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Derived Categories in Representation Theory

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We survey recent methods of derived categories in the representaion theory of algebras.

1 Triangulated Categories and Brown Representability

Definition 1.1 *A triangulated category $\mathcal{C}$ is an additive category together with*

(1) an autofunctor $T : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ (i.e. there is T^{-1} such that $T \circ T^{-1} = T^{-1} \circ T = 1_{\mathcal{C}}$) called the translation, and

(2) a collection $\mathcal{T}$ of sextuples (X, Y, Z, u, v, w):

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

called (distinguished) triangles. These data are subject to the following four axioms:

(TR1) *(1) Every sextuple (X, Y, Z, u, v, w) which is isomorphic to a (distinguished) triangle is a (distinguished) triangle.*

(2) Every morphism $u : X \to Y$ is embedded in a (distinguished) triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \qquad \begin{tikzcd} & Z \arrow[dl, "w"', "(1)"] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

(3) For any $X \in \mathcal{C}$,

$$X \xrightarrow{1} X \to 0 \to T(X)$$

is a (distinguished) triangle

(TR2) *A sextuple*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a (distinguished) triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is a (distinguished) triangle.

(TR3) *For any (distinguished) triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & T(X) \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

there exists $h : Z \to Z'$ which makes a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

(TR4) *(Octahedral axiom) For any two consecutive morphisms $u : X \to Y$ and $v : Y \to Z$, if we embed u, vu and v in (distinguished) triangles (X, Y, Z', u, i, i'), (X, Z, Y', vu, k, k') and (Y, Z, X', v, j, j'), respectively, then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commute*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "i'"] \arrow[d, dashed, "f"] & T(X) \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r, "k"] \arrow[d, "j"] & Y' \arrow[r, "k'"] \arrow[d, dashed, "g"] & T(X) \arrow[d, "T(u)"] \\
& X' \arrow[r, equal] \arrow[d, "j'"] & X' \arrow[r, "j'"] \arrow[d, "T(i)j'"] & TY \\
& T(Y) \arrow[r, "T(i)"] & T(Z') &
\end{tikzcd}$$

and the third column is a triangle.

Sometimes, we write $X[i]$ for $T^i(X)$.

Definition 1.2 (∂-functor) *Let $\mathcal{C}$, $\mathcal{C}'$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called a ∂-functor (sometimes exact functor) provided that there is a functorial isomorphism $\alpha : FT_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{C}'}F$ such that*

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\alpha_X F(w)} T_{\mathcal{C}'}(F(X))$$

is a triangle in $\mathcal{C}'$ whenever $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_{\mathcal{C}}(X)$ is a triangle in $\mathcal{C}$. Moreover, if a ∂-functor F is an equivalence, then F is called a triangulated equivalence . In this case, we denote by $\mathcal{C} \overset{\triangle}{\cong} \mathcal{C}'$.

For $(F, \alpha), (G, \beta) : \mathcal{C} \to \mathcal{C}'$ ∂-functors, a functorial morphism $\phi : F \to G$ is called a ∂-functorial morphism if

$$(T_{\mathcal{C}'}\phi) \circ \alpha = \beta \circ \phi T_{\mathcal{C}}$$

We denote by $\partial(\mathcal{C}, \mathcal{C}')$ the collection of all ∂-functors from $\mathcal{C}$ to $\mathcal{C}'$, and denote by $\partial \operatorname{Mor}(F, G)$ the collection of ∂-functorial morphisms from F to G.

Proposition 1.3 *Let $F : \mathcal{C} \to \mathcal{C}'$ be a ∂-functor between triangulated categories. If $G : \mathcal{C}' \to \mathcal{C}$ is a right (or left) adjoint of F, then G is also a ∂-functor.*

Definition 1.4 *A contravariant (resp., covariant) additive functor $H : \mathcal{C} \to \mathcal{A}$ from a triangulated category $\mathcal{C}$ to an abelian category $\mathcal{A}$ is called a homological functor (resp., a cohomological functor), if for any triangle (X, Y, Z, u, v, w) in $\mathcal{C}$ the sequence*

$$H(T(X)) \to H(Z) \to H(Y) \to H(X)$$
$$(\textit{resp.}, \ H(X) \to H(Y) \to H(Z) \to H(T(X))\ )$$

is exact. Taking $H(T^i(X)) = H^i(X)$, we have the long exact sequence:

$$\cdots \to H^{i+1}(X) \to H^i(Z) \to H^i(Y) \to H^i(X) \to \cdots$$
$$(\textit{resp.}, \ \cdots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \cdots)$$

Proposition 1.5 *The following hold.*

1. *If (X, Y, Z, u, v, w) is a triangle, then $vu = 0$, $wv = 0$ and $T(u)w = 0$.*
2. *For any $X \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \mathfrak{Ab}$ (resp., $\operatorname{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathfrak{Ab}$) is a homological functor (resp., a cohomological functor).*
3. *For any homomorphism of triangles*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

if two of f, g and h are isomorphisms, then the rest is also an isomorphism.

Definition 1.6 (Compact Object) *Let $\mathcal{C}$ be a triangulated category. An object $C \in \mathcal{C}$ is called a compact object in $\mathcal{C}$ if the canonical morphism*

$$\coprod_{i \in I} \operatorname{Hom}_{\mathcal{C}}(C, X_i) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(C, \coprod_{i \in I} X_i)$$

is an isomorphism for any set $\{X_i\}_{i \in I}$ of objects (if $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$).

For a triangulated category $\mathcal{C}$, a set $\mathcal{S}$ of compact objects is called a generating set if $\operatorname{Hom}_{\mathcal{C}}(\mathcal{S}, X) = 0 \Rightarrow X = 0$, and if $T(\mathcal{S}) = \mathcal{S}$. A triangulated category $\mathcal{C}$ is compactly generated if $\mathcal{C}$ contains arbitrary coproducts, and if it has a generating set.

Definition 1.7 (Homotopy Limit) *Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts (resp., products). For a sequence $\{X_i \to X_{i+1}\}_{i\in\mathbb{N}}$ (resp., $\{X_{i+1} \to X_i\}_{i\in\mathbb{N}}$) of morphisms in $\mathcal{C}$, the homotopy colimit (resp., homotopy limit) of the sequence is the third (resp., second) term of the triangle*

$$\coprod_i X_i \xrightarrow{1-\ shift} \coprod_i X_i \to \underrightarrow{\mathsf{hocolim}}\ X_i \to T\left(\coprod_i X_i\right)$$
$$(resp.,\ T^{-1}\left(\prod_i X_i\right) \to \underleftarrow{\mathsf{holim}}\ X_i \to \prod_i X_i \xrightarrow{1-\ shift} \prod_i X_i)$$

where the above shift morphism is the coproduct (resp., product) of $X_i \xrightarrow{f_i} X_{i+1}$ (resp., $X_{i+1} \xrightarrow{f_i} X_i$) ($i \in \mathbb{N}$).

Proposition 1.8 *Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts, $\{X_i \to X_{i+1}\}_{i\in\mathbb{N}}$ a sequence of morphisms in $\mathcal{C}$. For a compact object C in $\mathcal{C}$, we have*

$$\mathrm{Hom}(C, \underrightarrow{\mathsf{hocolim}} X_i) \cong \varinjlim \mathrm{Hom}(C, X_i)$$

Proof. We have an exact sequence

$$0 \to \coprod_i \mathrm{Hom}(C, X_i) \to \coprod_i \mathrm{Hom}(C, X_i) \to \mathrm{Hom}(C, \underrightarrow{\mathsf{hocolim}}\ X_i) \to 0 \quad \square$$

Theorem 1.9 (Brown Representability Theorem [Ne]) *Let $\mathcal{C}$ be a compactly generated triangulated category. If a homological functor $H : \mathcal{C} \to \mathfrak{Ab}$ sends coproducts to products, then it is representable, that is , there is an object $X \in \mathcal{C}$ such that $H \cong \mathrm{Hom}_{\mathcal{C}}(-, X)$.*

Sketch of Proof. Let $\mathcal{S}$ be a generating set of $\mathcal{C}$. There exist a coproduct X_1 of objects of $\mathcal{S}$ and a morphism $\mathrm{h}_{X_1} \to H$ such that $\mathrm{Hom}_{\mathcal{C}}(C, X_1) \twoheadrightarrow H(C)$ is surjective for any $C \in \mathcal{S}$. For a functor $K_1 = \mathrm{Ker}(\mathrm{h}_{X_1} \to H)$ there exists a coproduct Z_2 of objects in $\mathcal{S}$ and a morphism $\mathrm{h}_{Z_2} \to K_1$ such that $\mathrm{Hom}_{\mathcal{C}}(C, Z_2) \twoheadrightarrow K_1(C)$ is surjective for any $C \in \mathcal{S}$. Then we have a triangle $Z_2 \to X_1 \to X_2 \to Z_2[1]$. Since H is a homological functor, we have a commutative diagram

$$\begin{array}{ccccc}
H(X_2) & \longrightarrow & H(X_1) & \longrightarrow & H(Z_2) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathrm{Mor}(\mathrm{h}_{X_2}, H) & \longrightarrow & \mathrm{Mor}(\mathrm{h}_{X_1}, H) & \longrightarrow & \mathrm{Mor}(\mathrm{h}_{Z_2}, H)
\end{array}$$

Then there is a morphism $X_1 \to X_2$ satisfying a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{K}_1 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(-, X_1) & \longrightarrow & H \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathrm{K}_2 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(-, X_2) & \longrightarrow & H
\end{array}$$

and we have a morphism of exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C, X_1) & \longrightarrow & H(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_2(C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C, X_2) & \longrightarrow & H(C) & \longrightarrow & 0 \end{array}$$

for any $C \in \mathcal{S}$. By inductive step, we have a triangle

$$\coprod_i X_i \xrightarrow{1-\text{ shift}} \coprod_i X_i \to \underrightarrow{\mathsf{hocolim}}\ X_i \to T\left(\coprod_i X_i\right)$$

and we have an exact sequence

$$\begin{array}{ccccc} H(\underrightarrow{\mathsf{hocolim}}\ X_i) & \longrightarrow & \prod_i H(X_i) & \longrightarrow & \prod_i H(X_i) \\ \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\ \mathrm{Mor}(\mathrm{h}_{\underrightarrow{\mathsf{hocolim}}\ X_i}, H) & \longrightarrow & \prod_i \mathrm{Mor}(\mathrm{h}_{X_i}, H) & \longrightarrow & \prod_i \mathrm{Mor}(\mathrm{h}_{X_i}, H) \end{array}$$

Therefore there is a morphism $\mathrm{Hom}_{\mathcal{C}}(-, \underrightarrow{\mathsf{hocolim}}\ X_i) \to H$ such that

$$\mathrm{Hom}_{\mathcal{C}}(C, \underrightarrow{\mathsf{hocolim}} X_i) \cong H(C)$$

for any $C \in \mathcal{S}$. Hence we have $\mathrm{Hom}_{\mathcal{C}}(-, \underrightarrow{\mathsf{hocolim}} X_i) \cong H$. □

Corollary 1.10 (Adjoint Functor Theorem [Ne]) *Let $\mathcal{C}$ be a compactly generated triangulated category. If a ∂-functor $F : \mathcal{C} \to \mathcal{D}$ commutes with arbitrary coproducts, then there exists a ∂-functor $G : \mathcal{D} \to \mathcal{C}$ which is a right adjoint of F.*

Proof. For any $Y \in \mathcal{D}$, the functor

$$\mathrm{Hom}_{\mathcal{D}}(F(-), Y) : \mathcal{C} \to \mathfrak{Ab}$$

is a homological functor. By Brown representability theorem there is an object $GY \in \mathcal{C}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(F(-), Y) \cong \mathrm{Hom}_{\mathcal{C}}(-, GY) \quad \square$$

Definition 1.11 (Multiplicative System) *Let S be a multiplicative system in a triangulated category $\mathcal{C}$ which satisfies the following conditions:*

(FR0) *For a morphism s in $\mathcal{C}$, if there exist f, g such that $sf, gs \in \mathsf{S}$, then $s \in \mathsf{S}$.*

(FR1) *(1) $1_X \in \mathsf{S}$ for every $X \in \mathcal{C}$.*
(2) For $s, t \in \mathsf{S}$, if st is defined, then $st \in \mathsf{S}$.

(FR2) *Every diagram in $\mathcal{C}$*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \\ X' & \end{tikzcd}$$

with $s \in \mathsf{S}$, can be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "t"] & Y' \end{tikzcd}$$

with $s, t \in \mathsf{S}$. Ditto for the statement with all arrows reversed.

(FR3) *For $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the following are equivalent.*
(1) There exists $s \in \mathsf{S}$ such that $sf = sg$.
(2) There exists $t \in \mathsf{S}$ such that $ft = gt$.

(FR4) *For a morphism u in $\mathcal{C}$, $u \in \mathsf{S}$ if and only if $T(u) \in \mathsf{S}$.*

(FR5) *For triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and morphisms $f : X \to X'$, $g : Y \to Y'$ in S with $gu = u'f$, there exists $h : Z \to Z'$ in S such that (f, g, h) is a homomorphism of triangles.*

Definition 1.12 (Quotient Category) *We define the quotient category $\mathsf{S}^{-1}\mathcal{C}$ of $\mathcal{C}$, as follows:*

1. $\mathrm{Ob}(\mathsf{S}^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
2. *For $X, Y \in \mathrm{Ob}(\mathcal{C})$, let $V(X,Y) = \{(s, Y', f) | s : Y \to Y' \in \mathsf{S}, f : X \to Y\}$. In $V(X,Y)$, we define $(s, Y', f) \sim (s', Y'', f')$ if there is (s'', Y''', f') such that all triangles are commutative in the following diagram:*

$$\begin{tikzcd} & Y' \arrow[d, dashed] & \\ X \arrow[ur, "f"] \arrow[r, dashed, "f''"] \arrow[dr, "f'"'] & Y''' & Y \arrow[ul, "s"'] \arrow[l, dashed, "s''"'] \arrow[dl, "s'"] \\ & Y'' \arrow[u, dashed] & \end{tikzcd}$$

   Then we define a morphism from X to Y by an equivalence class $s^{-1}f$ of (s, Y', f).

3. *For $s^{-1}f : X \to Y, t^{-1}g : Y \to Z$, by (FR2) there are $s' : Z' \to Z'' \in \mathsf{S}$ and*

$g' : Y' \to Z''$ such that $s' o g = g' o s$. Then we define $(t^{-1}g)o(s^{-1}f) = (s'ot)^{-1}gof$.

$$\begin{tikzcd}
X \arrow[dr, "f"] & Y \arrow[d, "s"] \arrow[dr, "g"] & Z \arrow[d, "t"] \\
& Y' \arrow[dr, dashed, "g'"] & Z' \arrow[d, dashed, "s'"] \\
& & Z''
\end{tikzcd}$$

Moreover, we define the quotient functor $Q : \mathcal{C} \to \mathsf{S}^{-1}\mathcal{C}$ by

(Q1) *$Q(X) = X$ for $X \in \mathcal{C}$.*

(Q2) *$Q(f) = 1_Y^{-1} f$ for a morphism $f : X \to Y$ in $\mathcal{C}$.*

Remark 1.13 *Can we define (2) in the above?*

Definition 1.14 (Épaisse Subcategory) *Let $\mathcal{C}$ be a triangulated category. A full subcategory $\mathcal{U}$ of $\mathcal{C}$ is called a full triangulated subcategory if $X \to Y$ is a morphism in $\mathcal{U}$, then there is a triangle $X \to Y \to Z \to TX$ with $Z \in \mathcal{U}$.*

A full triangulated subcategory $\mathcal{U}$ is called an épaisse subcategory if it is closed under direct summands. In this case, let $\mathsf{S}(\mathcal{U})$ be the collection of morphisms s such that $X \xrightarrow{s} Y \to Z \to X[1]$ is a triangle with $Z \in \mathcal{U}$. Then $\mathsf{S}(\mathcal{U})$ is a multiplicative system satisfying (FR0) - (FR5). We write $\mathcal{C}/\mathcal{U} = \mathsf{S}(\mathcal{U})^{-1}\mathcal{C}$.

In the case that $\mathcal{C}$ contains arbitrary coproducts, a full triangulated subcategory $\mathcal{U}$ is called a localizing subcategory if it is closed under coproducts.

Remark 1.15 *The above definition of an épaisse subcategory $\mathcal{U}$ is the same as the origininal definition [Ve], that is, a full triangulated category satisfying that if $X \xrightarrow{u} Y$ factors through some object in $\mathcal{U}$ and if there is a triangle $X \xrightarrow{u} Y \to Z \to T(X)$ with $Z \in \mathcal{U}$, then $X, Y \in \mathcal{U}$.*

Proposition 1.16 ([BN]) *Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts. Then any localizing subcategory is an épaisse subcategory.*

Sketch of Proof. Let $\mathcal{U}$ be a localizing subcategory, and $X \in \mathcal{U}$ with $X = Y \oplus Z$ in $\mathcal{C}$. We take a morphism $e : X \twoheadrightarrow Y \hookrightarrow X$, and consider th sequence of morphisms

$$(*) \qquad X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$$

Then it is easy to see that $Y \cong \underrightarrow{\mathsf{hocolim}}(*) \in \mathcal{U}$. □

Proposition 1.17 *Let $\mathcal{C}$ be a triangulated category. For a multiplicative system S satisfying the conditions (FR0) - (FR5), let $\mathcal{U}(\mathsf{S})$ be the full triangulated subcategory consisting of objects Z which is in a triangle $X \xrightarrow{s} Y \to Z \to X[1]$ with $s \in \mathsf{S}$. Then the following hold.*

1. *$\mathsf{S}(\mathcal{U})$ and $\mathcal{U}(\mathsf{S})$ induce an 1 - 1 correspondence between the collection of multiplicative systems S satisfying the conditions (FR0) - (FR5) and the collection of épaisse subcategories $\mathcal{U}$.*

2. *For an épaisse subcategory $\mathcal{U}$, $\mathcal{C}/\mathcal{U}$ is a triangulated category whose (distinguished) triangles are defined to be isomorphic to (distinguished) triangles of $\mathcal{C}$.*

3. *Assume $\mathcal{C}$ contains arbitrary coproducts. For a localizing subcategory $\mathcal{U}$, $\mathcal{C}/\mathcal{U}$ also contains arbitrary coproducts.*

Definition 1.18 (stable t-structure) *For full subcategories $\mathcal{U}$ and $\mathcal{V}$ of a triangulated category $\mathcal{C}$, $(\mathcal{U}, \mathcal{V})$ is called a stable t-structure in $\mathcal{C}$ provided that*

1. *$\mathcal{U}$ and $\mathcal{V}$ are stable for translations.*

2. $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$.

3. *For every $X \in \mathcal{C}$, there exists a triangle $U \to X \to V \to TU$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.*

Proposition 1.19 ([BBD], c.f. [Mi]) *Let $\mathcal{C}$ be a triangulated category, $(\mathcal{U}, \mathcal{V})$ a stable t-structure in $\mathcal{C}$, and $i_* : \mathcal{U} \to \mathcal{C}, j_* : \mathcal{V} \to \mathcal{C}$ the canonical embeddings. Then the following hold.*

1. *$\mathcal{U}$ and $\mathcal{V}$ is épaisse subcategories of $\mathcal{C}$.*

2. *i_* (resp., j_*) has a right adjoint $i^!$ (resp., a left adjoint j^*).*

3. *The adjunction arrows induce a triangle*

$$i_*i^!X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} j_*j^*X \to i_*i^!X[1]$$

   for any $X \in \mathcal{C}$.

4. *$\mathcal{C}/\mathcal{U}$ (resp., $\mathcal{C}/\mathcal{V}$) exists , and it is triangulated equivalent to $\mathcal{V}$ (resp., $\mathcal{U}$).*

$$\begin{tikzcd}
\mathcal{C}/\mathcal{V} & & \\
\mathcal{U} \arrow[u, "\wr"] \arrow[r, shift left, "i_*"] & \mathcal{C} \arrow[ul] \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] \arrow[dr] & \mathcal{V} \arrow[l, shift left, "j_*"] \arrow[d, "\wr"] \\
& & \mathcal{C}/\mathcal{U}
\end{tikzcd}$$

Corollary 1.20 *Let $\mathcal{C}$ be a compactly generated triangulated category, and $\mathcal{U}$ a localizing subcategory of $\mathcal{C}$. Then $\mathcal{C}/\mathcal{U}$ can be defined if and only if there is a full triangulated subcategory $\mathcal{V}$ such that $(\mathcal{U}, \mathcal{V})$ a stable t-structure in $\mathcal{C}$.*

Proof. If $\mathcal{C}/\mathcal{U}$ can be defined, then the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{U}$ commutes with coproducts. By Adjoint Functor Theorem, Q has a right adjoint $F : \mathcal{C}/\mathcal{U} \to \mathcal{C}$. By Proposition 1.19, it is easy to see that $(\mathcal{U}, \mathrm{Im}\, F)$ is a stable t-structure in $\mathcal{C}$. □

2 Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ which is closed under isomorphisms.

Definition 2.1 (Complex) *A (cochain) complex is a collection $X^{\cdot} = (X^n, d_X^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d_X^{n+1}d_X^n = 0$. A complex $X^{\cdot} = (X^n, d_X^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ is called bounded below (resp., bounded above, bounded) if $X^n = 0$ for $n \ll 0$ (resp., $n \gg 0$, $n \ll 0$ and $n \gg 0$).*

A complex $X^{\cdot} = (X^n, d_X^n)$ is called a stalk complex if there exists an integer n_0 such that $X^i = O$ if $i \neq n_0$. We identify objects of $\mathcal{B}$ with a stalk complexes of degree 0.

A morphism $f : X^{\cdot} \to Y^{\cdot}$ of complexes is a collection of morphisms $f^n : X^n \to Y^n$ which makes a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

We denote by $\mathsf{C}(\mathcal{B})$ (resp., $\mathsf{C}^+(\mathcal{B})$, $\mathsf{C}^-(\mathcal{B})$, $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An autofunctor $T : \mathsf{C}(\mathcal{B}) \to \mathsf{C}(\mathcal{B})$ is called translation if $(TX^{\cdot})^n = X^{n+1}$ and $(Td_X)^n = -d_X^{n+1}$ for any complex $X^{\cdot} = (X^n, d_X^n)$.

In $\mathsf{C}(\mathcal{A})$, a morphism $u : X^{\cdot} \to Y^{\cdot}$ is called a quasi-isomorphism if $H_n(u)$ is an isomorphism for any n.

In this section, "$*$" means "nothing", "$+$", "$-$" or "b".

Definition 2.2 *For $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, Y^{\cdot})$, the mapping cone of u is a complex $\mathrm{M}^{\cdot}(u)$ with*

$$\mathrm{M}^n(u) = X^{n+1} \oplus Y^n,$$
$$d^n_{\mathrm{M}^{\cdot}(u)} = \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_Y^n \end{bmatrix} : X^{n+1}\oplus Y^n \to X^{n+2}\oplus Y^{n+1}.$$

Definition 2.3 (Homotopy Relation) *Two morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, Y^{\cdot})$ are said to be homotopic (denote by $f \underset{h}{\simeq} g$) if there is a collection of morphisms $h = (h^n)$, $h^n : X^n \to Y^{n+1}$ such that $f^n - g^n = d_Y^{n-1}h^n + h^{n+1}d_X^n$ for all $n \in \mathbb{Z}$.*

Definition 2.4 (Homotopy Category) *The homotopy category $\mathsf{K}^*(\mathcal{B})$ of $\mathcal{B}$ is defined by*

1. $\mathsf{Ob}(\mathsf{K}^*(\mathcal{B})) = \mathsf{Ob}(\mathsf{C}^*(\mathcal{B}))$,

2. $\mathrm{Hom}_{\mathsf{K}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{\mathsf{C}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) / \underset{h}{\simeq}$ *for* $X^{\cdot}, Y^{\cdot} \in \mathsf{Ob}(\mathsf{K}^*(\mathcal{B}))$.

Proposition 2.5 *A category* $\mathsf{K}^*(\mathcal{B})$ *is a triangulated category whose (distinguished) triangles are defined to be isomorphic to*

$$X^{\cdot} \xrightarrow{u} Y^{\cdot} \to \mathrm{M}^{\cdot}(u) \to T(X^{\cdot})$$

for any $u : X^{\cdot} \to Y^{\cdot}$ *in* $\mathsf{K}^*(\mathcal{B})$.

Definition 2.6 (Derived Category) *The derived category* $\mathsf{D}^*(\mathcal{A})$ *of an abelian category* $\mathcal{A}$ *is* $\mathsf{K}^*(\mathcal{A})/\mathsf{K}^{*,\phi}(\mathcal{A})$, *where* $\mathsf{K}^{*,\phi}(\mathcal{A})$ *is the full subcategory of* $\mathsf{K}^*(\mathcal{A})$ *consisting of null complexes, that is, complexes whose all cohomologies are* 0.

Proposition 2.7 *The following hold.*

1. $\mathsf{D}^*(\mathcal{A})$ *is a triangulated category, and the canonical functor* $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ *is a* ∂*-functor.*

2. *The i-th cohomology of complexes is a cohomological functor in the sense of Definition 1.4.*

Proposition 2.8 *If* $0 \to X^{\cdot} \xrightarrow{u} Y^{\cdot} \xrightarrow{v} Z^{\cdot} \to 0$ *is a exact sequence in* $\mathsf{C}(\mathcal{A})$, *then it can be embedded in a triangle in* $\mathsf{D}(\mathcal{A})$

$$Q(X^{\cdot}) \xrightarrow{Q(u)} Q(Y^{\cdot}) \xrightarrow{Q(v)} Q(Z^{\cdot}) \to TQ(X^{\cdot}).$$

Definition 2.9 (K-injective Complex) *A complex* $X^{\cdot}$ *of* $\mathsf{K}(\mathcal{B})$ *is called* K*-injective (resp.,* K*-projective) if*

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(N^{\cdot}, X^{\cdot}) = 0 \quad (\textit{ resp.}, \ \mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(X^{\cdot}, N^{\cdot}) = 0\ )$$

for any null complex $N^{\cdot}$.

Example 2.10 *Let* A *be a ring,* $\mathsf{Mod}\,A$ *the category of right* A*-modules, and* $\mathsf{Inj}\,A$ *(resp.,* $\mathsf{Proj}\,A$*) the category of injective (resp., projective) right* A*-modules. Then any complex* $I^{\cdot} \in \mathsf{K}^{+}(\mathsf{Inj}\,A)$ *(resp.,* $P^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,A)$*) is a* K*-injective (resp.,* K*-projective) complex in* $\mathsf{K}(\mathsf{Mod}\,A)$.

Example 2.11 *Let* k *be a field,* $A = k[x]/(x^2)$, *and*

$$X^{\cdot} : \cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots .$$

Then $X^{\cdot}$ *is a null complex of finitely generated projective-injective* A*-modules. But it is neither* K*-projective nor* K*-injective, because* $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(X^{\cdot}, X^{\cdot}) \neq 0$.

Theorem 2.12 ([Sp], [Ne], [LAM], [Fr]) *Let* $\mathsf{K}^{inj}(\mathsf{Mod}\,A)$ *(resp.,* $\mathsf{K}^{proj}(\mathsf{Mod}\,A)$*) be the category of* K*-injective (resp.,* K*-projective) complexes, then the following hold.*

1. $(\mathsf{K}^{proj}(\mathsf{Mod}\,A), \mathsf{K}^{\phi}(\mathsf{Mod}\,A))$ *is a stable t-structure in* $\mathsf{K}(\mathsf{Mod}\,A)$, *and hence* $\mathsf{D}(\mathsf{Mod}\,A)$ *exists and is triangulated equivalent to* $\mathsf{K}^{proj}(\mathsf{Mod}\,A)$.

2. $(\mathsf{K}^{\phi}(\mathsf{Mod}\,A), \mathsf{K}^{inj}(\mathsf{Mod}\,A))$ *is a stable t-structure in* $\mathsf{K}(\mathsf{Mod}\,A)$, *and hence* $\mathsf{D}(\mathsf{Mod}\,A)$ *is triangulated equivalent to* $\mathsf{K}^{inj}(\mathsf{Mod}\,A)$.

3. *For a Grothendieck category* $\mathcal{A}$, $(\mathsf{K}^{\phi}(\mathcal{A}), \mathsf{K}^{inj}(\mathcal{A}))$ *is a stable t-structure in* $\mathsf{K}(\mathcal{A})$, *and hence* $\mathsf{D}(\mathcal{A})$ *exists and is triangulated equivalent to* $\mathsf{K}^{inj}(\mathcal{A})$.

Proof. For a complex $X^{\cdot} = (X^i, d^i)$, we define the following truncations:

$$\begin{aligned} \sigma_{\leq n}X^{\cdot} &: \cdots \to X^{n-2} \to X^{n-1} \to \operatorname{Ker} d^n \to 0 \to \cdots \\ \sigma_{\geq n}X^{\cdot} &: \cdots \to 0 \to \operatorname{Cok} d^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots \end{aligned}$$

(1) For any n, there is a complex $P_n^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,A)$ which has a quasi-isomorphism $P_n^{\cdot} \to \sigma_{\leq n}X^{\cdot}$. Then we have the following quasi-isomorphisms (qis)

$$X^{\cdot} \cong \varinjlim \sigma_{\leq n}X^{\cdot} \xleftarrow{\text{qis}} \underrightarrow{\mathsf{hocolim}}\, \sigma_{\leq n}X^{\cdot} \xleftarrow{\text{qis}} \underrightarrow{\mathsf{hocolim}}\, P_n^{\cdot}$$

Since $\operatorname{Hom}_{\mathcal{C}}(\coprod_n P_n^{\cdot}, -) \cong \prod_n \operatorname{Hom}_{\mathcal{C}}(P_n^{\cdot}, -)$, $\coprod_n P_n^{\cdot}$ is K-projective. Here $\mathrm{h}^M = \operatorname{Hom}_{\mathcal{C}}(M, -)$ for any object M. It is easy to see that $\underrightarrow{\mathsf{hocolim}}\, P_n$ is K-projective by the following exact sequence

$$\mathrm{h}^{\coprod_n P_n^{\cdot}} \to \mathrm{h}^{\coprod_n P_n^{\cdot}} \to \mathrm{h}^{\underrightarrow{\mathsf{hocolim}} P_n^{\cdot}} \to \mathrm{h}^{T(\coprod_n P_n^{\cdot})} \to \mathrm{h}^{T(\coprod_n P_n^{\cdot})}$$

(2) For any n, there is a complex $I_n^{\cdot} \in \mathsf{K}^{+}(\mathsf{Inj}\,A)$ which has a quasi-isomorphism $\sigma_{\geq -n}X^{\cdot} \to I_n^{\cdot}$. Then we have the following quasi-isomorphisms (qis)

$$X^{\cdot} \cong \varprojlim \sigma_{\geq -n}X^{\cdot} \xrightarrow{\text{qis}} \underleftarrow{\mathsf{holim}}\, \sigma_{\geq -n}X^{\cdot} \xrightarrow{\text{qis}} \underleftarrow{\mathsf{holim}}\, I_n^{\cdot}$$

by the same reason of (1), we have the statement.

(3) Because there is a ring A such that $\mathcal{A}$ is a localization of $\mathsf{Mod}\,A$ (Gabriel-Popescu Theorem). See [LAM] or [Fr] for details. $\square$

Remark 2.13 *If* $P^{\cdot}$ *is a* K*-projective complex (e.g. a bounded above complex of projective* A*-modules), then we have*

$$\operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(P^{\cdot}, X^{\cdot}) \cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(P^{\cdot}, X^{\cdot})$$

for any complex $X^{\cdot}$. *Similarly, for a* K*-injective complex* $I^{\cdot}$ *(e.g. bounded below complex of injective* A*-modules), then we have*

$$\operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(X^{\cdot}, I^{\cdot}) \cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(X^{\cdot}, I^{\cdot})$$

for any complex $X^{\cdot}$. *In particular, given* A*-modules* M, N, *we have*

$$\operatorname{Ext}_A^i(M, N) \cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(M, N[i])$$

Definition 2.14 (Double Complex, Total complex) *A double complex $C^{\cdot\cdot}$ is a bi-graded object $(C^{p,q})_{p,q\in\mathbb{Z}}$ of $\mathcal{A}$ together with $d_{\mathrm{I}}^{p,q}: C^{p,q} \to C^{p+1,q}$ and $d_{\mathrm{II}}^{p,q}: C^{p,q} \to C^{p,q+1}$ such that*

$$C^{\cdot q} = (C^{p,q}, d_{\mathrm{I}}^{p,q}: C^{p,q} \to C^{p+1,q}), \quad C^{p\cdot} = (C^{p,q}, d_{\mathrm{II}}^{p,q}: C^{p,q} \to C^{p,q+1})$$

are complexes satisfying $d_{\mathrm{I}}^{p,q+1}d_{\mathrm{II}}^{p,q} - d_{\mathrm{II}}^{p+1,q}d_{\mathrm{I}}^{p,q} = 0$. For a double complex $C^{\cdot\cdot}$, we define the total complexes

$$\mathrm{Tot}^{\amalg} C^{\cdot\cdot} = (X^n, d^n), \text{ where } X^n = \coprod_{p+q=n} C^{p,q}, d^n = \coprod_{p+q=n} (d_{\mathrm{I}}^{p,q} + (-1)^p d_{\mathrm{II}}^{p,q})$$
$$\mathrm{Tot}^{\Pi} C^{\cdot\cdot} = (Y^n, d^n), \text{ where } Y^n = \prod_{p+q=n} C^{p,q}, d^n = \prod_{p+q=n} (d_{\mathrm{I}}^{p,q} + (-1)^p d_{\mathrm{II}}^{p,q}).$$

Definition 2.15 (Cartan-Eilenberg Resolution) *For a complex $X^{\cdot} \in \mathsf{D}(\mathsf{Mod}\, A)$, let*

$$\cdots \to P^{-1\cdot} \to P^{0\cdot} \to X^{\cdot} \to 0 \ (\textit{resp.}, 0 \to X^{\cdot} \to I^{0\cdot} \to I^{1\cdot} \to \cdots)$$

be an exact sequence with $P^{\cdot n}$ (resp., $I^{\cdot n}$) being a complex of projective (resp., injective) A-modules. We call $\cdots \to P^{-1\cdot} \to P^{0\cdot}$ (resp., $I^{0\cdot} \to I^{1\cdot} \to \cdots$) a Cartan-Eilenberg projective (resp., injective) resolution of $X^{\cdot}$ if the induced complexes $\cdots \to \mathrm{B}^n(P^{-1\cdot}) \to \mathrm{B}^n(P^{0\cdot})$ and $\cdots \to \mathrm{H}^n(P^{-1\cdot}) \to \mathrm{H}^n(P^{0\cdot})$ (resp.,$\mathrm{B}^n(I^{0\cdot}) \to \mathrm{B}^n(I^{1\cdot}) \to \cdots$ and $\mathrm{H}^n(I^{0\cdot}) \to \mathrm{H}^n(I^{1\cdot}) \to \cdots$) are also projective (resp., injective) resolutions of $\mathrm{B}^n(\mathrm{X}^{\cdot}), \mathrm{H}^n(\mathrm{X}^{\cdot})$, respectively.

Proposition 2.16 *Under the setting of Definition 2.15, the following hold.*

1. $\mathrm{Tot}^{\amalg} P^{\cdot\cdot}$ *is* K*-projective, and the induced morphism of complexes* $\mathrm{Tot}^{\amalg} P^{\cdot\cdot} \to X^{\cdot}$ *is a quasi-isomorphism.*

2. $\mathrm{Tot}^{\Pi} I^{\cdot\cdot}$ *is* K*-injective, and the induced morphism of complexes* $X^{\cdot} \to \mathrm{Tot}^{\Pi} I^{\cdot\cdot}$ *is a quasi-isomorphism.*

Sketch of Proof. We consider the following truncations

$$\sigma^{\mathrm{ii}}_{\leq n} P^{\cdot\cdot} : \cdots \to \sigma_{\leq n} P^{-1} \to \sigma_{\leq n} P^0, \quad \sigma^{\mathrm{ii}}_{\geq n} I^{\cdot\cdot} : \sigma_{\geq n} I^0 \to \sigma_{\geq n} I^1 \to \cdots$$

Then it is easy to see $\mathrm{Tot}^{\amalg} \sigma^{\mathrm{ii}}_{\leq n} P^{\cdot\cdot}$ (resp., $\mathrm{Tot}^{\Pi} \sigma^{\mathrm{ii}}_{\geq n} I^{\cdot\cdot}$) is K-projective (resp., K-injective), and that the induced morphism of complexes $\mathrm{Tot}^{\amalg} \sigma^{\mathrm{ii}}_{\leq n} P^{\cdot\cdot} \to \sigma_{\leq n} X^{\cdot}$ (resp., $\sigma_{\geq n} X^{\cdot} \to \mathrm{Tot}^{\Pi} \sigma^{\mathrm{ii}}_{\geq n} I^{\cdot\cdot}$) is a quasi-isomorphism. Therefore we have the following quasi-isomorphisms (qis)

$$X^{\cdot} \xleftarrow{\text{qis}} \underrightarrow{\mathsf{hocolim}}\ \sigma_{\leq n} X^{\cdot} \xleftarrow{\text{qis}} \underrightarrow{\mathsf{hocolim}}\ \mathrm{Tot}^{\amalg} \sigma^{\mathrm{ii}}_{\leq n} P^{\cdot\cdot} \xrightarrow{\text{qis}} \mathrm{Tot}^{\amalg} P^{\cdot\cdot}$$

$$(\text{resp., } X^{\cdot} \xrightarrow{\text{qis}} \underleftarrow{\mathsf{holim}}\ \sigma_{\geq -n} X^{\cdot} \xrightarrow{\text{qis}} \underleftarrow{\mathsf{holim}}\ \mathrm{Tot}^{\Pi} \sigma^{\mathrm{ii}}_{\geq -n} I^{\cdot\cdot} \xleftarrow{\text{qis}} \mathrm{Tot}^{\Pi} I^{\cdot\cdot})$$

and $\mathrm{Tot}^{\amalg} P^{\cdot\cdot}$ (resp., $\mathrm{Tot}^{\Pi} I^{\cdot\cdot}$) is K-projective (resp,m K-injecive). $\square$

Definition 2.17 (Right Derived Functor) *For a ∂-functor $F : \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$, the right derived functor of F is a ∂-functor*

$$\boldsymbol{R}^*F : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$$

together with a functorial morphism of ∂-functors

$$\xi \in \partial\operatorname{Mor}(Q_{\mathcal{A}'} \circ F, \boldsymbol{R}^*F \circ Q^*_{\mathcal{A}})$$

with the following property:
For $G \in \partial(\mathsf{D}^*(\mathcal{A}), \mathsf{D}(\mathcal{A}'))$ and $\zeta \in \partial\operatorname{Mor}(Q_{\mathcal{A}'} \circ F, G \circ Q^*_{\mathcal{A}})$, there exists a unique morphism $\eta \in \partial\operatorname{Mor}(\boldsymbol{R}^*F, G)$ such that

$$\zeta = (\eta Q^*_{\mathcal{A}})\xi.$$

In other words, we can simply write the above using functor categories. For triangulated categories $\mathcal{C}, \mathcal{C}'$, the ∂-functor category $\partial(\mathcal{C}, \mathcal{C}')$ is the category (?) consisting of ∂-functors from $\mathcal{C}$ to $\mathcal{C}'$ as objects and ∂-functorial morphisms as morphisms. Then we have

$$\partial\operatorname{Mor}(Q_{\mathcal{A}'} \circ F, -Q^*_{\mathcal{A}}) \cong \partial\operatorname{Mor}(\boldsymbol{R}^*F, -) \qquad \begin{tikzcd} \mathsf{K}^*(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathsf{K}(\mathcal{A}') \arrow[d, "Q_{\mathcal{A}'}"] \\ \mathsf{D}^*(\mathcal{A}) \arrow[r, shift left, "\boldsymbol{R}^*F"] \arrow[r, shift right, "G"'] & \mathsf{D}(\mathcal{A}') \end{tikzcd}$$

as functors from $\partial(\mathsf{D}^*(\mathcal{A}), \mathsf{D}(\mathcal{A}'))$ to $\mathfrak{Set}$.

Proposition 2.18 *Let $\mathcal{A}, \mathcal{A}'$ be abelian categories, $F : \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$ a ∂-functor. If $\mathcal{A}$ is a Grothendieck category, then we have the right derived functor $\boldsymbol{R}F : \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$ such that $F(X^\cdot) \cong \boldsymbol{R}F(X^\cdot)$ for any K-injective complex $X^\cdot$.*

Remark 2.19 *In the setting of Definition 2.17, the left derived functor $\boldsymbol{L}^*F : \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$ can be also defined by reversing arrows of ∂-functorial morphisms. Let $\mathrm{R}^n F(X^\cdot) = \mathrm{H}^n(\boldsymbol{R}F(X^\cdot))$, $\mathrm{L}^n F(X^\cdot) = \mathrm{H}^n(\boldsymbol{L}F(X^\cdot))$, then $\mathrm{R}^n F$ (resp., $\mathrm{L}^n F$) coincides with the ordinary definition of the n-th right (resp., left) derived functor. According to Proposition 2.16, if F commutes with products (resp., coproducts), then the n-th hypercohomology $\mathbb{R}^nF$ (resp., hyperhomology $\mathbb{L}^nF$) coincides with $\mathrm{R}^n F$ (resp., $\mathrm{L}^n F$) (c.f. [CE], [Mc], [We]).*

Definition 2.20 ($\operatorname{Hom}^\cdot_A, \dot{\otimes}_A$) *Let $X^\cdot, Y^\cdot$ be complexes in $\mathsf{C}(\operatorname{Mod} A)$, $Z^\cdot$ a complex in $\mathsf{C}(\operatorname{Mod} A^{op})$. We define the complex $\operatorname{Hom}^\cdot_A(X^\cdot, Y^\cdot)$ in $\mathsf{C}(\mathfrak{Ab})$ by*

$$\operatorname{Hom}^n_A(X^\cdot, Y^\cdot) = \prod_{j-i=n} \operatorname{Hom}_A(X^i, Y^j), \quad d^n_{\operatorname{Hom}^\cdot(X,Y)}(f) = d_X \circ f - (-1)^n f \circ d_Y$$

for $f \in \operatorname{Hom}^n_A(X^\cdot, Y^\cdot)$. And we define the complex $X^\cdot \dot{\otimes}_A Z^\cdot$ in $\mathsf{C}(\mathfrak{Ab})$ by

$$X^\cdot \overset{n}{\otimes}_A Z^\cdot = \coprod_{i+j=n} X^i \otimes_A Z^j, \quad d^n_{X \otimes Y} = d_X \otimes 1 + (-1)^n 1 \otimes d_Z$$

Proposition 2.21 *Let A be a ring. Then we have a right derived functor*

$$\boldsymbol{R}\operatorname{Hom}^{\cdot}_{A} : \mathsf{D}(\mathsf{Mod}\,A)^{op} \times \mathsf{D}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathfrak{Ab})$$

and a left derived functor

$$\dot{\otimes}^{\boldsymbol{L}}_{A} : \mathsf{D}(\mathsf{Mod}\,A) \times \mathsf{D}(\mathsf{Mod}\,A^{op}) \to \mathsf{D}(\mathfrak{Ab})$$

Proposition 2.22 *Let A be a ring. For complexes $X^{\cdot}, Y^{\cdot}$, we have isomorphisms*

$$H^{n}(\operatorname{Hom}^{\cdot}_{A}(X^{\cdot}, Y^{\cdot})) \cong \operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(X^{\cdot}, Y^{\cdot}[n])$$
$$H^{n}(\boldsymbol{R}\operatorname{Hom}^{\cdot}_{A}(X^{\cdot}, Y^{\cdot})) \cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(X^{\cdot}, Y^{\cdot}[n])$$

Definition 2.23 (Perfect Complex) *Let A be a ring. A complex $X^{\cdot} \in \mathsf{D}(\mathsf{Mod}\,A)$ is called a perfect complex if $X^{\cdot}$ is quasi-isomorphic to a bounded complex of finitely generated projective A-modules.*

Let X be a scheme, $\mathsf{D}(X)$ the derived category of sheaves of $\mathcal{O}_X$-modules. We denote by $\mathsf{D}_{qc}(X)$ the full subcategory of $\mathsf{D}(X)$ consisting of complexes whose cohomologies are quasi-coherent sheaves. A complex $X^{\cdot} \in \mathsf{D}_{qc}(X)$ is called a perfect complex if $X^{\cdot}$ is locally quasi-isomorphic to a bounded complex of vector bundles (See [TT]).

We denote by $\mathsf{D}_{pf}(\mathcal{A})$ the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ consisting of perfect complexes.

Proposition 2.24 ([Rd1], [Ne]) *For a ring A, the following hold.*

1. *A complex $X^{\cdot} \in \mathsf{D}(\mathsf{Mod}\,A)$ is perfect if and only if it is a compact object in $\mathsf{D}(\mathsf{Mod}\,A)$.*
2. *$\mathsf{D}(\mathsf{Mod}\,A)$ is compactly generated.*

Theorem 2.25 ([BV]) *Let X be a quasi-compact quasi-separated scheme, then the following hold.*

1. *A complex $X^{\cdot} \in \mathsf{D}_{qc}(X)$ is perfect if and only if it is a compact object in $\mathsf{D}_{qc}(X)$.*
2. *$\mathsf{D}_{qc}(X)$ is compactly generated.*

Theorem 2.26 ([BN]) *Let X be a quasi-compact separated scheme, then the canonical functor $\mathsf{D}(\mathsf{Qcoh}\,X) \to \mathsf{D}_{qc}(X)$ is a triangulated equivalence, where $\mathsf{Qcoh}\,X$ is the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules.*

Corollary 2.27 ([BV]) *Let X be smooth over a field, then we have*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\,X) \overset{\triangle}{\cong} \mathsf{D}_{pf}(X).$$

where $\mathsf{coh}\,X$ is the category of coherent sheaves of $\mathcal{O}_X$-modules.

For a ring A, we denote by $\mathsf{proj}\,A$ the category of finitely generated projective A-modules.

Theorem 2.28 (**[Rd1]**, **[Rd2]**) *Let A, B be algebras over a field k. The following are equivalent.*

1. $\mathsf{D}(\mathsf{Mod}\,A) \overset{\Delta}{\cong} \mathsf{D}(\mathsf{Mod}\,B)$.
2. $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,A) \overset{\Delta}{\cong} \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,B)$.
3. *There is a perfect complex $T^{\cdot} \in \mathsf{D}(\mathsf{Mod}\,A)$ such that*
   (a) $B \cong \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\,A)}(T^{\cdot})$,
   (b) $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(T^{\cdot}, T^{\cdot}[i]) = 0$ *for* $i \neq 0$,
   (c) $\{T^{\cdot}[i] | i \in \mathbb{Z}\}$ *is a generating set in* $\mathsf{D}(\mathsf{Mod}\,A)$.
4. *There is a complex $V^{\cdot}$ of B-A-bimodules such that*

$$\boldsymbol{R}\mathrm{Hom}^{\cdot}_{A}(V^{\cdot}, -) : \mathsf{D}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathsf{Mod}\,B)$$

   is an equivalence.

In this case, $T^{\cdot}$ is called a tilting complex for A, $V^{\cdot}$ is called a two-sided tilting complex, and $\boldsymbol{R}\mathrm{Hom}^{\cdot}_{A}(V^{\cdot}, -)$ is called a standard equivalence.

Theorem 2.29 (**[BO]**) *Let X be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If X' is a smooth algebraic variety such that $\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,X) \overset{\Delta}{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,X')$, then X' is isomorphic to X.*

Theorem 2.30 (**[Be]**) *Let $\mathbf{P} = \mathbf{P}^n_k$ be the n-dimensional projective space over a field k, and let $\mathcal{T}_1 = \bigoplus_{i=0}^{n} \mathcal{O}(i)$, $\mathcal{T}_2 = \bigoplus_{i=0}^{n} \Omega(-i)$, and $B_1 = \mathrm{End}_{\mathbf{P}}(\mathcal{T}_1)$, $B_2 = \mathrm{End}_{\mathbf{P}}(\mathcal{T}_2)$. Then B_i is a finite dimensional k-algebra of finite global dimension, and*

$$\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathbf{P}) \overset{\Delta}{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,B_i)$$

where $\mathsf{mod}\,B_i$ is the category of finitely generated B_i-modules ($i = 1, 2$).

Definition 2.31 *Let A be an algebra over a field k. The* derived Picard group *of A (relative to k) is*

$$\mathrm{DPic}_k(A) := \frac{\{\textit{two-sided tilting complexes } T \in \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\,A^{\mathsf{e}})\}}{\textit{isomorphism}}$$

with identity element A, product $(T_1, T_2) \mapsto T_1 \otimes^{\mathsf{L}}_{A} T_2$ and inverse $T \mapsto T^{\vee} := \mathrm{R}\,\mathrm{Hom}_A(T, A)$. Given any k-linear triangulated category $\mathcal{C}$ we let

$$\mathrm{Out}^{\Delta}_{k}(\mathcal{C}) := \frac{\{\textit{k-linear triangulated self-equivalences of } \mathcal{C}\}}{\textit{isomorphism}}.$$

Theorem 2.32 ([MY]) *Let k be an algebraically closed field, and A a finite dimensional hereditary k-algebra. Then we have*

$$\mathrm{DPic}_k(A) = \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\,A)) = \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,A))$$

M. Kontsevich and A. Rosenberg introduced the notion of non-commutative projective spaces $\mathbf{NP}^n$ [KR], and showed that

$$\mathsf{D}^{\mathsf{b}}(\mathsf{Qcoh}\,\mathbf{NP}^n) \overset{\triangle}{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\,kQ_n)$$
$$\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathbf{NP}^n) \overset{\triangle}{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,kQ_n)$$

where Q_n is the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha_0"] \arrow[r, bend right, "\alpha_n"'] \arrow[r, phantom, "\vdots"] & \bullet \end{tikzcd}$$

Corollary 2.33 ([MY]) *For non-commutative projective spaces $\mathbf{NP}^n$, we have*

$$\begin{aligned} \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathsf{b}}(\mathsf{Qcoh}\,\mathbf{NP}^n)) &\cong \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathbf{NP}^n)) \\ &\cong \mathbb{Z} \times (\mathbb{Z} \ltimes \mathrm{PGL}_{n+1}(k)) \end{aligned}$$

Theorem 2.34 ([BO]) *Let X be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,X))$ is generated by the automorphisms of variety, the twists by invertible sheaves and the translations, and hence $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,X)) \cong (\mathrm{Aut}_k X \ltimes \mathrm{Pic}\,X) \times \mathbb{Z}$.*

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