tilde{\sigma}_{\alpha_{i, a}}$.

It is well-known (cf. [C, Theorem 10.2.3 and Proposition 10.2.5]) that

$$(2.25) \quad P_{S_{n+1}}(q) = \prod_{r=1}^n \frac{q^{r+1} - 1}{q - 1},$$

$$(2.26) \quad P_{W(B_n)}(q) = \prod_{r=1}^n \frac{q^{2r} - 1}{q - 1},$$

$$(2.27) \quad P_{W(D_n)}(q) = \frac{q^n - 1}{q - 1} \prod_{r=1}^{n-1} \frac{q^{2r} - 1}{q - 1}.$$

§ 2.5. Iwahori-Hecke type algebra $H_q(A(m, n))$ associated with the Lie superalgebra $A(m, n)$

Let

$$\triangleright : S_{m+n+2} \times \mathcal{D}_{m+1|n+1} \rightarrow \mathcal{D}_{m+1|n+1}$$

denote the usual (left) action of the symmetric group S_{m+n+2} on $\mathcal{D}_{m+1|n+1}$ by permutations, that is, for $\sigma \in S_{m+n+2}$,

$$\sigma \triangleright (p_1, \dots, p_{m+n+2}) = (p_{\sigma^{-1}(1)}, \dots, p_{\sigma^{-1}(m+n+2)}).$$

Let $\sigma_i := (i, i+1) \in S_{m+n+2}$. Let W be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $N = \{1, 2, \dots, m+n+1\}$, $A = \mathcal{D}_{m+1|n+1}$, $i \triangleright d = \sigma_i \triangleright d$, $V_0 = V_0^{(m+n+2),'}$, $R_d^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq m+n+1\}$, $\alpha_{i,d} = \varepsilon_i - \varepsilon_{i+1}$ and $\sigma_{i,d} = \tilde{\sigma}_{\alpha_{i,d}}$. Denote $H_q(W)$ by $H_q(A(m, n))$. Then $H_q(A(m, n))$ is the $\mathbb{C}$ -algebra (with 1) generated by

$$(2.28) \quad \{E_d \mid d \in \mathcal{D}_{m+1|n+1}\} \cup \{T_{i,d} \mid 1 \leq i \leq m+n+1, d \in \mathcal{D}_{m+1|n+1}\}$$

and defined by the relations (2.6)-(2.11) and the relations

$$(2.29) \quad T_{i, \sigma_j \sigma_i \triangleright d} T_{j, \sigma_i \triangleright d} T_{i,d} = T_{j, \sigma_i \sigma_j \triangleright d} T_{i, \sigma_j \triangleright d} T_{j,d} \quad \text{if } |i - j| = 1,$$

$$(2.30) \quad T_{i, \sigma_j \triangleright d} T_{j,d} = T_{j, \sigma_i \triangleright d} T_{i,d} \quad \text{if } |i - j| \geq 2.$$

Define $d_e, d_o \in \mathcal{D}_{m+1|n+1}$ by

$$(2.31) \quad d_e := (\overbrace{0, \dots, 0}^{m+1}, \overbrace{1, \dots, 1}^{n+1}), \quad d_o := (\overbrace{1, \dots, 1}^{n+1}, \overbrace{0, \dots, 0}^{m+1}).$$

For $d = (p_1, \dots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$, define the two elements

$$(2.32) \quad \tau_{+,d}, \tau_{-,d} \in S_{m+n+2}$$

by

$$(2.33) \quad p_{\tau_{\pm,d}(i)} = \frac{1 \mp 1}{2} \quad \text{and} \quad \tau_{\pm,d}(i) \leq \tau_{\pm,d}(j) \quad \text{if } 1 \leq i \leq j \leq m+1,$$

$$(2.34) \quad p_{\tau_{\pm,d}(i)} = \frac{1 \pm 1}{2} \quad \text{and} \quad \tau_{\pm,d}(i) \leq \tau_{\pm,d}(j) \quad \text{if } m+2 \leq i \leq j \leq m+n+2.$$

Then $\tau_{+,d}$ (resp. $\tau_{-,d}$) is the minimal length element among the elements $\sigma \in S_{m+n+2}$ satisfying the condition that for any i , i -th component of d_e (resp. d_o) is the same as $\sigma(i)$ -th component $p_{\sigma(i)}$ of d .

Example 2.3. Assume that $m = n = 1$. Then $\mathcal{D}_{2|2} = \{d_e = (0, 0, 1, 1), d_1 = (0, 1, 0, 1), d_2 = (1, 0, 0, 1), d_3 = (0, 1, 1, 0), d_4 = (1, 0, 1, 0), d_o = (1, 1, 0, 0)\}$. Then $\tau_{+,d_e} = \begin{bmatrix} 1234 \\ 1234 \end{bmatrix}$, $\tau_{-,d_e} = \begin{bmatrix} 1234 \\ 3412 \end{bmatrix}$, $\tau_{+,d_1} = \begin{bmatrix} 1234 \\ 1324 \end{bmatrix}$, $\tau_{-,d_1} = \begin{bmatrix} 1234 \\ 2413 \end{bmatrix}$, $\tau_{+,d_2} = \begin{bmatrix} 1234 \\ 2314 \end{bmatrix}$, $\tau_{-,d_2} = \begin{bmatrix} 1234 \\ 1423 \end{bmatrix}$, $\tau_{+,d_3} = \begin{bmatrix} 1234 \\ 1423 \end{bmatrix}$, $\tau_{-,d_3} = \begin{bmatrix} 1234 \\ 2314 \end{bmatrix}$, $\tau_{+,d_4} = \begin{bmatrix} 1234 \\ 2413 \end{bmatrix}$, $\tau_{-,d_4} = \begin{bmatrix} 1234 \\ 1324 \end{bmatrix}$, $\tau_{+,d_o} = \begin{bmatrix} 1234 \\ 3412 \end{bmatrix}$, $\tau_{-,d_o} = \begin{bmatrix} 1234 \\ 1234 \end{bmatrix}$. See also Figure 2.

Now we consider $|W|$. Recall ρ and d_e from Theorem 1.5 and (2.32) respectively. It is easy to see that $P_{d_e}\rho(e_{d_e}We_{d_e})\iota_{d_e} \subset \{(\sum_{i=1}^{m+n+2} \mathbf{E}_{\sigma(i)i})|_{V_0^{(m+n+2),\prime}} | \sigma \in S_{m+n+2}, \sigma(\{1, \dots, m+1\}) = \{1, \dots, m+1\})\}$. Hence $|e_{d_e}We_{d_e}| \leq (m+1)!(n+1)!$ by Theorem 1.5, so $|W \setminus \{0\}| = |\mathcal{D}_{m+1|n+1}|^2 |e_{d_e}We_{d_e}| \leq \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}$. Hence by (2.16), we conclude

$$(2.35) \quad \dim H_q(A(m, n)) \leq \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}.$$

Proposition 2.4. Let V and W be finite dimensional $\mathbb{C}$ -linear spaces, and let $\mathbf{l} : H_q(S_{m+1}) \rightarrow \text{End}_{\mathbb{C}}(V)$ and $\mathbf{r} : H_q(S_{n+1}) \rightarrow \text{End}_{\mathbb{C}}(W)$ be $\mathbb{C}$ -algebra homomorphisms, i.e., representations. Let $\mathbf{l} \otimes \mathbf{r} : H_q(S_{m+1}) \otimes H_q(S_{n+1}) \rightarrow \text{End}_{\mathbb{C}}(V \otimes W)$ denote the tensor representation of $\mathbf{l}$ and $\mathbf{r}$ in the ordinary sense. Let $C_{V \otimes W; d}$ be copies of the $\mathbb{C}$ -linear space $V \otimes W$, indexed by $d \in \mathcal{D}_{m+1|n+1}$. Let $C_{V \otimes W} := \bigoplus_{d \in \mathcal{D}_{m+1|n+1}} C_{V \otimes W; d}$. Let $P_d : C_{V \otimes W} \rightarrow C_{V \otimes W; d}$ and $\iota_d : C_{V \otimes W; d} \rightarrow C_{V \otimes W}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique $\mathbb{C}$ -algebra homomorphism $\mathbf{l} \boxtimes^{A(m,n)} \mathbf{r} : H_q(A(m, n)) \rightarrow \text{End}_{\mathbb{C}}(C_{V \otimes W})$ satisfying the following conditions:

- (i) For each $d \in \mathcal{D}_{m+1|n+1}$, one has $(\mathbf{l} \boxtimes^{A(m,n)} \mathbf{r})(E_d) = \iota_d \circ P_d$,
- (ii) For each $i \in \{1, \dots, m+n+1\}$ and each $d = (p_1, \dots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$, one has

$$(2.36) \quad (\mathbf{l} \boxtimes^{A(m,n)} \mathbf{r})(T_{i,d}) = \begin{cases} P_{\sigma_i \triangleright d} \circ \iota_d & \text{if } p_i \neq p_{i+1}, \\ \iota_d \circ (\mathbf{l}(T_{\tau_{+,d}^{-1}(i)}) \otimes \text{id}_W) \circ P_d & \text{if } p_i = p_{i+1} = 0, \\ \iota_d \circ (\text{id}_V \otimes \mathbf{r}(T_{\tau_{-,d}^{-1}(i)})) \circ P_d & \text{if } p_i = p_{i+1} = 1. \end{cases}$$

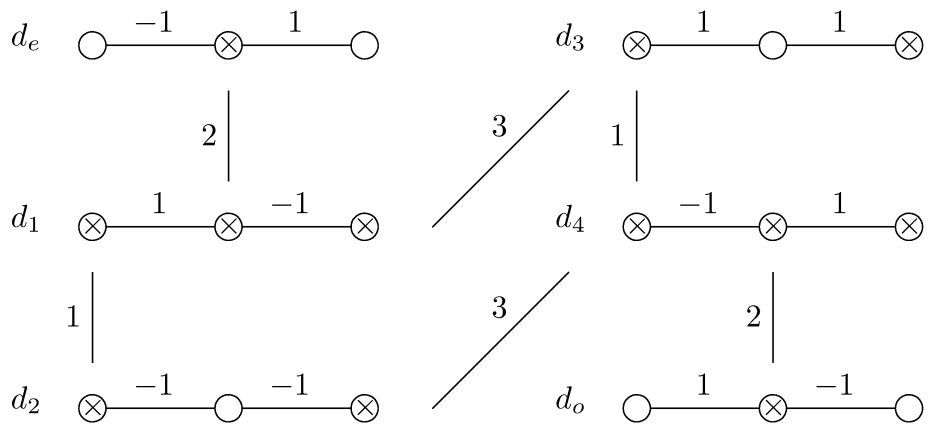
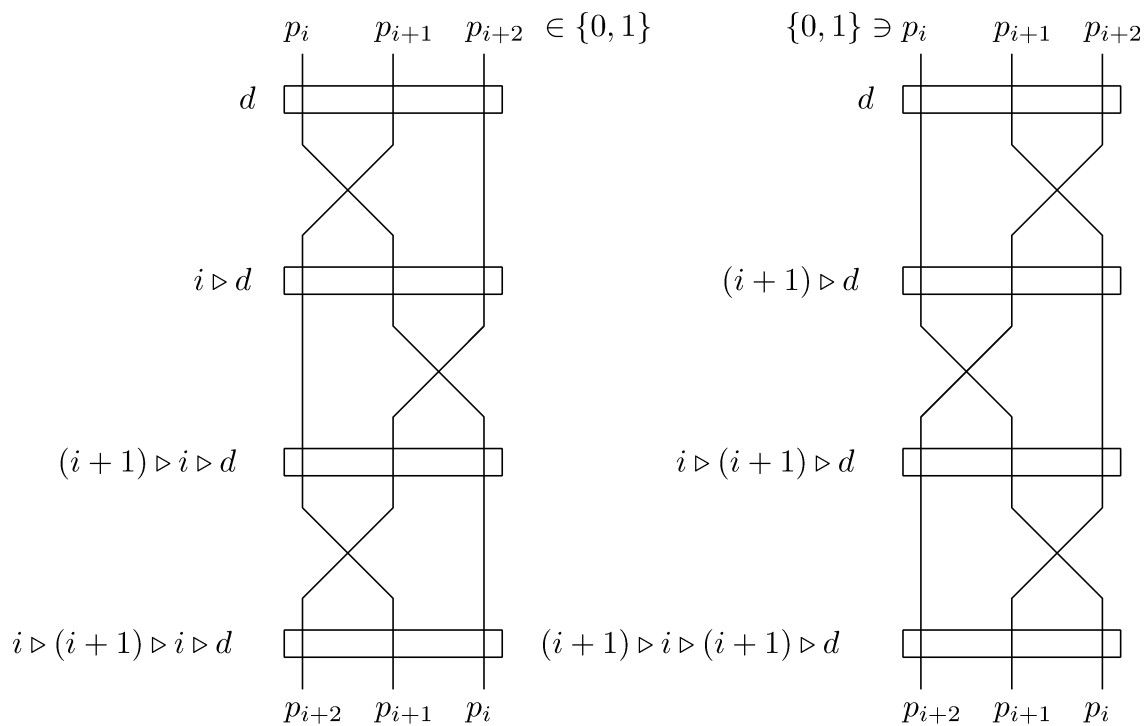
Figure 2. Dynkin diagrams of the Lie superalgebra $A(1,1)$ 

Figure 3. Braid relation

Proof. This can be checked directly. Refer to Figure 3. We explain by using an example. Denote $(1 \boxtimes^{A(m,n)} \mathbf{r})(T_{i',d'})$ by $S_{i',d'}$ for any d' and i' . Let $d = (p_1, \dots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$ and $i \in \{1, \dots, m+n\}$ and assume $p_i = p_{i+1} = 0$ and $p_{i+2} = 1$. Let $d_1 := i \triangleright d (= \sigma_i \triangleright d)$, $d_2 := (i+1) \triangleright d_1$, $d_3 := i \triangleright d_2$, $d_4 := (i+1) \triangleright d$, $d_5 := i \triangleright d_4$ and $d_6 := (i+1) \triangleright d_4$. Then

$$(2.37) \quad d = d_1 = (p_1, \dots, p_{i-1}, 0, 0, 1, p_{i+2}, \dots, p_{m+n+2}),$$

$$(2.38) \quad d_2 = d_4 = (p_1, \dots, p_{i-1}, 0, 1, 0, p_{i+2}, \dots, p_{m+n+2}),$$

$$(2.39) \quad d_3 = d_5 = d_6 = (p_1, \dots, p_{i-1}, 1, 0, 0, p_{i+2}, \dots, p_{m+n+2}).$$

Note that $\tau_{+,d_5} = \sigma_i \sigma_{i+1} \tau_{+,d}$. Hence $\tau_{+,d_5}^{-1}(i+1) = \tau_{+,d}^{-1}(i)$. Then we have $S_{i,d} = \iota_d \circ (\mathbf{1}(T_{\tau_{+,d}^{-1}(i)}) \otimes \text{id}_W) \circ P_d$, $S_{i+1,d_1} = \iota_{d_2} \circ P_d$, $S_{i,d_2} = \iota_{d_3} \circ P_{d_2}$, $S_{i+1,d} = \iota_{d_2} \circ P_d$, $S_{i,d_4} = \iota_{d_3} \circ P_{d_2}$ and $S_{i+1,d_5} = \iota_{d_3} \circ (\mathbf{1}(T_{\tau_{+,d}^{-1}(i)}) \otimes \text{id}_W) \circ P_{d_3}$. Hence we have $S_{i,d_2} S_{i+1,d_1} S_{i,d} = S_{i+1,d_5} S_{i,d_4} S_{i+1,d} = \iota_{d_3} \circ (\mathbf{1}(T_{\tau_{+,d}^{-1}(i)}) \otimes \text{id}_W) \circ P_d$, as desired. $\square$

For $\lambda \in \Lambda_{H_q(S_{m+1})}$ and $\mu \in \Lambda_{H_q(S_{n+1})}$, we denote $\rho_{H_q(S_{m+1}),\lambda} \boxtimes^{A(m,n)} \rho_{H_q(S_{n+1}),\mu}$ by $\rho_{q;\lambda,\mu}^{A(m,n)}$ and we denote $C_{V \otimes W}$, P_d , ι_d for $V = V_{H_q(S_{m+1}),\lambda}$ and $W = V_{H_q(S_{n+1}),\mu}$ by $C_{q;\lambda,\mu}^{A(m,n)}$, $P_d^{\lambda,\mu}$, $\iota_d^{\lambda,\mu}$ respectively.

Theorem 2.5. *Let $q \in \mathbb{C}$ and assume that*

$$(2.40) \quad q P_{S_{m+1}}(q) P_{S_{n+1}}(q) \neq 0.$$

Then the $\mathbb{C}$ -algebra homomorphism

$$(2.41) \quad \bigoplus_{(\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} \rho_{q;\lambda,\mu}^{A(m,n)} : \\ H_q(A(m,n)) \rightarrow \bigoplus_{(\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} \text{End}_{\mathbb{C}}(C_{q;\lambda,\mu}^{A(m,n)})$$

is an isomorphism. Further we have

$$(2.42) \quad \dim H_q(A(m,n)) = \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}.$$

Moreover $H_q(A(m,n))$ is a semisimple $\mathbb{C}$ -algebra and a complete set of non-equivalent irreducible representations of $H_q(A(m,n))$ is given by $\{\rho_{q;\lambda,\mu}^{A(m,n)} | (\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}\}$.

Proof. Define the $\mathbb{C}$ -algebra homomorphism

$$f_1 : H_q(S_{m+1}) \otimes H_q(S_{n+1}) \rightarrow H_q(A(m,n))$$

by $f_1(T_i \otimes 1) = T_{i,e}$ and $f_1(1 \otimes T_j) = T_{m+1+j,e}$. Let

$$R_{\lambda,\mu} := (\iota_{d_e}^{\lambda,\mu} \circ P_{d_e}^{\lambda,\mu}) \text{End}_{\mathbb{C}}(C_{q;\lambda,\mu}^{A(m,n)})(\iota_{d_e}^{\lambda,\mu} \circ P_{d_e}^{\lambda,\mu}).$$

Let f_2 denote the homomorphism of (2.41). It follows from (2.40) that $H_q(S_{m+1}) \otimes H_q(S_{n+1})$ is a semisimple $\mathbb{C}$ -algebra. This implies

$$\text{Im}(f_2 \circ f_1) = \bigoplus_{(\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} R_{\lambda,\mu}.$$

On the other hand, we have

$$\text{End}_{\mathbb{C}}(C_{q;\lambda,\mu}^{A(m,n)}) = \bigoplus_{d_1, d_2 \in \mathcal{D}_{m+1|n+1}} (\iota_{d_1}^{\lambda,\mu} \circ P_{d_1}^{\lambda,\mu}) R_{\lambda,\mu} (\iota_{d_2}^{\lambda,\mu} \circ P_{d_2}^{\lambda,\mu}).$$

Hence by (2.36) we can easily see that f_2 is surjective. In particular, we have

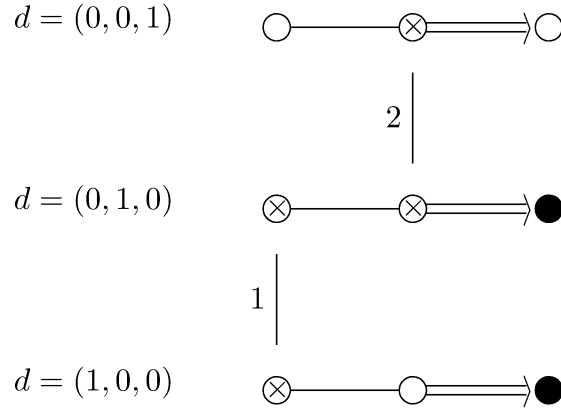
$$\begin{aligned} & \dim H_q(A(m,n)) \\ & \geq \sum_{(\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} |\mathcal{D}_{m+1|n+1}|^2 \dim R_{\lambda,\mu} \\ & = |\mathcal{D}_{m+1|n+1}|^2 \sum_{(\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} \dim R_{\lambda,\mu} \\ & = \left(\frac{(m+n+2)!}{(m+1)!(n+1)!} \right)^2 \\ & \quad \sum_{(\lambda,\mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} (\dim V_{H_q(S_{m+1}),\lambda})^2 (\dim V_{H_q(S_{n+1}),\mu})^2 \\ & = \left(\frac{(m+n+2)!}{(m+1)!(n+1)!} \right)^2 \dim H_q(S_{m+1}) \dim H_q(S_{n+1}) \\ & = \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}. \end{aligned}$$

Hence by (2.35), we have (2.42). Hence f_2 is an isomorphism. Then the rest of the statement follows from well-known facts concerning semisimple algebras (cf. [CR, (25.22) and (27.4)]). $\square$

§ 2.6. Iwahori-Hecke type algebra associated with the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$

Let $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Let $\ell := m+n$. For $1 \leq i \leq \ell$, define $\hat{\sigma}_i \in S_\ell$ by $\hat{\sigma}_i := \sigma_i$ ($1 \leq i \leq \ell-1$), and $\hat{\sigma}_\ell := \text{id}$.

Let W be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $N = \{1, 2, \dots, \ell\}$, $A = \mathcal{D}_{m|n}$, $i \triangleright d = \hat{\sigma}_i \triangleright d$, $V_0 = V_0^{(m+n)}$, $R_d^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq$

Figure 4. Dynkin diagrams of the Lie superalgebra $B(1, 2)$

$\ell\} \cup \{\varepsilon_i | 1 \leq i \leq \ell\}$, $\alpha_{i,d} = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq \ell - 1$), $\alpha_{\ell,d} = \varepsilon_{\ell}$ and $\sigma_{i,d} = \tilde{\sigma}_{\alpha_{i,d}}$. Denote $H_q(W)$ by $H_q(B(m, n))$. Then $H_q(B(m, n))$ is the $\mathbb{C}$ -algebra (with 1) generated by

$$(2.43) \quad \{E_d | d \in \mathcal{D}_{m|n}\} \cup \{T_{i,d} | 1 \leq i \leq \ell, d \in \mathcal{D}_{m|n}\}$$

and defined by the relations (2.6)-(2.11) and the relations

$$(2.44)$$

$$(2.45) \quad T_{\ell-1, \hat{\sigma}_{\ell-1} \triangleright d} T_{\ell, \hat{\sigma}_{\ell-1} \triangleright d} T_{\ell-1, d} T_{\ell, d} = T_{\ell, d} T_{\ell-1, \hat{\sigma}_{\ell-1} \triangleright d} T_{\ell, \hat{\sigma}_{\ell-1} \triangleright d} T_{\ell-1, d}$$

$$(2.46) \quad T_{i, \hat{\sigma}_{i+1} \triangleright d} T_{i+1, \hat{\sigma}_i \triangleright d} T_{i, d} = T_{i+1, \hat{\sigma}_i \hat{\sigma}_{i+1} \triangleright d} T_{i, \hat{\sigma}_{i+1} \triangleright d} T_{i+1, d} \quad \text{if } 1 \leq i \leq \ell - 1,$$

$$(2.46) \quad T_{i, \hat{\sigma}_j \triangleright d} T_{j, d} = T_{j, \hat{\sigma}_i \triangleright d} T_{i, d} \quad \text{if } |i - j| \geq 2.$$

Recall ρ and $d_e \in \mathcal{D}_{m|n}$ from Theorem 1.5 and (2.31) respectively. Then

$$P_{d_e} \rho(e_{d_e} W e_{d_e}) \iota_{d_e} \subset \{(\sum_{i=1}^{m+n} z_i \mathbf{E}_{\sigma(i)i}) | \sigma \in S_{m+n}, z_i \in \{-1, 1\}, \sigma(\{1, \dots, m\}) = \{1, \dots, m\}\}.$$

Hence $|e_{d_e} W e_{d_e}| \leq 2^{m+n} m! n!$ by Theorem 1.5, so $|W \setminus \{0\}| = |\mathcal{D}_{m|n}|^2 |e_{d_e} W e_{d_e}| \leq \frac{2^{m+n} ((m+n)!)^2}{m! n!}$. Hence by (2.16), we conclude

$$(2.47) \quad \dim H_q(B(m, n)) \leq \frac{2^{m+n} ((m+n)!)^2}{m! n!}.$$

Proposition 2.6. *Let $V_{\mathbf{l}}$ and $V_{\mathbf{r}}$ be finite dimensional $\mathbb{C}$ -linear spaces, and let $\mathbf{l} : H_q(W(B_m)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{l}})$ and $\mathbf{r} : H_q(W(B_n)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{r}})$ be $\mathbb{C}$ -algebra homomorphisms, i.e., representations. Let $\mathbf{l} \otimes \mathbf{r} : H_q(W(B_m)) \otimes H_q(W(B_n)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{l}} \otimes V_{\mathbf{r}})$ denote the tensor representation of $\mathbf{l}$ and $\mathbf{r}$ in the ordinary sense. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}, d}$ be copies*

of the $\mathbb{C}$ -linear space $V_1 \otimes V_r$, indexed by $d \in \mathcal{D}_{m|n}$. Let $C_{V_1 \otimes V_r} := \bigoplus_{d \in \mathcal{D}_{m|n}} C_{V_1 \otimes V_r; d}$. Let $P_d : C_{V_1 \otimes V_r} \rightarrow C_{V \otimes W; d}$ and $\iota_d : C_{V \otimes W; d} \rightarrow C_{V_1 \otimes V_r}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique $\mathbb{C}$ -algebra homomorphism $\mathbf{1} \boxtimes \mathbf{r} = \mathbf{1} \boxtimes^{B(m,n)} \mathbf{r} : H_q(B(m, n)) \rightarrow \text{End}_{\mathbb{C}}(C_{V_1 \otimes V_r})$ satisfying the following conditions:

- (i) For each $d \in \mathcal{D}_{m|n}$, one has $(\mathbf{1} \boxtimes \mathbf{r})(E_d) = \iota_d \circ P_d$,
- (ii) For each $i \in \{1, \dots, \ell = m + n\}$ and each $d = (p_1, \dots, p_\ell) \in \mathcal{D}_{m|n}$, one has

(2.48)

$$(\mathbf{1} \boxtimes \mathbf{r})(T_{i,d}) = \begin{cases} P_{\hat{\sigma}_i \triangleright d} \circ \iota_d & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i \neq p_{i+1}, \\ \iota_d \circ (\mathbf{1}(T_{\tau_{+,d}^{-1}(i)}) \otimes \text{id}_{V_r}) \circ P_d & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i = p_{i+1} = 0, \\ \iota_d \circ (\text{id}_{V_1} \otimes \mathbf{r}(T_{\tau_{-,d}^{-1}(i)})) \circ P_d & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i = p_{i+1} = 1, \\ \iota_d \circ (\mathbf{1}(T_m) \otimes \text{id}_{V_r}) \circ P_d & \text{if } i = \ell \text{ and } p_\ell = 0, \\ \iota_d \circ (\text{id}_{V_1} \otimes \mathbf{r}(T_n)) \circ P_d & \text{if } i = \ell \text{ and } p_\ell = 1, \end{cases}$$

where $\tau_{\pm,d}$ are the ones of (2.32).

Proof. We can check out this directly in a way similar to that for Proof of Proposition 2.4. $\square$

For $\lambda \in \Lambda_{H_q(W(B_m))}$ and $\mu \in \Lambda_{H_q(W(B_n))}$, we denote $\rho_{H_q(W(B_m)), \lambda} \boxtimes^{B(m,n)} \rho_{W(B_n), \mu}$ by $\rho_{q; \lambda, \mu}^{B(m,n)}$ and we denote $C_{V \otimes W}$ for $V = V_{H_q(W(B_m)), \lambda}$ and $W = V_{H_q(W(B_n)), \mu}$ by $C_{q; \lambda, \mu}^{B(m,n)}$.

Theorem 2.7. Let $q \in \mathbb{C}$ and assume that

$$(2.49) \quad q P_{W(B_m)}(q) P_{W(B_n)}(q) \neq 0.$$

Then the $\mathbb{C}$ -algebra homomorphism

$$(2.50) \quad \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}} \rho_{q; \lambda, \mu}^{B(m,n)} : H_q(B(m, n)) \rightarrow \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}} \text{End}_{\mathbb{C}}(C_{q; \lambda, \mu}^{B(m,n)})$$

is an isomorphism. Further we have

$$(2.51) \quad \dim H_q(B(m, n)) = \frac{2^{m+n}((m+n)!)^2}{m!n!}.$$

Moreover $H_q(B(m, n))$ is a semisimple $\mathbb{C}$ -algebra and a complete set of non-equivalent irreducible representations of $H_q(B(m, n))$ is given by $\{\rho_{q; \lambda, \mu}^{B(m,n)} | (\lambda, \mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}\}$.

Proof. Let $\mathbf{l} : H_q(W(B_m)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{l}})$ and $\mathbf{r} : H_q(W(B_n)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{r}})$ be irreducible representations. Further, let $\mathbf{l} \boxtimes \mathbf{r} : H_q(B(m, n)) \rightarrow \text{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}})$ be the representation introduced in Proposition 2.6 for these $\mathbf{l}$ and $\mathbf{r}$. By (2.48), we can easily see that

$$(2.52) \quad \forall d', \forall d'' \in \mathcal{D}_{m|n}, \quad P_{d'} \circ \iota_{d''} \in \text{Im}(\mathbf{l} \boxtimes \mathbf{r}).$$

Define the representation $f_1 : H_q(W(B_m)) \otimes H_q(W(B_n)) \rightarrow \text{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}; d_e})$ by $f_1(T_i \otimes 1) = (P_{d_e} \circ \iota_{d_o})(\mathbf{l} \boxtimes \mathbf{r})(T_{n+i, d_o})(P_{d_o} \circ \iota_{d_e})$ and $f_1(1 \otimes T_j) = (\mathbf{l} \boxtimes \mathbf{r})(T_{m+j, d_e})$. The condition (2.49) implies that f_1 is an irreducible representation of $H_q(W(B_m)) \otimes H_q(W(B_n))$. Moreover, using (2.52), we can easily see that $\mathbf{l} \boxtimes \mathbf{r}$ is an irreducible representation of $H_q(B(m, n))$.

By the above argument, together with (2.47), in the same way as that for Proof of Theorem 2.5, we can complete the proof of this theorem. $\square$

§ 2.7. Iwahori-Hecke type algebra associated with the Lie superalgebra $\mathfrak{osp}(2m|2n)$

Let $m, n \in \mathbb{N}$. Define the set $\mathcal{D}_{m|n}^{CD}$ by

$$(2.53) \quad \begin{aligned} \mathcal{D}_{m|n}^{CD} := & \{d^D | d = (p_1, \dots, p_{m+n}) \in \mathcal{D}_{m|n}, p_{m+n} = 0\} \\ & \cup \{d_+^C, d_-^C | d = (p_1, \dots, p_{m+n}) \in \mathcal{D}_{m|n}, p_{m+n} = 1\}, \end{aligned}$$

so that

$$(2.54) \quad |\mathcal{D}_{m|n}^{CD}| = \frac{(m+n-1)!}{(m-1)!n!} + 2 \frac{(m+n-1)!}{m!(n-1)!} = \frac{(m+n-1)!(m+2n)}{m!n!}.$$

Let $\ell := m+n$ and $N = \{1, \dots, \ell\}$. Define the action $\triangleright$ of $F_2(N)$ on $\mathcal{D}_{m|n}^{CD}$ by

$$(2.55) \quad i \triangleright a = \begin{cases} (\sigma_i \triangleright d)^D & \text{if } a = d^D, 1 \leq i \leq \ell-2 \text{ and } p_i \neq p_{i+1}, \\ (\sigma_i \triangleright d)_+^C & \text{if } a = d^D, i = \ell-1 \text{ and } p_i \neq p_{i+1}, \\ (\sigma_{i-1} \triangleright d)_-^C & \text{if } a = d^D, i = \ell \text{ and } p_{i-1} \neq p_i, \\ (\sigma_i \triangleright d)_{\pm}^C & \text{if } a = d_{\pm}^C, 1 \leq i \leq \ell-2 \text{ and } p_i \neq p_{i+1}, \\ (\sigma_i \triangleright d)^D & \text{if } a = d_+^C, i = \ell-1 \text{ and } p_i \neq p_{i+1}, \\ (\sigma_{i-1} \triangleright d)^D & \text{if } a = d_-^C, i = \ell \text{ and } p_{i-1} \neq p_i, \\ a & \text{otherwise.} \end{cases}$$

Now we define $R = (R, N, \mathcal{D}_{m|n}^{CD}, \triangleright) \in \mathcal{R}$ as follows. Let N be as above. Let $A = \mathcal{D}_{m|n}^{CD}$. Let $V_0 = V_0^{(\ell)}$. Let $a = d^D, d_+^C$ or $d_-^C \in \mathcal{D}_{m|n}^{CD}$ with $d = (p_1, \dots, p_{m+n}) \in \mathcal{D}_{m|n}$. Let

R_a^+ be the subset of V_0 formed by the elements $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, ($1 \leq i < j \leq \ell$) and $(\pm 1)^{\delta_{k\ell}} 2\varepsilon_k$ ($1 \leq k \leq \ell$ and $p_k = 1$, and $\pm$ is the one of $d_{\pm}^C$). Define

$$(2.56) \quad \alpha_{i,a} := \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } a = d^D \text{ or } d_+^C \text{ and } 1 \leq i \leq \ell - 1, \\ \varepsilon_i - \varepsilon_{i+1} & \text{if } a = d_-^C \text{ and } 1 \leq i \leq \ell - 2, \\ \varepsilon_{\ell-1} + \varepsilon_{\ell} & \text{if } a = d^D \text{ and } i = \ell, \\ 2\varepsilon_{\ell} & \text{if } a = d_+^C \text{ and } i = \ell, \\ -2\varepsilon_{\ell} & \text{if } a = d_-^C \text{ and } i = \ell - 1, \\ \varepsilon_{\ell-1} + \varepsilon_{\ell} & \text{if } a = d_-^C \text{ and } i = \ell. \end{cases}$$

Define $\sigma_{i,a} := \tilde{\sigma}_{\alpha_{i,a}}$. Let W be the Coxeter groupoid associated with R . Recall ρ and $d_e \in \mathcal{D}_{m|n}$ from Theorem 1.5 and (2.31) respectively. It is easy to show that

$$\rho(e_{(d_e)^D} W e_{(d_e)^D}) = \left\{ \sum_{j=1}^{\ell} z_j \mathbf{E}_{\sigma(j)_j} \mid \sigma \in S_{\ell}, z_j \in \{-1, 1\}, \prod_{j=n+1}^{\ell} z_j = 1, \sigma(\{1, \dots, n\}) = \{1, \dots, n\} \right\},$$

so $|e_{(d_e)^D} W e_{(d_e)^D}| \leq m!n!2^{\ell-1}$ by Theorem 1.5. Hence $|W \setminus \{0\}| \leq |\mathcal{D}_{m|n}^{CD}|^2 m!n!2^{\ell-1}$. Denote $H_q(W)$ by $H_q(\mathfrak{osp}(2m|2n))$. By (2.16) and (2.54), we have

$$(2.57) \quad \dim H_q(\mathfrak{osp}(2m|2n)) \leq \frac{2^{m+n-1}((m+n-1)!(m+2n))^2}{m!n!}.$$

Recall that $H_q(\mathfrak{osp}(2m|2n))$ is the $\mathbb{C}$ -algebra (with 1) generated by

$$(2.58) \quad \{E_a \mid a \in \mathcal{D}_{m|n}^{CD}\} \cup \{T_{i,a} \mid 1 \leq i \leq m+n, a \in \mathcal{D}_{m|n}^{CD}\}$$

and defined by the relations (2.6)-(2.11) and the relations

$$(2.59) \quad (T_{i,a} T_{j,a})^2 = (T_{j,a} T_{i,a})^2 \text{ if } a = d_{\pm}^C, p_{\ell-1} = p_{\ell} \text{ and } i = \ell - 1, j = \ell,$$

$$(2.60) \quad T_{i,a} T_{j,a} = T_{j,a} T_{i,a} \text{ if } a = d^D, p_{\ell-1} = p_{\ell} \text{ and } i = \ell - 1, j = \ell,$$

$$(2.61) \quad T_{i,j \triangleright d} T_{j,i \triangleright d} T_{i,d} = T_{j,i j \triangleright d} T_{i,j \triangleright d} T_{j,d} \text{ if } p_{\ell-1} \neq p_{\ell} \text{ and } i = \ell - 1, j = \ell,$$

$$(2.62) \quad T_{i,j \triangleright d} T_{j,i \triangleright d} T_{i,d} = T_{j,i j \triangleright d} T_{i,j \triangleright d} T_{j,d} \quad \text{if } 1 \leq i \leq \ell - 3, j = i + 1,$$

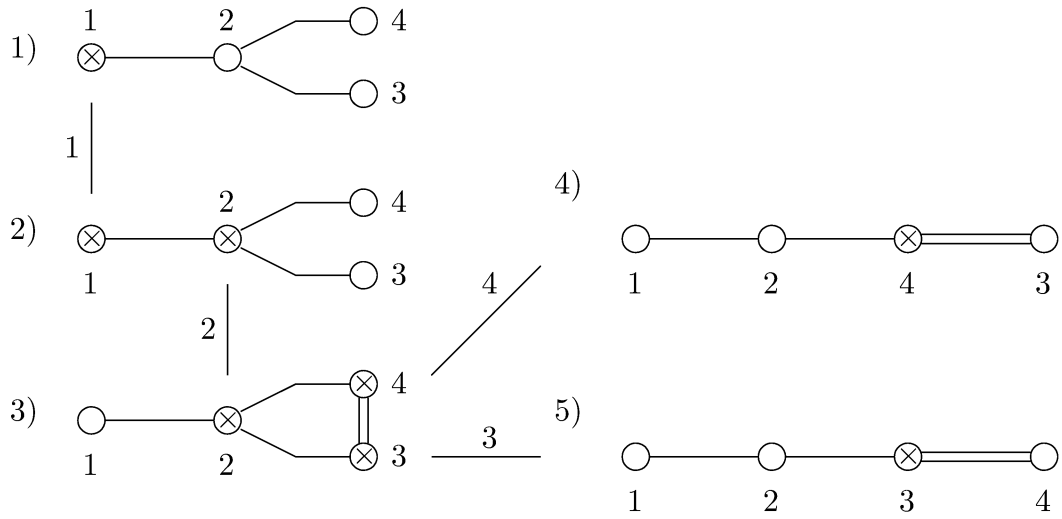
$$(2.63) \quad T_{i,j \triangleright d} T_{j,i \triangleright d} T_{i,d} = T_{j,i j \triangleright d} T_{i,j \triangleright d} T_{j,d} \quad \text{if } a = d_+^C \text{ and } i = \ell - 2, j = \ell - 1,$$

$$(2.64) \quad T_{i,j \triangleright d} T_{j,i \triangleright d} T_{i,d} = T_{j,i j \triangleright d} T_{i,j \triangleright d} T_{j,d} \quad \text{if } a = d_-^C \text{ and } i = \ell - 2, j = \ell,$$

$$(2.65) \quad T_{i,j \triangleright d} T_{j,i \triangleright d} T_{i,d} = T_{j,i j \triangleright d} T_{i,j \triangleright d} T_{j,d} \text{ if } a = d^D \text{ and } i = \ell - 2, j \in \{\ell - 1, \ell\},$$

$$(2.66) \quad T_{j,i \triangleright d} T_{i,d} = T_{i,j \triangleright d} T_{j,d} \text{ if } i < j, \text{ and } i, j \text{ are not the ones in (2.64)-(2.65).}$$

Recall that $W(C_k) = W(B_k)$ and $H_q(W(C_k)) = H_q(W(B_k))$.



where 1) $(1, 0, 0, 0)^D$, 2) $(0, 1, 0, 0)^D$, 3) $(0, 0, 1, 0)^D$, 4) $(0, 0, 0, 1)^C_-$, 5) $(0, 0, 0, 1)^C_+$

Figure 5. Dynkin diagrams of the Lie superalgebra $D(3, 1)$

Proposition 2.8. *Let $V_{\mathbf{l}}$ and $V_{\mathbf{r}}$ be finite dimensional $\mathbb{C}$ -linear spaces, and let $\mathbf{l} : H_q(W(D_m)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{l}})$ and $\mathbf{r} : H_q(W(C_n)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{r}})$ be $\mathbb{C}$ -algebra homomorphisms, i.e., representations. Let $\mathbf{l} \otimes \mathbf{r} : H_q(W(D_m)) \otimes H_q(W(C_n)) \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbf{l}} \otimes V_{\mathbf{r}})$ denote the tensor representation of $\mathbf{l}$ and $\mathbf{r}$ in the ordinary sense. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};a}$ be copies of the $\mathbb{C}$ -linear space $V_{\mathbf{l}} \otimes V_{\mathbf{r}}$, indexed by $a \in \mathcal{D}_{m|n}^{CD}$. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}} := \bigoplus_{d \in \mathcal{D}_{m|n}^{CD}} C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};d}$. Let $P_a : C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}} \rightarrow C_{V \otimes W;a}$ and $\iota_a : C_{V \otimes W;a} \rightarrow C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique $\mathbb{C}$ -algebra homomorphism $\mathbf{l} \boxtimes \mathbf{r} = \mathbf{l} \boxtimes^{CD} \mathbf{r} : H_q(\mathfrak{osp}(2m|2n)) \rightarrow \text{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}})$ satisfying the following conditions:*

- (i) *For each $a \in \mathcal{D}_{m|n}^{CD}$, one has $(\mathbf{l} \boxtimes \mathbf{r})(E_a) = \iota_a \circ P_a$.*
- (ii) *For each $i \in \{1, \dots, \ell = m+n\}$ and each $a \in \mathcal{D}_{m|n}^{CD}$ with $d = (p_1, \dots, p_{\ell}) \in \mathcal{D}_{m|n}$ such that $a = d_+^C, d_-^C$ or d^D , one has*

$$(2.67) \quad (\mathbf{l} \boxtimes \mathbf{r})(T_{i,a}) = \begin{cases} P_{i \triangleright a} \circ \iota_a & \text{if } 1 \leq i \leq \ell \text{ and } i \triangleright a \neq a, \\ \iota_a \circ (\mathbf{l}(T_{\tau_{+,d}^{-1}(i)}) \otimes \text{id}_{V_{\mathbf{r}}}) \circ P_a & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i = p_{i+1} = 0, \\ \iota_a \circ (\mathbf{l}(T_m) \otimes \text{id}_{V_{\mathbf{r}}}) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell} = 0, \\ \iota_a \circ (\text{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{\tau_{-,d}^{-1}(i)})) \circ P_a & \text{if } 1 \leq i \leq \ell - 2 \text{ and } p_i = p_{i+1} = 1, \\ \iota_a \circ (\text{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{n-1})) \circ P_a & \text{if } i = \ell - 1 \text{ and } p_{\ell-1} = p_{\ell} = 1, a = d_+^C. \\ \iota_a \circ (\text{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_n)) \circ P_a & \text{if } i = \ell - 1 \text{ and } p_{\ell} = 1, a = d_-^C. \\ \iota_a \circ (\text{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_n)) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell} = 1, a = d_+^C. \\ \iota_a \circ (\text{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{n-1})) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell-1} = p_{\ell} = 1, a = d_-^C, \end{cases}$$

where $\tau_{\pm,d} \in S_{m+n}$ are the ones of (2.32).

Proof. We can check out this directly in a way similar to that for Proof of Proposition 2.4. $\square$

For $\lambda \in \Lambda_{H_q(W(D_m))}$ and $\mu \in \Lambda_{H_q(W(C_n))}$, we denote $\rho_{H_q(W(D_m)),\lambda} \boxtimes^{CD} \rho_{W(C_n),\mu}$ by $\rho_{q;\lambda,\mu}^{CD}$ and we denote $C_{V \otimes W}$ for $V = V_{H_q(W(D_m)),\lambda}$ and $W = V_{H_q(W(C_n)),\mu}$ by $C_{q;\lambda,\mu}^{CD}$.

Theorem 2.9. *Let $q \in \mathbb{C}$ and assume that*

$$(2.68) \quad qP_{W(D_m)}(q)P_{W(C_n)}(q) \neq 0.$$

Then the $\mathbb{C}$ -algebra homomorphism

$$(2.69) \quad \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(D_m))} \times \Lambda_{H_q(W(C_n))}} \rho_{q; \lambda, \mu}^{CD} : \\ H_q(\mathfrak{osp}(2m|2n)) \rightarrow \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(D_m))} \times \Lambda_{H_q(W(C_n))}} \text{End}_{\mathbb{C}}(C_{q; \lambda, \mu}^{CD})$$

is an isomorphism. Further we have

$$(2.70) \quad \dim H_q(\mathfrak{osp}(2m|2n)) = \frac{2^{m+n-1}((m+n-1)!(m+2n))^2}{m!n!}.$$

Moreover $H_q(\mathfrak{osp}(2m|2n))$ is a semisimple $\mathbb{C}$ -algebra and a complete set of non-equivalent irreducible representations of $H_q(\mathfrak{osp}(2m|2n))$ is given by $\{\rho_{q; \lambda, \mu}^{CD} | (\lambda, \mu) \in \Lambda_{H_q(W(D_m))} \times \Lambda_{H_q(W(C_n))}\}$.

Proof. Note that $W(D_m) \times W(C_n) = 2^{m+n-1}m!n!$. Then we can prove this theorem in the same way as that for Proof of Theorem 2.7. $\square$

Remark 1. Now, by (2.3), (2.4) and Theorems 2.5, 2.7 and 2.9, it has turned out that if q is non-zero and not any primitive root of unity, then as a $\mathbb{C}$ -algebra, $H_q(\mathfrak{g}) = H_q(W)$ introduced in this section for the Lie superalgebra $\mathfrak{g} = A(m, n)$ or $\mathfrak{osp}(m|2n)$ is very similar to the Iwahori-Hecke algebra $H_q(W_0)$ associated with the Weyl group W_0 of the Lie algebra $\mathfrak{g}(0)$ given as the even part of $\mathfrak{g}$, that is, Morita equivalence.

Remark 2. Assume q to be an element of $\mathbb{C}$ transcendental over $\mathbb{Q}$. Then the $\mathbb{Z}$ -subalgebra (with identity) of $\mathbb{C}$ generated by q can also be regarded as the polynomial ring $\mathbb{Z}[q]$ in the variable q over $\mathbb{Z}$. Let W be one of the Coxeter groupoids treated in Subsections 2.5, 2.6 and 2.7. By Lemma 2.2 and (2.42), (2.51), (2.70), one see that $\{f(w) | w \in W \setminus \{0\}\}$ is a $\mathbb{C}$ -basis of $H_q(W)$, that is, $H_q(W) = \bigoplus_{w \in W \setminus \{0\}} \mathbb{C}f(w)$. Define $H_{\mathbb{Z}[q], q}(W)$ to be the $\mathbb{Z}[q]$ -submodule generated by $\{f(w) | w \in W \setminus \{0\}\}$. Using Theorem 1.7 and Corollary 1.8, one see that $H_{\mathbb{Z}[q], q}(W)$ is also a $\mathbb{Z}[q]$ -subalgebra of $H_q(W)$. Let $\mathbb{A}$ be any commutative ring (with identity). Let ζ be any element of $\mathbb{A}$. Regard $\mathbb{A}$ as a $\mathbb{Z}[q]$ -algebra via the $\mathbb{Z}$ -algebra homomorphism $\mathbb{Z}[q] \rightarrow \mathbb{A}$ that takes q to ζ . Let $H_{\mathbb{A}, \zeta}(W)$ be the $\mathbb{A}$ -algebra (with identity) defined by $H_{\mathbb{A}, \zeta}(W) := H_{\mathbb{Z}[q], q}(W) \otimes_{\mathbb{Z}[q]} \mathbb{A}$. For $X \in H_{\mathbb{Z}[q], q}(W)$, we also denote the element $X \otimes 1$ of $H_{\mathbb{A}, \zeta}(W)$ by X . Then $H_{\mathbb{A}, \zeta}(W)$ is a free $\mathbb{A}$ -module with an $\mathbb{A}$ -basis $\{f(w) | w \in W \setminus \{0\}\}$, that is,

$$(2.71) \quad \text{rank}_{\mathbb{A}} H_{\mathbb{A}, \zeta}(W) = |W| - 1.$$

Using Theorem 1.7 and Corollary 1.8 again, one see that $H_{\mathbb{A}, \zeta}(W)$ can also be defined by the same generators as (2.5) and the same relations as (2.6)-(2.13) with ζ in place of q .

The same properties as above seem to be true for many Coxeter groupoids, which might be able to be proved in a way similar to that of the proof of [L, Proposition 3.3].

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