

Distributions of exponential growth with support in a proper convex cone

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1 Introduction

In this talk we treated the space $H'(\mathbb{R}^n, K)$ of distributions of exponential growth. The spaces of distributions of exponential growth for the 1-dimensional case, direct product case or global case were investigated by many authors ([3], [5], [6], [10], [11], [12], [13], [15], [17]). In [3] M.Hasumi studied the space $H(\mathbb{R}^n, \mathbb{R}^n)$ and the dual space $H'(\mathbb{R}^n, \mathbb{R}^n)$ (see Definition 3.2 and Definition 3.5). In [10] M.Morimoto studied the space $H(\mathbb{R}^n, K)$ and the dual space $H'(\mathbb{R}^n, K)$ (see Definition 3.2 and Definition 3.5). The purpose of this talk was to treat the space of distributions of exponential growth supported by a proper convex cone $\bar{\Gamma} \subset \mathbb{R}^n$, (denote by $H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$).

In §3 we shall state the base space $H(\mathbb{R}^n, K)$ and its dual space $H'(\mathbb{R}^n, K)$. The main purpose in this section is to introduce the structure theorem for $H'_A(\mathbb{R}^n, K)$, the space of distributions of exponential growth supported by a set $\bar{A} \subset \mathbb{R}^n$ (Theorem 3.7). Therefore as corollary we obtain the structure theorem for $H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$, where $\bar{\Gamma} \subset \mathbb{R}^n$ is a proper convex cone, (Corollary 3.8), and the result which G.Lysik obtained for the case of direct product support of half lines ([6]). Furthermore we have the decomposition theorem for distributions of exponential growth with support in $\bar{\Gamma}_+ \cup \bar{\Gamma}_-$, (Corollary 3.10).

In §4 we shall characterize the space $H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$ by the heat kernel method (Theorem 4.1), which T.Matsuzawa introduced for the spaces of distributions, ultradistributions and hyperfunctions [2], [7], [8], [9].

In §5 we shall introduce the Paley-Wiener theorem for $H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$. Then we showed that the Fourier-Laplace transform of $T \in H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$ is a holomorphic function constructed by a finite sum of functions which are holomorphic on the domains whose imaginary parts are proper convex cones with

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vertex at the elements of K and with some polynomial growth conditions and conversely such a holomorphic function can be represented by the Fourier-Laplace transform of a distribution of exponential growth $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then we can see that T is constructed by a finite sum of distributions of exponential growth supported by a proper convex cone $\overline{\Gamma}$ (Theorem 5.5). As corollary we have the result which M.Morimoto showed for the 1-dimensional case [10].

In §6 we shall state the space of the image by the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then by using the Paley-Wiener theorem given in §5, we can obtain the Edge-of-the-Wedge theorem for this space (Theorem 6.10). These results are generalizations of the work which M.Morimoto showed for the case of direct product ([11], Theorem 2).

2 Preliminaries

Definition 2.1. We define some notations:

$$\begin{aligned} x &= (x_1, \dots, x_n) \in \mathbb{R}^n, \quad \langle x, \xi \rangle = \sum_{j=1}^n x_j \xi_j \quad \text{for } x, \xi \in \mathbb{R}^n, \quad x^2 = \langle x, x \rangle. \\ z &= (z_1, \dots, z_n) \in \mathbb{C}^n, \quad z_j = x_j + iy_j, \quad j = 1, \dots, n. \\ \alpha &= (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}. \end{aligned}$$

$$E(x, t) = (4\pi t)^{-\frac{n}{2}} \exp(-x^2/4t), \quad t > 0.$$

For $\zeta \in \mathbb{C}^n$, $\zeta = (\zeta_1, \dots, \zeta_n)$, we put $|\zeta| = \sqrt{|\zeta_1|^2 + \dots + |\zeta_n|^2}$.

Definition 2.2. Let K be a convex compact set in $\mathbb{R}^n$. Then we define supporting function of K by $h_K(x) = \sup_{\xi \in K} \langle x, \xi \rangle$.

Definition 2.3. Let Ω be an open set in $\mathbb{C}^n$. We denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions on Ω and by $\mathcal{C}(\Omega)$ the space of continuous functions on Ω .

Definition 2.4. $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Definition 2.5. Let A be a set in $\mathbb{R}^n$. Then we denote by A° the interior of A , $\bar{A}$ the closure of A , for $\varepsilon > 0$, $A_\varepsilon = \{x \in \mathbb{R}^n; \text{dis}(x, A) \leq \varepsilon\}$ and by $\text{ch}(A)$ convex hull of A .

Definition 2.6. Let Γ be a cone with vertex at 0. If $\overline{\text{ch}\Gamma}$ contains no straight line, then we call Γ proper cone.

Definition 2.7 ([4],[16]). Let Γ be a cone. We put

$$\Gamma' := \{\xi \in \mathbb{R}^n; \langle y, \xi \rangle \geq 0 \text{ for all } y \in \Gamma\}.$$

Then we call Γ' dual cone of Γ .

Definition 2.8. Let Γ be a cone. Then we denote by $\text{pr}\Gamma$ the intersection of Γ and the unit sphere. The cone Γ_1 is said to be a compact cone in the cone Γ_2 if $\text{pr}\bar{\Gamma}_1 \subset \text{pr}\Gamma_2$ and we write $\Gamma_1 \Subset \Gamma_2$.

Proposition 2.9 ([16]). *Following conditions are equivalent:*

1. Γ is proper cone.
2. $(\Gamma')^\circ \neq \emptyset$.
3. For any $C \Subset (\Gamma')^\circ$, there exists a number $\sigma = \sigma(C) > 0$ such that $\langle \xi, x \rangle \geq \sigma \|\xi\| \|x\|$, $\xi \in C$, $x \in \text{ch}\bar{\Gamma}$.

Proposition 2.10 ([16]). $(\Gamma')' = \overline{\text{ch}\Gamma}$ and $(\Gamma_1 \cap \Gamma_2)' = \text{ch}(\Gamma'_1 \cup \Gamma'_2)$. Furthermore for a convex cone Γ , we have $\Gamma = \Gamma + \Gamma$.

Definition 2.11. Let Γ_+ be a cone with vertex at 0. Then we put $\Gamma_- = -\Gamma_+$.

Definition 2.12. Let A be a set in $\mathbb{R}^n$. We put $\mathcal{S}'_A := \{T \in \mathcal{S}'(\mathbb{R}^n); \text{supp } T \subset \bar{A}\}$.

3 Distributions of exponential growth

In this section, we shall introduce $H'(\mathbb{R}^n, K)$, the space of distributions of exponential growth, and give the structure theorem of $H'(\mathbb{R}^n, K)$.

Definition 3.1. Let K be a convex compact set in $\mathbb{R}^n$ and $\varepsilon > 0$. Then we define $H_b(\mathbb{R}^n, K_\varepsilon)$ as follows:

$$H_b(\mathbb{R}^n, K_\varepsilon) := \{\varphi \in C^\infty(\mathbb{R}^n); \sup_{x \in \mathbb{R}^n} |D^p \varphi(x) e^{h_K(x) + \varepsilon|x|}| < +\infty, \text{ for } \forall p \in \mathbb{N}^n\}.$$

Definition 3.2. We define the spaces $H(\mathbb{R}^n, \mathbb{R}^n)$ and $H(\mathbb{R}^n, K)$ as follows:

$$H(\mathbb{R}^n, \mathbb{R}^n) := \varprojlim_{\varepsilon > 0} H_b(\mathbb{R}^n, K_\varepsilon), \quad H(\mathbb{R}^n, K) := \varinjlim_{\varepsilon > 0} H_b(\mathbb{R}^n, K_\varepsilon),$$

where $\varprojlim_{\varepsilon > 0}$ means projective limit and $\varinjlim_{\varepsilon > 0}$ means inductive limit.

Remark 3.3. Now we give the relations of $H(\mathbb{R}^n, K)$ and the other function spaces:

- (i) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}$.
- (ii) Let $r \geq 0$, $s \geq 0$, $\mathcal{S}_r^s(\mathbb{R}^n)$ be Gel'fand-Shilov space and $\mathcal{S}_r(\mathbb{R}^n) = \varinjlim_{s \rightarrow \infty} \mathcal{S}_r^s(\mathbb{R}^n)$. Then it is known that

$$\mathcal{S}_1(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n); \exists \delta > 0 \forall \alpha \sup_{x \in \mathbb{R}^n} |D_x^\alpha f(x)| e^{\delta|x|} < \infty\},$$

(for details we refer the reader [12]). Therefore

- (a) If $K = \{0\}$, then $H(\mathbb{R}^n, K) = \mathcal{S}_1(\mathbb{R}^n)$.
- (b) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}_1(\mathbb{R}^n)$.
- (iii) The space $H(\mathbb{R}^n, K)$ is slightly different from $\mathfrak{A}_E$ in [1]. In fact

$$\varphi(x) \in H(\mathbb{R}^n, K) \Leftrightarrow \exists \varepsilon > 0 \forall p \in \mathbb{N}^n \text{ s.t. } \sup_{x \in \mathbb{R}^n} |D^p \varphi(x) e^{h_K(x) + \varepsilon|x|}| < \infty.$$

$$\varphi(x) \in \mathfrak{A}_E \Leftrightarrow \forall p \in \mathbb{N}^n \exists k > 0 \text{ s.t. } \sup_{x \in \mathbb{R}^n} |D^p \varphi(x)| e^{k|x|} < \infty.$$

Therefore if $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathfrak{A}_E$.

Remark 3.4. L.Hörmander treated the base space $\mathcal{S}_f$ so that $\mathcal{D} \subset \mathcal{S}_f \subset H(\mathbb{R}^n, K)$ and the Fourier-Laplace transform of $\mathcal{S}_f$. For the details we refer the reader to [5].

Definition 3.5. We denote by $H'(\mathbb{R}^n, \mathbb{R}^n)$ the dual space of $H(\mathbb{R}^n, \mathbb{R}^n)$ and by $H'(\mathbb{R}^n, K)$ the dual space of $H(\mathbb{R}^n, K)$. The elements of $H'(\mathbb{R}^n, \mathbb{R}^n)$ and $H'(\mathbb{R}^n, K)$ are called distributions of exponential growth.

Definition 3.6. We put $H'_A(\mathbb{R}^n, K) := \{T \in H'(\mathbb{R}^n, K); \text{supp } T \subset \bar{A}\}$.

Now we have the structure theorem for distributions of exponential growth with support $\bar{A} \subset \mathbb{R}^n$:

Theorem 3.7 ([14]). Let A be a set in $\mathbb{R}^n$ and $T \in H'_A(\mathbb{R}^n, K)$. Then for every $\varepsilon > 0$ there exist $S(x) \in \mathcal{S}'_A$, $n_0 \in \mathbb{N}$ and $t_j \in K$, $j = 1, 2, \dots, n_0$ such that

$$T = S(x)e^{\varepsilon\sqrt{1+x^2}} \sum_{1 \leq j \leq n_0} e^{t_j x}.$$

For $H'_\Gamma(\mathbb{R}^n, K)$, we have the following corollaries:

Corollary 3.8 ([14]). Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_\Gamma(\mathbb{R}^n, K)$. Then for any $\varepsilon > 0$ there exist $m_\varepsilon \in \mathbb{N}$ and bounded continuous functions $F_{\varepsilon, \alpha}(x)$, $|\alpha| \leq m_\varepsilon$, $\text{supp}(F_{\varepsilon, \alpha}(x)) \subset \bar{\Gamma}$ such that

$$T = \sum_{|\alpha| \leq m_\varepsilon} \left(\frac{\partial}{\partial x} \right)^\alpha \{e^{h_K(x) + \varepsilon|x|} F_{\varepsilon, \alpha}(x)\}.$$

Corollary 3.9 ([14]). Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_\Gamma(\mathbb{R}^n, K)$. Then for any $\varepsilon > 0$ there exist n_0 , a partial differential operator with finite order $P_\varepsilon(D)$ and a polynomially bounded continuous function $G_\varepsilon(x)$, $\text{supp}(G_\varepsilon(x)) \subset \bar{\Gamma}$ such that

$$T = P_\varepsilon(D)G_\varepsilon(x) \times F^*(x), \quad F^*(x) = e^{\varepsilon\sqrt{1+x^2}} \sum_{1 \leq n \leq n_0} e^{t_n x},$$

where $t_n \in K$, $(n = 1, \dots, n_0)$.

Corollary 3.10 ([14]). Let $T \in H'_{\Gamma_+ \cup \Gamma_-}(\mathbb{R}^n, K)$. Then there exist $T_+ \in H'_{\Gamma_+}(\mathbb{R}^n, K)$ and $T_- \in H'_{\Gamma_-}(\mathbb{R}^n, K)$ such that

$$T = T_+ + T_-.$$

Remark 3.11. M.Morimoto obtained this result for the 1-dimensional case in [10].

Example 3.12 (Example for Corollary 3.8). Let $n = 2$, $K = \{(x_1, x_2) \in \mathbb{R}^2; |x| \leq 1\}$ and $\Gamma := \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0, x_1 > 0\}$. We define $T(x)$ by

$$T(x) = \begin{cases} \sqrt{x_1^2 - x_2^2} e^{|x|}, & x_1^2 - x_2^2 > 0, x_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $h_K(x) = |x|$, $T(x) \in H'_{\Gamma}(\mathbb{R}^2, K)$ and for $\varepsilon > 0$,

$$T(x) = \sqrt{x_1^2 - x_2^2} e^{-\varepsilon|x|} e^{|x|} e^{\varepsilon|x|} = F_{\varepsilon}(x) e^{h_K(x) + \varepsilon|x|},$$

where

$$F_{\varepsilon}(x) = \begin{cases} \sqrt{x_1^2 - x_2^2} e^{-\varepsilon|x|}, & x_1^2 - x_2^2 > 0, x_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $F_{\varepsilon}(x)$ is a bounded continuous function and $\text{supp}(F_{\varepsilon}) \subset \bar{\Gamma}$.

Example 3.13. Let $n = 1$, $K = \{1\}$ and $\Gamma := (0, \infty)$. We define $T(x)$ by

$$T(x) = \begin{cases} e^x, & x \in \Gamma = (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then $T \in H'_{\Gamma}(\mathbb{R}, K)$ and for $\varepsilon > 0$

$$T = \sum_{k=0}^1 \left(\frac{\partial}{\partial x} \right)^k \{F_{\varepsilon,k}(x) e^{x+\varepsilon|x|}\},$$

where $F_{\varepsilon,k}(x) = (-1)^{k+1} \chi_+(x) e^{-\varepsilon|x|}$ and

$$\chi_+(x) = \begin{cases} x, & x \in \Gamma = (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then $F_{\varepsilon,k}(x)$ is a bounded continuous function and $\text{supp}(F_{\varepsilon,k}) \subset \bar{\Gamma}$.

4 Distributions of exponential growth supported by a proper convex cone

In this section, we shall characterize $H'_{\Gamma}(\mathbb{R}^n, K)$ by the heat kernel method.

Theorem 4.1 ([14]). Let $\Gamma \subset \mathbb{R}^n$ be a proper open convex cone, $T \in H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$ and $U(x, t) = \langle T_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfying the following conditions:

$$\left(\frac{\partial}{\partial t} - \Delta \right) U(x, t) = 0, \quad (1)$$

$$U(x, t) \rightarrow T, \quad (t \rightarrow 0_+), \text{ in } H'(\mathbb{R}^n, K), \quad (2)$$

$$\forall \varepsilon > 0 \exists N_\varepsilon \geq 0 \exists C_\varepsilon \geq 0$$

$$\text{s.t. } |U(x, t)| \leq C_\varepsilon t^{-N_\varepsilon} e^{-\frac{\text{dis}(x, \bar{\Gamma})^2}{16t}} e^{h_K(x) + \varepsilon|x|}, \quad 0 < t < 1, \quad x \in \mathbb{R}^n. \quad (3)$$

Conversely, for a function $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfying (1) and (3), there exists a unique $T \in H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$ such that $\langle T_y, E(x - y, t) \rangle = U(x, t)$.

Corollary 4.2 ([14]). Let $T \in H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$ and $U(x, t) = \langle T_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfies the following conditions:

$$\left(\frac{\partial}{\partial t} - \Delta \right) U(x, t) = 0, \quad (4)$$

$$U(x, t) \rightarrow T, \quad (t \rightarrow 0_+), \text{ in } H'(\mathbb{R}^n, K), \quad (5)$$

$$\forall \varepsilon > 0 \exists N \exists C \geq 0 \text{ s.t. } |U(x, t)| \leq C t^{-N} e^{h_K(x) + \varepsilon|x|}, \quad 0 < t < 1, \quad x \in \mathbb{R}^n$$

and $U(x, t) \rightarrow 0, (t \rightarrow 0_+)$, uniformly for all compact sets in $\mathbb{R}^n \setminus \bar{\Gamma}$. (6)

Conversely, for a function $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfying (4) and (6), there exists a unique $T \in H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$ such that $\langle T_y, E(x - y, t) \rangle = U(x, t)$.

5 Paley-Wiener theorem for $H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$

In this section, we shall see the Paley-Wiener theorem for $H'_{\bar{\Gamma}}(\mathbb{R}^n, K)$. For the 1-dimentional case, it is given in [10].

Definition 5.1. Let Γ be a proper open convex cone, K be a compact set and $\varepsilon' > 0$. Then we denote L by

$$L = \left\{ \bigcap_{u \in K} (\{u\} + (\bar{\Gamma}')^\circ) \right\}^\circ.$$

Proposition 5.2. $L \neq \emptyset$.

Definition 5.3 ([10], [16]). For $T \in H'_{\Gamma}(\mathbb{R}^n, K)$, we define the Fourier-Laplace transform $\mathcal{LF}(T)$ of T by

$$\mathcal{LF}(T)(\zeta) := \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_x, e^{i\zeta x} \rangle, \quad \zeta \in \mathbb{C}^n.$$

The right hand side means

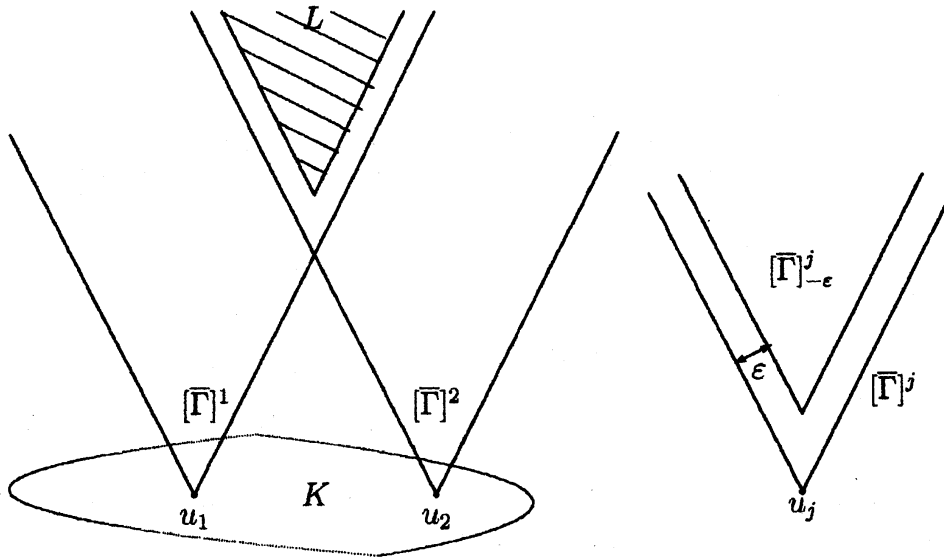
$$\langle T_x, e^{i\zeta x} \rangle = \langle T_x, \chi(x) e^{i\zeta x} \rangle,$$

where $\chi(x) \in C^\infty(\mathbb{R}^n)$ which satisfies

$$\chi(x) = \begin{cases} 1 & , x \in \bar{\Gamma}_\varepsilon \\ 0 & , x \notin \bar{\Gamma}_{2\varepsilon}, \quad \varepsilon > 0. \end{cases}$$

Definition 5.4. Let Γ be a proper open convex cone and K be a compact set. For $\varepsilon > 0$ and $u_j \in K$, $j = 1, \dots, j_0$, we set the following notations:

$$[\bar{\Gamma}]^j = (\{u_j\} + \bar{\Gamma})^\circ, \quad [\bar{\Gamma}]^j_{-\varepsilon} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus [\bar{\Gamma}]^j)_\varepsilon.$$



Theorem 5.5 ([14]). Let Γ be a proper open convex cone, K be a convex compact set, $T \in H'_\Gamma(\mathbb{R}^n, K)$ and $f(\zeta) = \mathcal{LF}(T)(\xi + \eta)$. Then for every $\varepsilon > 0$ there exist $j_0 \in \mathbb{N}$, $l_\varepsilon \geq 0$ and the families $\{u_j\}_{j=1}^{j_0} \subset K$, $\{f_j(\zeta)\}_{j=1}^{j_0}$ satisfying the conditions (7), (8), (9):

$$f_j(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath[\bar{\Gamma}']^j). \quad (7)$$

$\forall \bar{\Gamma}_C \in (\bar{\Gamma}')^\circ \exists M_{\varepsilon, \bar{\Gamma}_C} \geq 0$ such that

$$|f_j(\zeta)| \leq M(1 + |\zeta|)^{l_\varepsilon}, \quad \zeta \in \mathbb{R}^n + \imath[\bar{\Gamma}_C]_{-2\varepsilon}^j. \quad (8)$$

$$f(\zeta) = \sum_{1 \leq j \leq j_0} f_j(\zeta). \quad (9)$$

In particular, $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$.

Conversely if $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ satisfies the conditions (7), (8) and (9), then there exists a unique $T \in H'_\Gamma(\mathbb{R}^n, K)$ such that $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_x, e^{\imath \zeta x} \rangle$.

Furthermore T is given by the following formula:

$$T = \sum_{1 \leq j \leq j_0} T_j, \quad T_j \in H'_\Gamma(\mathbb{R}^n, \{u_j\}), \quad (10)$$

$$f_j(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_{j,x}, e^{\imath \zeta x} \rangle. \quad (11)$$

Corollary 5.6 ([14]). Let Γ be a proper open convex cone, $T \in H'_\Gamma(\mathbb{R}^n, \{0\})$ and $f(\zeta) = \mathcal{LF}(T)(\xi + \eta)$. Then for $\varepsilon > 0$ there exists $l_\varepsilon \geq 0$ satisfying the conditions (12), (13):

$$f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L). \quad (12)$$

$\forall \bar{\Gamma}_C \in (\bar{\Gamma}')^\circ \exists M_{\varepsilon, \bar{\Gamma}_C} \geq 0$ such that

$$|f(\zeta)| \leq M(1 + |\zeta|)^{l_\varepsilon}, \quad \zeta \in \mathbb{R}^n + \imath[\bar{\Gamma}_C]_{-2\varepsilon}. \quad (13)$$

Conversely if $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L)$ satisfies the conditions (12) and (13), then there exists a unique $T \in H'_\Gamma(\mathbb{R}^n, \{0\})$ such that $f(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \langle T_x, e^{\imath \zeta x} \rangle$.

Remark 5.7 (Remark for Corollary 5.6). Now we consider more general Fourier-Laplace transforms. That is, if $T \in \mathcal{D}'$ and $e^{-\eta x}T \in \mathcal{S}'$, then we can define the Fourier-Laplace transform $\mathcal{LF}(T)(\zeta)$ of T . Furthermore it is known that we can obtain the Paley-Wiener theorem for $T \in \mathcal{D}'$ if Γ_T° is not empty where $\Gamma_T := \{\eta \in \mathbb{R}^n; e^{-\langle \cdot, \eta \rangle}T \in \mathcal{S}'\}$ (see Theorem 7.4.2 in [4]).

So we can assert that for the Paley-Wiener theorem for $T \in \mathcal{D}'$ (that is, for Theorem 7.4.2 in [4]) we can take the element of the space $H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\})$ as $T \in \mathcal{D}'$ if and only if the conditions of Corollary 5.6 are satisfied.

Example 5.8 (Example for Theorem 5.5). Let $n = 2$, $K = \{0\} \times [-1, 1]$ and $\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0, x_1 > 0\} = (\overline{\Gamma}')^\circ$. We define $T(x)$ by

$$T(x) = \begin{cases} e^{|x_2|}, & x_1^2 - x_2^2 > 0, x_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We can see $T \in H'_{\overline{\Gamma}}(\mathbb{R}^2, K)$ and if $\eta \in L := \{\eta = (\eta_1, \eta_2); \{(1, 0)\} + (\overline{\Gamma}')^\circ\}$, then

$$\begin{aligned} \langle T_x, e^{i\zeta x} \rangle &= \frac{1}{i\zeta_1(i\zeta_1 + i\zeta_2 + 1)} - \frac{1}{i\zeta_1(i\zeta_1 - i\zeta_2 + 1)} \\ &= f_1(\zeta) + f_2(\zeta). \end{aligned}$$

Then we can see $f_1(\zeta) \in \mathcal{H}(\mathbb{R}^2 + iL_1)$ and $f_2(\zeta) \in \mathcal{H}(\mathbb{R}^2 + iL_2)$, where

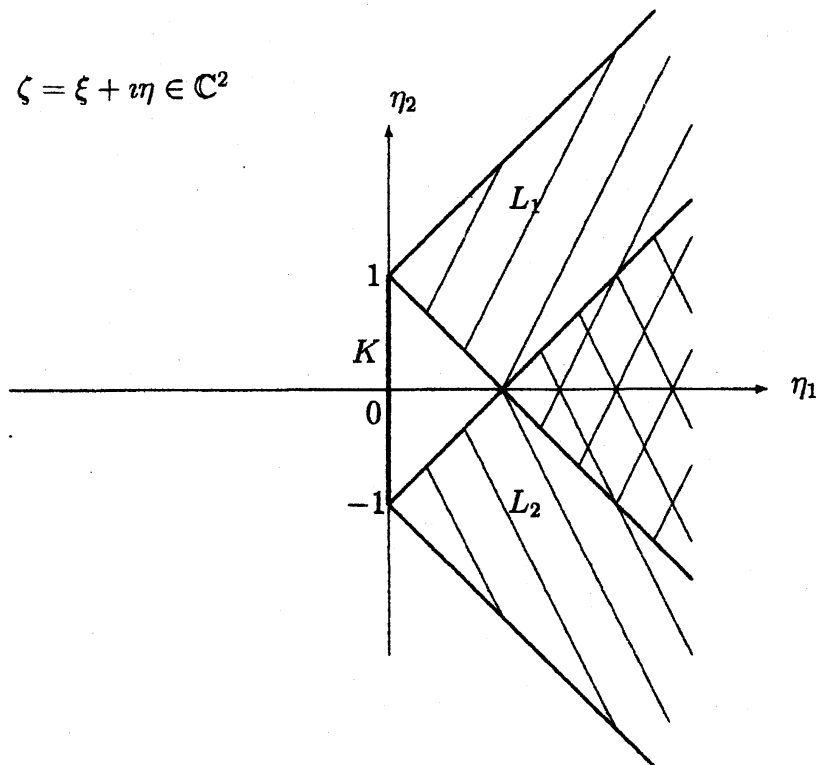
$$L_1 := \{\eta = (\eta_1, \eta_2); \{(0, 1)\} + (\overline{\Gamma}')^\circ\}, \quad L_2 := \{\eta = (\eta_1, \eta_2); \{(0, -1)\} + (\overline{\Gamma}')^\circ\},$$

and $L = L_1 \cap L_2$. Now we define

$$T_1 = \begin{cases} e^{x_2}, & x_1 > x_2, \quad x_2 > 0, \\ 0, & \text{otherwise,} \end{cases} \quad T_2 = \begin{cases} e^{-x_2}, & x_1 > -x_2, \quad x_2 < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $T_1 \in H'_{\overline{\Gamma}}(\mathbb{R}^2, \{(0, 1)\})$, $T_2 \in H'_{\overline{\Gamma}}(\mathbb{R}^2, \{(0, -1)\})$ and

$$\langle T_{1x}, e^{i\zeta x} \rangle = f_1(\zeta), \quad \langle T_{2x}, e^{i\zeta x} \rangle = f_2(\zeta), \quad T = T_1 + T_2.$$



6 Edge-of-the-Wedge theorem

In this section we shall see the Edge-of-the-Wedge theorem for the space of the image by the Fourier-Laplace transform of $T \in H'_{\Gamma}(\mathbb{R}^n, K)$. First we introduce some spaces of holomorphic functions. For details we refer the reader to [10], [11].

Definition 6.1. For a subset A of $\mathbb{R}^n$, we define a set $\mathcal{T}(A)$ by $\mathcal{T}(A) = \mathbb{R}^n \times iA$.

Definition 6.2. For a convex compact set K of $\mathbb{R}^n$ and $\varepsilon > 0$,

$$\begin{aligned} \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)) \\ := \{ \varphi(\zeta) \in \mathcal{H}(\mathcal{T}(K_\varepsilon^\circ)) \cap \mathcal{C}(\mathcal{T}(K_\varepsilon)); \sup_{\zeta \in \mathcal{T}(K_\varepsilon)} |\zeta^\alpha \varphi(\zeta)| < \infty \text{ for } \forall \alpha \in \mathbb{N}^n \}, \end{aligned}$$

$$\mathcal{Q}(\mathcal{T}(K)) := \varinjlim_{\varepsilon > 0} \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)).$$

Definition 6.3. The dual space $\mathcal{Q}'(\mathcal{T}(K))$ of $\mathcal{Q}(\mathcal{T}(K))$ is called tempered ultrahyperfunctions [10], [11].

We have the following theorem for the spaces $H(\mathbb{R}^n, K)$ and $\mathcal{Q}(\mathcal{T}(K))$:

Theorem 6.4 ([10]). Let $\varphi(x) \in H(\mathbb{R}^n, K)$. The Fourier inverse transform

$$\mathcal{F}^{-1}(\varphi)(\zeta) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x) e^{-i\zeta x} dx$$

establishes a topological isomorphism of $H(\mathbb{R}^n, K)$ onto $\mathcal{Q}(\mathcal{T}(K))$. The inverse mapping $\mathcal{F}$ is given by

$$\mathcal{F}(\psi)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \psi(\xi + i\eta) e^{i(\xi + i\eta)x} d\xi, \quad \eta \in K_\varepsilon^\circ, \quad \psi \in \mathcal{Q}_b(\mathcal{T}(K_\varepsilon)). \quad (14)$$

Remark 6.5. In (14), we notice that $\mathcal{F}(\psi)(x)$ is independent of $\eta \in K_\varepsilon^\circ$ by Cauchy's integral theorem.

Definition 6.6 ([10]). For $T \in H'(\mathbb{R}^n, K)$, we define the dual Fourier transform $\mathcal{F}(T)$ as a continuous linear functional on $\mathcal{Q}(\mathcal{T}(K))$ by the formula

$$\langle \mathcal{F}(T), \psi \rangle = \langle T, \mathcal{F}(\psi) \rangle, \quad \text{for } \psi \in \mathcal{Q}(\mathcal{T}(K)). \quad (15)$$

As a consequence of Theorem 6.4, we have the following theorem:

Theorem 6.7 ([10]). The dual Fourier transform (15) gives topological isomorphisms

$$\mathcal{F} : H'(\mathbb{R}^n, K) \rightarrow \mathcal{Q}'(\mathcal{T}(K)).$$

Definition 6.8. Let $K = \{u\}$, $\psi \in \mathcal{Q}_b(\mathcal{T}(K_{\varepsilon_1}))$ and assume that $f(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL)$ satisfies

$$\forall \varepsilon > 0 \exists l_\varepsilon \geq 0 \forall \bar{\Gamma}_C \in (\bar{\Gamma}')^\circ \exists M_{\varepsilon, \bar{\Gamma}_C} \geq 0 \text{ s.t.}$$

$$|f(\zeta)| \leq M(1 + |\zeta|)^{l_\varepsilon}, \quad \zeta \in \mathbb{R}^n + i[\bar{\Gamma}_C]_{-\varepsilon}.$$

Then we define $\langle f(\zeta), \psi(\zeta) \rangle$ by

$$\begin{aligned} \langle f(\zeta), \psi(\zeta) \rangle &:= \langle f(\xi + i\eta_0), \psi(\xi + i\eta_0) \rangle \\ &= \int_{\mathbb{R}^n} f(\xi + i\eta_0) \psi(\xi + i\eta_0) d\xi, \end{aligned}$$

where $\eta_0 \in (\{u\} + (\bar{\Gamma}')^\circ) \cap (K_{\varepsilon_1}^\circ)$.

Definition 6.9. Let $K = \{u\}$, $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ and $\psi \in \mathcal{Q}(\mathcal{T}(K))$, $\psi \in \mathcal{Q}_b(\mathcal{T}(K_{\varepsilon_1}))$. By Theorem 5.5 and Definition 6.8, we define $\langle \mathcal{LF}(T)(\zeta), \psi(\zeta) \rangle$ by

$$\langle \mathcal{LF}(T)(\zeta), \psi(\zeta) \rangle := \langle \mathcal{LF}(T)(\xi + \eta_0), \psi(\xi + \eta_0) \rangle, \quad (16)$$

where $\eta_0 \in (\{u\} + (\overline{\Gamma}')^\circ) \cap (K_{\varepsilon_1})^\circ$.

Now we can show Edge-of-the-Wedge theorem. For the direct product case, it is given in [11].

Theorem 6.10 (Edge-of-the-Wedge Theorem [14]). Let Γ_1, Γ_2 be proper open convex cones in $\mathbb{R}^n$,

$$L_m = \{u_m\} + (\overline{\Gamma}'_m)^\circ, \quad m = 1, 2.$$

Assume that $F_1(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L_1)$ and $F_2(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath L_2)$ satisfy

$$\begin{aligned} \forall \varepsilon > 0 \exists l_{m\varepsilon} \geq 0 \forall \overline{\Gamma}_{C_m} \in (\overline{\Gamma}'_m)^\circ \exists M_{\varepsilon, \overline{\Gamma}_{C_m}} \geq 0 \text{ s.t.} \\ |F_m(\zeta)| \leq M_{\varepsilon, \overline{\Gamma}_{C_m}} (1 + |\zeta|)^{l_{m\varepsilon}}, \quad \zeta \in \mathbb{R}^n + \imath [\overline{\Gamma}_{C_m}]_{-2\varepsilon}, \quad m = 1, 2, \end{aligned} \quad (17)$$

where $[\overline{\Gamma}_{C_m}]_{-\varepsilon} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus (\{u_m\} + \overline{\Gamma}_{C_m})^\circ)_\varepsilon$.

Let K be a convex compact set which contains the segment with $\{u_1\}$ and $\{u_2\}$ as extremal point. Assume that

$$\langle F_1(\zeta), \psi(\zeta) \rangle = \langle F_2(\zeta), \psi(\zeta) \rangle \quad \forall \psi(\zeta) \in \mathcal{Q}(\mathcal{T}(K)). \quad (18)$$

Then there exists $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(L'_1 \cup L'_2))$ such that

$$F(\zeta)|_{(\mathbb{R}^n + \imath L_1)} = F_1(\zeta), \quad F(\zeta)|_{(\mathbb{R}^n + \imath L_2)} = F_2(\zeta),$$

where $L'_1 = \{u_1\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ$ and $L'_2 = \{u_2\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ$. Furthermore

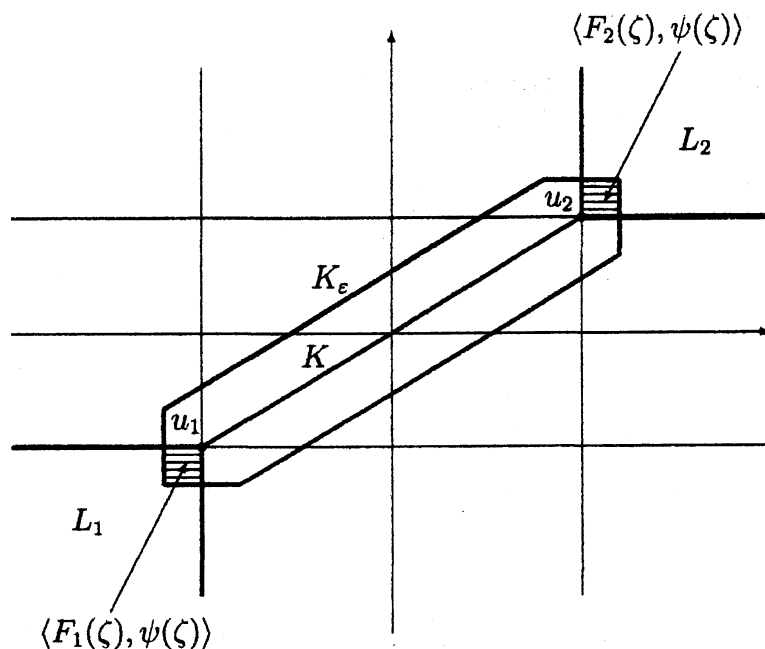
- (i) if $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \{0\}$, then $F(\zeta)$ is polynomial,
- (ii) if $\{u_1\} = \{u_2\} (= \{u\})$, then we have

$$F(\zeta) \in \mathcal{H}(\mathbb{R}^n + \imath(\{u\} + (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ)) \quad (19)$$

and

$$\begin{aligned} \forall \varepsilon > 0 \exists l_\varepsilon \geq 0 \forall \overline{\Gamma}_C \in (\overline{\Gamma}'_1 \cup \overline{\Gamma}'_2)^\circ \exists M_{\varepsilon, \overline{\Gamma}_C} \geq 0 \\ |F(\zeta)| \leq M(1 + |\zeta|)^{l_\varepsilon}, \quad \zeta \in \mathbb{R}^n + \imath [\overline{\Gamma}_C]_{-\varepsilon}, \end{aligned} \quad (20)$$

where $[\overline{\Gamma}_C]_{-\varepsilon} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus (\{u\} + \overline{\Gamma}_C)^\circ)_\varepsilon$.



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