

On the structure of multiple-scale solutions of the Painlevé equations with a large parameter

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§0. Introduction.

The purpose of this talk is to report that our conjecture on multiple-scale solutions of Painlevé equations (P_J) ($J = \text{I, II}, \dots, \text{VI}$; cf. Table 0.1 below) with a large parameter η has been proved; each 2-parameter multiple-scale solution of (P_J) is locally reduced to a suitably chosen 2-parameter multiple-scale solution of the first Painlevé equation (P_I) . (See Theorem 2.1 for the precise statement.)

This is a natural generalization of the result on 0-parameter solutions. ([KT1, Theorem 2.3.])

The details of this report will appear in [KT3].

Although we use the same notations as in [AKT], we list up basic equations and related symbols for the sake of definiteness. In what follows, J ranges over $\{I, II, \dots, VI\}$ unless otherwise stated.

Table 0.1. *Painlevé equations with a large parameter η .*

$$\begin{aligned}
(P_I) \quad & \frac{d^2 \lambda}{dt^2} = \eta^2(6\lambda^2 + t). \\
(P_{II}) \quad & \frac{d^2 \lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \alpha). \\
(P_{III}) \quad & \frac{d^2 \lambda}{dt^2} = \frac{1}{\lambda} \left(\frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + 8\eta^2 \left[2\alpha_\infty \lambda^3 + \frac{\alpha'_\infty}{t} \lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda} \right]. \\
(P_{IV}) \quad & \frac{d^2 \lambda}{dt^2} = \frac{1}{2\lambda} \left(\frac{d\lambda}{dt} \right)^2 - \frac{2}{\lambda} + 2\eta^2 \left[\frac{3}{4} \lambda^3 + 2t\lambda^2 + (t^2 + 4\alpha_1)\lambda - \frac{4\alpha_0}{\lambda} \right]. \\
(P_V) \quad & \frac{d^2 \lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1} \right) \left(\frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2} \left(2\lambda - \frac{1}{2\lambda} \right) \\
& + \eta^2 \frac{2\lambda(\lambda-1)^2}{t^2} \left[(\alpha_0 + \alpha_\infty) - \alpha_0 \frac{1}{\lambda^2} \right. \\
& \left. - \alpha_2 \frac{t}{(\lambda-1)^2} - \alpha_1 t^2 \frac{\lambda+1}{(\lambda-1)^3} \right]. \\
(P_{VI}) \quad & \frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) \left(\frac{d\lambda}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \frac{d\lambda}{dt} \\
& + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2} \right. \\
& + \eta^2 \left\{ (\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty) - \alpha_0 \frac{t}{\lambda^2} \right. \\
& \left. \left. + \alpha_1 \frac{t-1}{(\lambda-1)^2} - \alpha_t \frac{t(t-1)}{(\lambda-t)^2} \right\} \right].
\end{aligned}$$

Table 0.2. *Painlevé Hamiltonian systems with a large parameter η .*

$$(H_J) \quad \begin{cases} \frac{d\lambda}{dt} = \eta \frac{\partial K_J}{\partial \nu} \\ \frac{d\nu}{dt} = -\eta \frac{\partial K_J}{\partial \lambda} \end{cases},$$

where K_J is tabulated below:

$$\begin{aligned} K_I &= \frac{1}{2} [\nu^2 - (4\lambda^3 + 2t\lambda)] . \\ K_{II} &= \frac{1}{2} [\nu^2 - (\lambda^4 + t\lambda^2 + 2\alpha\lambda)] . \\ K_{III} &= \frac{2\lambda^2}{t} \left[\nu^2 - \eta^{-1} \frac{3\nu}{2\lambda} - \left(\frac{\alpha_0 t^2}{\lambda^4} + \frac{\alpha'_0 t}{\lambda^3} + \frac{\alpha'_\infty t}{\lambda} + \alpha_\infty t^2 \right) \right] . \\ K_{IV} &= 2\lambda \left[\nu^2 - \eta^{-1} \frac{\nu}{\lambda} - \left(\frac{\alpha_0}{\lambda^2} + \alpha_1 + \left(\frac{\lambda + 2t}{4} \right)^2 \right) \right] . \\ K_V &= \frac{\lambda(\lambda - 1)^2}{t} \\ &\quad \times \left[\nu^2 - \eta^{-1} \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1} \right) \nu \right. \\ &\quad \left. - \left(\frac{\alpha_0}{\lambda^2} + \frac{\alpha_1 t^2}{(\lambda - 1)^4} + \frac{\alpha_2 t}{(\lambda - 1)^3} + \frac{\alpha_\infty}{(\lambda - 1)^2} \right) \right] . \\ K_{VI} &= \frac{\lambda(\lambda - 1)(\lambda - t)}{t(t - 1)} \\ &\quad \times \left[\nu^2 - \eta^{-1} \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1} \right) \nu \right. \\ &\quad \left. - \left(\frac{\alpha_0}{\lambda^2} + \frac{\alpha_1}{(\lambda - 1)^2} + \frac{\alpha_\infty}{\lambda(\lambda - 1)} + \frac{\alpha_t}{(\lambda - t)^2} \right) \right] . \end{aligned}$$

Table 0.3. *Relevant Schrödinger equations with a large parameter η .*

$$(SL_J) \quad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_J(x, t, \eta) \right) \psi_J(x, t, \eta) = 0,$$

where Q_J is given below. (See the preceding Table 0.2 for the symbol K_J used there.)

$$Q_I = 4x^3 + 2tx + 2K_I - \eta^{-1} \frac{\nu}{x - \lambda} + \eta^{-2} \frac{3}{4(x - \lambda)^2}.$$

$$Q_{II} = x^4 + tx^2 + 2\alpha x + 2K_{II} - \eta^{-1} \frac{\nu}{x - \lambda} + \eta^{-2} \frac{3}{4(x - \lambda)^2}.$$

$$Q_{III} = \frac{\alpha_0 t^2}{x^4} + \frac{\alpha'_0 t}{x^3} + \frac{\alpha'_\infty t}{x} + \alpha_\infty t^2 + \frac{tK_{III}}{2x^2} \\ + \eta^{-1} \left(\frac{1}{2x^2} - \frac{1}{x(x - \lambda)} \right) \lambda \nu + \eta^{-2} \frac{3}{4(x - \lambda)^2}.$$

$$Q_{IV} = \frac{\alpha_0}{x^2} + \alpha_1 + \left(\frac{x + 2t}{4} \right)^2 + \frac{K_{IV}}{2x} - \eta^{-1} \frac{\lambda \nu}{x(x - \lambda)} + \eta^{-2} \frac{3}{4(x - \lambda)^2}.$$

$$Q_V = \frac{\alpha_0}{x^2} + \frac{\alpha_1 t^2}{(x - 1)^4} + \frac{\alpha_2 t}{(x - 1)^3} + \frac{\alpha_\infty}{(x - 1)^2} + \frac{tK_V}{x(x - 1)^2} \\ - \eta^{-1} \frac{\lambda(\lambda - 1)\nu}{x(x - 1)(x - \lambda)} + \eta^{-2} \frac{3}{4(x - \lambda)^2}.$$

$$Q_{VI} = \frac{\alpha_0}{x^2} + \frac{\alpha_1}{(x - 1)^2} + \frac{\alpha_\infty}{x(x - 1)} + \frac{\alpha_t}{(x - t)^2} + \frac{t(t - 1)K_{VI}}{x(x - 1)(x - t)} \\ - \eta^{-1} \frac{\lambda(\lambda - 1)\nu}{x(x - 1)(x - \lambda)} + \eta^{-2} \frac{3}{4(x - \lambda)^2}.$$

Table 0.4. *Deformation equations.*

$$(D_J) \quad \frac{\partial \psi}{\partial t} = A_J \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial A_J}{\partial x} \psi,$$

where A_J denotes the function given below:

$$A_I = A_{II} = \frac{1}{2(x - \lambda)}.$$

$$A_{III} = \frac{2\lambda x}{t(x - \lambda)} + \frac{x}{t}.$$

$$\begin{aligned}
A_{\text{IV}} &= \frac{2x}{x-\lambda}. \\
A_{\text{V}} &= \frac{\lambda-1}{t} \frac{x(x-1)}{x-\lambda}. \\
A_{\text{VI}} &= \frac{\lambda-t}{t(t-1)} \frac{x(x-1)}{x-\lambda}.
\end{aligned}$$

We note that (SL_J) and (D_J) are in involution (i.e., compatible) if (λ, ν) obeys the Hamiltonian system (H_J) , which is known to be equivalent to (P_J) .

§1. A canonical form of (SL_J) and (D_J) near the double turning point.

Let us consider the following pair of equations (Can) and (D_{can}) , where ρ and σ are functions of t and η :

$$(Can) \quad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\text{can}}(x, \rho, \sigma, \eta) \right) \psi = 0$$

with

$$(1.1) \quad Q_{\text{can}} = 4x^2 + \eta^{-1}E + \frac{\eta^{-3/2}\rho}{x - \eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(x - \eta^{-1/2}\sigma)^2},$$

where $E = \rho^2 - 4\sigma^2$,

and

$$(D_{\text{can}}) \quad \frac{\partial \psi}{\partial t} = A_{\text{can}} \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial A_{\text{can}}}{\partial x} \psi,$$

where

$$(1.2) \quad A_{\text{can}} = \frac{1}{2(x - \eta^{-1/2}\sigma)}.$$

One can readily verify that equations (Can) and (D_{can}) are in involution if ρ and σ satisfy the following equation:

$$(H_{\text{can}}) \quad \begin{cases} \frac{d\rho}{dt} = -4\eta\sigma \\ \frac{d\sigma}{dt} = -\eta\rho \end{cases}.$$

For the sake of clarity of notations, we use the symbol $(\rho_{\text{can}}, \sigma_{\text{can}})$ to denote a solution of (H_{can}) ; note that ρ_{can} and σ_{can} are hyperbolic functions.

As is shown in Proposition 1.3 of [KT1], the top term $\lambda_0(t)$ of a multiple-scale solution of (P_J) gives rise to a double turning point of (SL_J) . An important fact proved in Theorem 1.1 of [KT1] is that S_{odd} , the odd part of a solution S of the Riccati equation associated with (SL_J) , is holomorphic near $x = \lambda_0(t)$ as far as we are concerned with 0-parameter solutions. Furthermore this regularity result leads to a very simple canonical form of the equation near the double turning point. (Cf. Theorem 1.2 of [KT1].) Although such a clear-cut result cannot be expected for 2-parameter multiple-scale solutions, we can still confirm the following Proposition 1.1 concerning the structure of simultaneous equations (SL_J) and (D_J) near the double turning point $x = \lambda_0(t)$. The relation (1.8.a) below is the counterpart of the canonical form for 0-parameter solutions, and the proposition plays a crucially important role in the proof of Theorem 2.1. For the sake of clarity of presentation we put $\sim$ over the variables and functions relevant to (SL_J) , like $\tilde{x}, \tilde{t}$, etc., in the proposition. We also fix a point $\tilde{t}_*$ at a generic point as in Proposition 1.1 of [KT1], and we choose and fix sufficiently small neighborhoods $\tilde{U}$ and $\tilde{V}$ of $\tilde{x} = \tilde{\lambda}_0(\tilde{t}_*)$ and $\tilde{t}_*$, respectively.

Proposition 1.1. *For each $J = \text{I, II}, \dots, \text{VI}$, there exist holomorphic functions $x_{j/2}(\tilde{x}, \tilde{t}, \eta)$ and $t_{j/2}(\tilde{t}, \eta)$ ($j = 0, 1, 2, \dots; (\tilde{x}, \tilde{t}) \in \tilde{U} \times \tilde{V}$) which satisfy the following relations:*

$$(1.3) \quad x_0 \text{ and } t_0 \text{ are independent of } \eta,$$

$$(1.4) \quad \frac{\partial x_0}{\partial \tilde{x}} \text{ never vanishes on } \tilde{U} \times \tilde{V},$$

$$(1.5) \quad x_0(\tilde{\lambda}_0(\tilde{t}), \tilde{t}) = 0 \text{ holds on } \tilde{V},$$

$$(1.6) \quad t_0(\tilde{t}) = \tilde{\phi}_J(\tilde{t})/2 \text{ holds on } \tilde{V}, \text{ where}$$

$$\tilde{\phi}_J(\tilde{t}) = \int_{\tilde{r}}^{\tilde{t}} \sqrt{\frac{\partial \tilde{F}_J}{\partial \tilde{\lambda}}(\tilde{\lambda}_0(\tilde{s}), \tilde{s})} d\tilde{s}$$

with $\tilde{r}$ being a turning point for $\tilde{\lambda}_J^{(0)}$ and with $\tilde{F}_J$ denoting the coefficient of η^2 in the equation (P_J) ,

$$(1.7) \quad x_{1/2}(\tilde{x}, \tilde{t}, \eta) \text{ and } t_{1/2}(\tilde{t}, \eta) \text{ identically vanish,}$$

$$(1.8) \quad \text{If we set } x(\tilde{x}, \tilde{t}, \eta) = \sum_{j \geq 0} x_{j/2}(\tilde{x}, \tilde{t}, \eta) \eta^{-j/2} \text{ and } t(\tilde{t}, \eta) = \sum_{j \geq 0} t_{j/2}(\tilde{t}, \eta) \eta^{-j/2}, \text{ then}$$

$$(1.8.a) \quad \tilde{Q}_J(\tilde{x}, \tilde{t}, \eta) = \left(\frac{\partial x}{\partial \tilde{x}} \right)^2 Q_{\text{can}}(x(\tilde{x}, \tilde{t}, \eta), \rho_{\text{can}}(t(\tilde{t}, \eta), \eta), \sigma_{\text{can}}(t(\tilde{t}, \eta), \eta), \eta) - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \tilde{t}, \eta); \tilde{x}\},$$

$$\text{where } \{x(\tilde{x}, \tilde{t}, \eta); \tilde{x}\} = \frac{\frac{\partial^3 x}{\partial \tilde{x}^3}}{\frac{\partial x}{\partial \tilde{x}}} - \frac{3}{2} \left(\frac{\frac{\partial^2 x}{\partial \tilde{x}^2}}{\frac{\partial x}{\partial \tilde{x}}} \right)^2,$$

$$(1.8.b) \quad \sigma_{\text{can}}(t(\tilde{t}, \eta), \eta) = \eta^{1/2} x(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta),$$

$$(1.8.c) \quad \rho_{\text{can}}(t(\tilde{t}, \eta), \eta) = -\frac{\eta^{1/2} \tilde{\nu}(\tilde{t}, \eta)}{\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta)} - \frac{3\eta^{-1/2} \frac{\partial^2 x}{\partial \tilde{x}^2}(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta)}{4 \left(\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta) \right)^2},$$

$$(1.9) \quad x_{j/2} \text{ and } t_{j/2} \ (j \geq 2) \text{ respectively have the form}$$

$$\sum_{k=-(j-2)}^{j-2} y_k(\tilde{x}, \tilde{t}) e^{k\tilde{\phi}_J(\tilde{t})\eta} \quad \text{and} \quad \sum_{k=-(j-2)}^{j-2} s_k(\tilde{t}) e^{k\tilde{\phi}_J(\tilde{t})\eta};$$

that is, $x_{j/2}$ and $t_{j/2}$ ($j \geq 2$) consist of k -instanton terms with $|k| \leq j-2$.

Note that (1.8.b) and (1.8.c) implicitly give relations between constants contained in $(\rho_{\text{can}}, \sigma_{\text{can}})$ and $(\tilde{\lambda}, \tilde{\nu})$, although they cannot establish a unique correspondence between them.

Actually, after introducing $x(\tilde{x}, \tilde{t}, \eta)$ by that given in Theorem 3.1 of [AKT], we try to construct $t(\tilde{t}, \eta)$ by first requiring

$$(1.10) \quad \rho_{\text{can}}^2 - 4\sigma_{\text{can}}^2 = \eta \left(\frac{\tilde{\nu}}{\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}, \tilde{t}, \eta)} + \frac{3\eta^{-1} \frac{\partial^2 x}{\partial \tilde{x}^2}(\tilde{\lambda}, \tilde{t}, \eta)}{4(\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}, \tilde{t}, \eta))^2} \right)^2 - 4\eta x(\tilde{\lambda}, \tilde{t}, \eta)^2;$$

Surprisingly enough, both sides of (1.10) are independent of $\tilde{t}$, and requiring (1.10) amounts to requiring relations among constants contained in $(\rho_{\text{can}}, \sigma_{\text{can}})$ and $(\tilde{\lambda}, \tilde{\nu})$. (See the proof of Lemma 1.1 of [KT2].) The construction of $t(\tilde{t}, \eta)$ is, then, achieved by the induction on j making full use of (1.10). We note that in the course of our argument $t_{j/2}$ (j : an even integer ≥ 2) is determined only modulo an additive constant. This freedom of $t_{j/2}$ is effectively used in our proof of Theorem 2.1. Still more important is the fact that fixing $t_{j/2}$ leads to a unique correspondence between the constants contained in $(\rho_{\text{can}}, \sigma_{\text{can}})$ and $(\tilde{\lambda}, \tilde{\nu})$; this is a key relation for the description of the connection formula for general Painlevé transcendents. (See [AKT], [KT3], and [T] for details.)

Although Proposition 1.1 is concerned with the relation between (SL_J) and (Can) , we can further verify the following:

Proposition 1.2. *Let $\psi(x, t, \eta)$ be a WKB solution of (Can) that satisfies (D_{can}) also, and let $\tilde{\psi}(\tilde{x}, \tilde{t}, \eta)$ denote*

$$(1.11) \quad \left(\frac{\partial x(\tilde{x}, \tilde{t}, \eta)}{\partial \tilde{x}} \right)^{-1/2} \psi(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta).$$

Then $\tilde{\psi}(\tilde{x}, \tilde{t}, \eta)$ satisfies both (SL_J) and (D_J) near the double turning point $\tilde{x} = \tilde{\lambda}_0(\tilde{t})$.

The proof of this proposition is attained by verifying

$$(1.12) \quad \tilde{A}_J \frac{\partial x}{\partial \tilde{x}} - \frac{\partial x}{\partial \tilde{t}} - A_{\text{can}} \frac{\partial t}{\partial \tilde{t}} = 0;$$

one can readily verify (1.12) guarantees that $\tilde{\psi}$ satisfies not only (SL_J) but also (D_J) . (Cf. the proof of Proposition 2.2 of [KT1].)

Remark 2.1. Although $t(\tilde{t}, \eta)$ cannot be uniquely determined by (1.8) when 2-parameters of $(\tilde{\lambda}, \tilde{\nu})$ vanish, $t(\tilde{t}, \eta)$ can be uniquely determined by the limit as the 2-parameters tend to 0. Proposition 1.2 continues to hold for the choice of $t(\tilde{t}, \eta)$ in this degenerate case.

§2. Local equivalence of 2-parameter multiple-scale solutions.

The local equivalence of the simultaneous equations (SL_J) and (D_J) and the simultaneous equations (Can) and (D_{can}) established in the precedent section automatically entails the local equivalence of (SL_J) & (D_J) and (SL_I) & (D_I) near the double turning point. As one may naturally expect in view of the results in [KT1], this local equivalence can be “matched” with the local equivalence between (SL_J) and (SL_I) near the simple turning point (that merges with the double turning point at the turning point for $\tilde{\lambda}_J^{(0)}$ in question). The ‘matching’ is achieved this time by making use of the freedom in the choice of $t(\tilde{t}, \eta)$ in Proposition 1.1. Once such a semi-global equivalence is constructed, it gives rise to the required local reduction of $\tilde{\lambda}_J$ to λ_I . To state the result in a precise manner, let us clarify the geometric situation in which we analyze the problem. (Cf. §2 of [KT1].) Let $\tilde{t}_*$ be a point in a Stokes curve for $\tilde{\lambda}_J^{(0)}$ emanating from a simple turning point $\tilde{r}$ for $\tilde{\lambda}_J^{(0)}$. In what follows, we assume $\tilde{t}_* \neq \tilde{r}$. Then there exist a simple turning point $\tilde{a}(\tilde{t})$ and a Stokes curve $\tilde{\gamma}$ of (SL_J) such that $\tilde{\gamma}$ joins $\tilde{a}(\tilde{t})$ and the double turning point $\tilde{\lambda}_{J,0}(\tilde{t})$. (See Corollary 2.1 of [KT1].) Having this

configuration in mind, we obtain the following Theorem 2.1, which is a natural generalization of the local equivalence of 0-parameter Painlevé transcendents (Theorem 2.3 of [KT1]):

Theorem 2.1. *For each 2-parameter formal solution $(\tilde{\lambda}_J, \tilde{\nu}_J)$ of (H_J) that is obtained by multiple-scale analysis ([AKT, §1]), there exists a 2-parameter formal solution (λ_I, ν_I) of (H_I) for which the following holds: There exist a neighborhood $\tilde{U}$ of $\tilde{\gamma}$, a neighborhood $\tilde{V}$ of $\tilde{t}_*$ and holomorphic functions $x_{j/2}(\tilde{x}, \tilde{t}, \eta)$ and $t_{j/2}(\tilde{t}, \eta)$ ($j = 0, 1, 2, \dots$, $\tilde{x} \in \tilde{U}$ and $\tilde{t} \in \tilde{V}$) which satisfy the following:*

(2.1) *The functions x_0 and t_0 are independent of η ,*

$$(2.2.1) \quad x_0(\tilde{\lambda}_{J,0}(\tilde{t}), \tilde{t}) = \lambda_{I,0}(t_0(\tilde{t})),$$

$$(2.2.ii) \quad x_0(\tilde{a}(\tilde{t}), \tilde{t}) = -2\lambda_{I,0}(t_0(\tilde{t})) (= a(t_0(\tilde{t}))),$$

$$(2.3) \quad \frac{\partial x_0}{\partial \tilde{x}} \text{ never vanishes on } \tilde{U} \times \tilde{V},$$

$$(2.4) \quad \tilde{\phi}_J(\tilde{t}) = \phi_I(t_0(\tilde{t})),$$

$$(2.5) \quad x_{1/2} \text{ and } t_{1/2} \text{ vanish identically,}$$

$$(2.6) \quad \text{Setting } x(\tilde{x}, \tilde{t}, \eta) = \sum_{j \geq 0} x_{j/2} \eta^{-j/2} \text{ and } t(\tilde{t}, \eta) = \sum_{j \geq 0} t_{j/2} \eta^{-j/2}, \text{ we find the following:}$$

$$(2.6.a) \quad \tilde{Q}_J(\tilde{x}, \tilde{t}, \eta) = \left(\frac{\partial x}{\partial \tilde{x}} \right)^2 Q_I(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta) - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \tilde{t}, \eta); \tilde{x}\},$$

$$(2.6.b) \quad x(\tilde{\lambda}_J(\tilde{t}, \eta), \tilde{t}, \eta) = \tilde{\lambda}_I(t(\tilde{t}, \eta), \eta).$$

The relation (2.6.a) implies the transformation $(\tilde{x}, \tilde{t}) \mapsto (x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta))$ brings (SL_I) into (SL_J) , and the transformation gives rise to the required transformation of Painlevé transcendents as is stated in (2.6.b). See [KT2] for the core idea of the proof. The detailed proof will appear in [KT3].

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