

Convergence of approximating solutions of the Navier-Stokes equations

Yuta Koizumi

Department of Pure and Applied Mathematics, Waseda University

This is the joint work with Mr. Toya Taniguchi[Waseda University, toya@moegi.waseda.jp]. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1 Introduction

Let us consider the Cauchy problem of the Navier-Stokes equations in $\mathbb{R}^n (n \geq 2)$;

$$(N-S) \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u|_{t=0} = a & \text{in } \mathbb{R}^n \end{cases}$$

where $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ and $\pi = \pi(x, t)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and the time $t \in (0, T)$, respectively, while $a = a(x) = (a_1(x), \dots, a_n(x))$ is the given initial data of velocity. In the famous paper of Kato [1], he proved that for every $a \in L_\sigma^n(\mathbb{R}^n) \equiv PL^n(\mathbb{R}^n)$, there exist $0 < T < \infty$ and a unique solution $u \in BC([0, T]; L_\sigma^n(\mathbb{R}^n))$ of integral equation

$$(IE) \quad u(t) = u_0(t) - \int_0^t P \nabla \cdot e^{-(t-s)A} (u \otimes u)(s) ds, \quad 0 < t < T, \\ \text{with } u_0(t) = e^{-tA} a$$

satisfying properties

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} u(t) \in BC([0, T]; L^p(\mathbb{R}^n)) \text{ for all } n \leq p \leq \infty, \quad (1.1)$$

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})+\frac{1}{2}} \nabla u(t) \in BC([0, T]; L^p(\mathbb{R}^n)) \text{ for all } n \leq p < \infty \quad (1.2)$$

where P is the Helmholtz projection onto the solenoidal vector fields and $A = -P\Delta$ denotes the Stokes operator. We call such a solution u the *mild solution* of (N-S) on

$(0, T)$. It is known that the mild solution u necessarily satisfies $u \in C((0, T); W^{2,n}(\mathbb{R}^n)) \cap C^1((0, T); L_\sigma^n(\mathbb{R}^n))$ and fulfills the abstract evolution equation

$$(E) \begin{cases} \frac{du}{dt} + Au + P(u \cdot \nabla)u = 0 \text{ in } L_\sigma^n(\mathbb{R}^n), & 0 < t < T, \\ u(0) = a \end{cases}$$

due to the theory of the holomorphic semigroup. We call such a solution u the *strong solution* of (N-S) on $(0, T)$. Kozono-Okada-Shimizu [3] constructed the strong solution of (N-S) for more general initial data in homogeneous Besov space as well as its analyticity. Let us define the approximating solutions $\{u_j\}_{j=0}^\infty$ of (IE).

$$\begin{cases} u_0(t) = e^{-tA}a, \\ u_{j+1}(t) = u_0(t) - \int_0^t P \nabla \cdot e^{-(t-s)A} (u_j \otimes u_j)(s) ds, & j = 0, 1, 2, \dots \end{cases}$$

In this work, we define a *very mild solution* u as a solution of (IE) only satisfying (1.1) and show that it necessarily becomes the strong solution of (N-S) on $(0, T)$. We also show that the convergence of the approximating solutions $\{u_j\}_{j=0}^\infty$ corresponding to the norm in (1.1)

$$\sup_{0 < t < T} t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{p})} \|u_j(t) - u(t)\|_p \rightarrow 0 \text{ as } j \rightarrow \infty \quad (1.3)$$

for some $n \leq p < \infty$, necessarily yields that $\{u_j\}_{j=0}^\infty$ converge to u in the topology of $C((0, T); W^{2,n}(\mathbb{R}^n))$ and $C^1((0, T); L_\sigma^n(\mathbb{R}^n))$. It should be noted that (1.3) is the most fundamental scaling invariant norm of u in L^p with the time weight. Throughout this paper, we denote by $\|\cdot\|_p$ the usual L^p -norm on $\mathbb{R}^n$.

2 Main Results

Before stating our main theorems, let us define a function space X_p by

$$X_p = \{u \in C((0, T); L_\sigma^p(\mathbb{R}^n)); t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{p})}u \in BC([0, T]; L^p(\mathbb{R}^n))\}$$

for $n \leq p < \infty$. X_p is a Banach space with the norm

$$\|u\|_{X_p} = \sup_{0 < t < T} t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{p})} \|u(t)\|_p. \quad (2.1)$$

Our main results now read:

Theorem 2.1. (Koizumi-Taniguchi [2]) *Let $a \in L_\sigma^n(\mathbb{R}^n)$. Suppose that u is a very mild solution of (N-S) on $(0, T)$. Then, it necessarily holds that*

- (i) $t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}}\nabla u \in BC([0, T]; L^q(\mathbb{R}^n))$ for all $n \leq q < \infty$;
(ii) $u \in C^1((0, T); L^n(\mathbb{R}^n)) \cap C((0, T); W^{2,n}(\mathbb{R}^n))$ with

$$\sup_{0 < t < T} t \|Au(t)\|_n + \sup_{0 < t < T} t \|\partial_t u(t)\|_n < \infty;$$

- (iii) u satisfies the differential equation (E) with $\|u(t) - a\|_n \rightarrow 0$ as $t \rightarrow +0$.

Theorem 2.2. (Koizumi-Taniguchi [2]) Let $a \in L^n_\sigma(\mathbb{R}^n)$. Suppose that u is a very mild solution of (N-S) on $(0, T)$. Let $\{u_j\}_{j=0}^\infty$ be approximating solutions of (IE). Then, it holds that $u_j \in X_p$ for all $n \leq p < \infty$ and all $j = 0, 1, \dots$. If

$$\|u_j - u\|_{X_p} \rightarrow 0 \quad (2.2)$$

for some $n \leq p < \infty$, then we have the following properties (i) and (ii).

(i)

$$\sup_{0 < t < T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}} \|\nabla u_j(t) - \nabla u(t)\|_q \rightarrow 0 \text{ as } j \rightarrow \infty \quad (2.3)$$

for all $n \leq q < \infty$;

(ii)

$$\sup_{0 < t < T} t \|Au_j(t) - Au(t)\|_n \rightarrow 0 \text{ as } j \rightarrow \infty, \quad (2.4)$$

$$\sup_{0 < t < T} t \|\partial_t u_j(t) - \partial_t u(t)\|_n \rightarrow 0 \text{ as } j \rightarrow \infty. \quad (2.5)$$

Remark. (i) It should be noted that convergences (2.4) and (2.5) are obtained only in terms of (2.2). Furthermore, we will clarify that even (2.6) is a consequence of (2.2). We emphasize that (2.2) is closely related to a scaling invariant norm. Indeed, for the norm $\|\cdot\|_{X_p}$ defined in (2.1) with $T = \infty$, it holds that

$$\|u\|_{X_p} = \|u_\lambda\|_{X_p} \text{ for all } \lambda > 0,$$

where $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$. Hence, our theorem exhibits that once approximating solutions $\{u_j\}_{j=0}^\infty$ converge to the very mild solution u like (2.2) in such a scaling invariant norm in X_p as (2.1), the convergence in higher ordered Sobolev spaces like (2.4) and (2.5) necessarily holds.

(ii) As a further result of Theorem 2, Koizumi [3] proved that (1.3) necessarily implies that

$$\sup_{0 < t < T} t^{m+\frac{|\alpha|}{2}+\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \|\partial_t^m D^\alpha u_j(t) - \partial_t^m D^\alpha u(t)\|_q \rightarrow 0 \text{ as } j \rightarrow \infty \quad (2.6)$$

for all $n \leq q < \infty$, all $m \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^n$ with $D^\alpha = \partial_x^\alpha$. Moreover, if we define the approximating solutions of the pressure $\{p_j(t)\}_{j=1}^\infty$ of (N-S) by $p_j(t) = (-\Delta)^{-1} \operatorname{div} \operatorname{div}(u_{j-1} \otimes u_{j-1})(t)$, as a consequence of (2.6), $\{p_j(t)\}_{j=1}^\infty$ satisfy

$$\sup_{0 < t < T} t^{m+\frac{|\alpha|}{2}+\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}} \|\partial_t^m D^\alpha p_j(t) - \partial_t^m D^\alpha p(t)\|_q \rightarrow 0 \text{ as } j \rightarrow \infty \quad (2.7)$$

for all $n \leq q < \infty$, all $m \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^n$, where p is the pressure determined by (E) i.e., $p(t) = (-\Delta)^{-1} \operatorname{div} \operatorname{div}(u \otimes u)(t)$.

3 Outline of the Proof

In what follows we use the symbols $\varepsilon_j(t) = u_j(t) - u(t)$, $\mathcal{E}_j(t) = (u_j \otimes u_j)(t) - (u \otimes u)(t)$ ($j = 0, 1, \dots$). The following lemma is essential for the proof of our main theorems.

Lemma. *Let $a \in L^n_\sigma(\mathbb{R}^n)$. Suppose that $\{u_j\}_{j=0}^\infty$ and u are approximating solutions of (IE) and a very mild solution of (N-S) on $(0, T)$, respectively. Then we have the following estimates (i), (ii) and (iii).*

(i)

$$\|u\|_{Y_q} \leq C(\|u_0\|_{X_q} + \|u\|_{X_{2q}}^2 + \|u\|_{X_{2r}}^2 + \|u\|_{X_{2q}}(\|u_0\|_{X_{2q}} + \|u\|_{X_{2q}}^2)) \quad (3.1)$$

with $n \leq q < \infty$ and $n/2 < r < n$.

(ii)

$$\|\varepsilon_j\|_{Y_q} \leq C\|\varepsilon_{j-1}\|_{X_{q_2}}(\|u_0\|_{X_{q_1}} + \|u\|_{X_{q_1}}^2 + \|u_{j-2}\|_{X_q}^2 + (\|u\|_{X_{q_1}} + \|u_{j-1}\|_{X_{q_1}})(\|u\|_{X_{q_2}} + \|u_{j-2}\|_{X_{q_2}})) \quad (3.2)$$

with $n \leq q, q_1, q_2 < \infty$ and $1/q = 1/q_1 + 1/q_2$.

(iii)

$$\sup_{0 < t < T} t\|A\varepsilon_j(t)\|_n + \sup_{0 < t < T} t\|\partial_t \varepsilon_j(t)\|_n \quad (3.3)$$

$$\begin{aligned} \leq C & \left(\|\varepsilon_{j-1}\|_{Y_q}(\|u_0\|_{X_p} + \|u\|_{X_p} + \|u\|_{X_p}^2) + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_{r_1}} + \|\varepsilon_{j-2}\|_{Y_{r_2}}\|u\|_{X_p}\|u\|_{X_{r_1}} \right. \\ & + \|\varepsilon_{j-1}\|_{X_p}(\|u_0\|_{X_q} + \|u_{j-1}\|_{Y_q} + \|u_{j-2}\|_{X_{r_1}}\|u_{j-2}\|_{Y_{r_2}}) + \|\varepsilon_{j-2}\|_{X_p}\|u_{j-1}\|_{Y_q}(\|u\|_{X_p} + \|u_{j-2}\|_{X_p}) \\ & \left. + \|\varepsilon_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-2}\|_{X_{r_1}}\|u\|_{X_p}\|u_{j-2}\|_{Y_{r_2}} \right) \end{aligned}$$

with $n < p, q < \infty, 1/n = 1/p + 1/q, n \leq r_1, r_2 < \infty$ and $1/n < 1/r = 1/r_1 + 1/r_2 < 1/q + 1/n$.

The proof of the lemma is based on Hölder continuity in time of $u(t)$, $\varepsilon_j(t)$ and $\mathcal{E}_j(t)$ as L^p -valued functions. We only prove (3.3) using the following proposition which shows Hölder continuity in time of $\nabla \mathcal{E}_j(t)$ as an L^n -valued function.

Proposition. *Let $1/n = 1/p + 1/q$ with $n < p, q < \infty$. Assume that $1/n < 1/r = 1/r_1 + 1/r_2 < 1/q + 1/n$ with $n \leq r_1, r_2 < \infty$. Suppose that $\{u_j\}_{j=0}^\infty$ and u are approximating solutions of (IE) and a very mild solution of (N-S) on $(0, T)$, respectively. Assume that $u_j \in X_{\tilde{p}}$ for all $j = 0, 1, \dots$ and all $n \leq \tilde{p} < \infty$. Then it holds that*

$$\|\nabla \mathcal{E}_j(t+h) - \nabla \mathcal{E}_j(t)\|_n$$

$$\begin{aligned}
&\leq C \left((\|\varepsilon_j\|_{Y_q} \|u_0\|_{X_p} + \|\varepsilon_{j-1}\|_{Y_{r_2}} \|u\|_{X_p} \|u\|_{X_{r_1}} \right. \\
&\quad + \|\varepsilon_{j-1}\|_{X_p} \|u_j\|_{Y_q} (\|u\|_{X_p} + \|u_{j-1}\|_{X_p}) + \|\varepsilon_{j-1}\|_{X_{r_1}} \|u\|_{X_p} \|u_{j-1}\|_{Y_{r_2}} \\
&\quad + \|\varepsilon_j\|_{X_p} (\|u_0\|_{X_q} + \|u_{j-1}\|_{X_{r_1}} \|u_{j-1}\|_{Y_{r_2}})) h^\alpha t^{-\alpha-1} \\
&\quad + (\|\varepsilon_j\|_{Y_q} \|u\|_{X_p}^2 + \|\varepsilon_{j-1}\|_{X_p} \|u_j\|_{Y_q} (\|u\|_{X_p} + \|u_{j-1}\|_{X_p})) h^{\beta_{n,p}} t^{-1-\beta_{n,p}} \\
&\quad + (\|\varepsilon_j\|_{X_p} \|u_{j-1}\|_{X_{r_1}} \|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{X_{r_1}} \|u\|_{X_p} \|u_{j-1}\|_{Y_{r_2}} \\
&\quad + \|\varepsilon_{j-1}\|_{Y_{r_2}} \|u\|_{X_p} \|u\|_{X_{r_1}}) h^{\beta_{n,q,r}} t^{-1-\beta_{n,q,r}} \Big), \\
&\quad h > 0, \quad 0 < t < T, \quad \beta_{n,p} \equiv \frac{1}{2} - \frac{n}{2p}, \quad \beta_{n,q,r} \equiv \frac{1}{2} - \frac{n}{2} \left(\frac{1}{r} - \frac{1}{q} \right)
\end{aligned}$$

for all $0 < \alpha < \beta_{n,q,r}$, where $C = C(n, p, q, r, \alpha)$ is a constant independent of h and t .

Proof of (3.3). By (IE) and the definition of u_j we have

$$\begin{aligned}
A\varepsilon_j(t) &= P(e^{-\frac{t}{2}A} - 1)((u_{j-1} \cdot \nabla)u_{j-1})(t) - ((u \cdot \nabla)u)(t) \\
&\quad - \int_0^{\frac{t}{2}} PAe^{-(t-s)A}((u_{j-1} \cdot \nabla)u_{j-1})(s) - ((u \cdot \nabla)u)(s)ds \\
&\quad + \int_{\frac{t}{2}}^t PAe^{-(t-s)A}(\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(s))ds \\
&\equiv J_1(t) + J_2(t) + J_3(t), \quad 0 < t < T.
\end{aligned} \tag{3.4}$$

It holds by the Hölder inequality and the bounds of $\{e^{-tA}\}_{t>0}$ and P in L^p that

$$\begin{aligned}
\|J_1(t)\|_n &\leq C\|(u_{j-1} \cdot \nabla)u_{j-1})(t) - ((u \cdot \nabla)u)(t)\|_n \\
&\leq C(\|\varepsilon_{j-1}\|_p \|\nabla u_{j-1}(t)\|_q + \|u(t)\|_p \|\nabla \varepsilon_{j-1}(t)\|_q) \\
&\leq Ct^{-1}(\|\varepsilon_{j-1}\|_{X_p} \|u_{j-1}\|_{Y_q} + \|\varepsilon_{j-1}\|_{Y_q} \|u\|_{X_p})
\end{aligned} \tag{3.5}$$

for all $0 < t < T$ with $C = C(n)$. It holds by the Hölder inequality, L^p - L^q estimate of Stokes semigroup and the bounds of P in L^p that

$$\begin{aligned}
\|J_2(t)\|_n &\leq \int_0^{\frac{t}{2}} \|Ae^{-(t-s)A}((u_{j-1} \cdot \nabla)u_{j-1})(s) - ((u \cdot \nabla)u)(s)\|_n ds \\
&\leq C \int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})} \|((u_{j-1} \cdot \nabla)u_{j-1})(s) - ((u \cdot \nabla)u)(s)\|_r ds \\
&\leq C \int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})} (\|\varepsilon_{j-1}(s)\|_{r_1} \|\nabla u_{j-1}(s)\|_{r_2} + \|u(s)\|_{r_1} \|\nabla \varepsilon_{j-1}(s)\|_{r_2}) ds \\
&\leq C(\|\varepsilon_{j-1}\|_{X_{r_1}} \|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{Y_{r_2}} \|u\|_{X_{r_1}}) \int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})} s^{-1-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} ds \\
&\leq C(\|\varepsilon_{j-1}\|_{X_{r_1}} \|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{Y_{r_2}} \|u\|_{X_{r_1}}) t^{-1-\frac{n}{2}(\frac{1}{r}-\frac{1}{n})} \int_0^{\frac{t}{2}} s^{-1-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} ds
\end{aligned} \tag{3.6}$$

$$= Ct^{-1}(\|\varepsilon_{j-1}\|_{X_{r_1}}\|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{Y_{r_2}}\|u\|_{X_{r_1}})$$

for all $0 < t < T$ with $C = C(n, r)$. By the analiticity of $\{e^{-tA}\}_{t>0}$ we have

$$\begin{aligned}\|J_3(t)\|_n &\leq \int_{\frac{t}{2}}^t \|Ae^{-(t-s)A}(\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(s))\|_n ds \\ &\leq C \int_{\frac{t}{2}}^t (t-s)^{-1} \|\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(s)\|_n ds\end{aligned}$$

for all $0 < t < T$ with $C = C(n)$. Set $\alpha = \beta_{n,q,r}/2$. Changing variable $s \rightarrow \tau = t - s$ of integration and using the Proposition, from the above estimate we obtain

$$\begin{aligned}\|J_3(t)\|_n & \\ &\leq C \int_0^{\frac{t}{2}} \tau^{-1} \|\nabla\mathcal{E}_{j-1}(t) - \nabla\mathcal{E}_{j-1}(t-\tau)\|_n d\tau \\ &\leq C \left(\|\varepsilon_j\|_{Y_q} \|u_0\|_{X_p} + \|\varepsilon_{j-1}\|_{Y_{r_2}} \|u\|_{X_p} \|u\|_{X_{r_1}} + \|\varepsilon_{j-1}\|_{X_p} \|u_j\|_{Y_q} (\|u\|_{X_p} + \|u_{j-1}\|_{X_p}) \right. \\ &\quad + \|\varepsilon_{j-1}\|_{X_{r_1}} \|u\|_{X_p} \|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_j\|_{X_p} (\|u_0\|_{X_q} + \|u_{j-1}\|_{X_{r_1}} \|u_{j-1}\|_{Y_{r_2}}) t^{-1-\alpha} \int_0^{\frac{t}{2}} \tau^{-1+\alpha} d\tau \\ &\quad + (\|\varepsilon_j\|_{Y_q} \|u\|_{X_p}^2 + \|\varepsilon_{j-1}\|_{X_p} \|u_j\|_{Y_q} (\|u\|_{X_p} + \|u_{j-1}\|_{X_p})) t^{-1-\beta_{n,p}} \int_0^{\frac{t}{2}} \tau^{-1+\beta_{n,p}} d\tau \\ &\quad + (\|\varepsilon_j\|_{X_p} \|u_{j-1}\|_{X_{r_1}} \|u_{j-1}\|_{Y_{r_2}} + \|\varepsilon_{j-1}\|_{X_{r_1}} \|u\|_{X_p} \|u_{j-1}\|_{Y_{r_2}} \\ &\quad + \|\varepsilon_{j-1}\|_{Y_{r_2}} \|u\|_{X_p} \|u\|_{X_{r_1}}) t^{-1-\beta_{n,q,r}} \int_0^{\frac{t}{2}} \tau^{-1+\beta_{n,q,r}} d\tau \Big) \\ &\leq Ct^{-1} \left(\|\varepsilon_{j-1}\|_{Y_q} (\|u_0\|_{X_p} + \|u\|_{X_p}^2) + \|\varepsilon_{j-2}\|_{Y_{r_2}} \|u\|_{X_p} \|u\|_{X_{r_1}} \right. \\ &\quad + \|\varepsilon_{j-1}\|_{X_p} (\|u_0\|_{X_q} + \|u_{j-2}\|_{X_{r_1}} \|u_{j-2}\|_{Y_{r_2}}) \\ &\quad \left. + \|\varepsilon_{j-2}\|_{X_p} \|u_{j-1}\|_{Y_q} (\|u\|_{X_p} + \|u_{j-2}\|_{X_p}) + \|\varepsilon_{j-2}\|_{X_{r_1}} \|u\|_{X_p} \|u_{j-2}\|_{Y_{r_2}} \right)\end{aligned}\tag{3.7}$$

for all $0 < t < T$ with $C = C(n, p, q, r)$. Now the desired estimate for $A\varepsilon_j(t)$ follows from (3.4)-(3.7). The estimate for $\partial_t \varepsilon_j(t)$ is easily deduced from the one for $A\varepsilon_j(t)$. In fact, due to the semigroup argument we have

$$\partial_t \varepsilon_j(t) = A\varepsilon_j(t) + (P(u_{j-1} \cdot \nabla)u_{j-1})(t) - P((u \cdot \nabla)u)(t).$$

The second term of the above equality is estimated as

$$\begin{aligned}\|P(u_{j-1} \cdot \nabla)u_{j-1}(t) - P((u \cdot \nabla)u)(t)\|_n & \\ &\leq C(\|\varepsilon_{j-1}(t)\|_p \|\nabla u_{j-1}(t)\|_q + \|u(t)\|_p \|\nabla \varepsilon_{j-1}(t)\|_q) \\ &\leq Ct^{-1}(\|\varepsilon_{j-1}\|_{X_p} \|u_{j-1}\|_{Y_q} + \varepsilon_{j-1}\|_{Y_q} \|u\|_{X_p}).\end{aligned}\tag{3.8}$$

Combining (3.5)-(3.7) and (3.8), we have (3.3). $\square$

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Department of Pure and Applied Mathematics

Waseda University

Tokyo 169-8555

JAPAN

E-mail address: `ytkzm376@akane.waseda.jp`