

2-Microlocal Boundary Value Problems and Their Applications

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1. Let M be a real analytic manifold, X a complex neighbourhood of M , and Y a complex hypersurface of X . Let Λ be a regular involutive conic submanifold of $T^*_M X$, Λ^C (resp. $\tilde{\Lambda}$) a complexification (resp. a partial complexification) of Λ in T^*X . Let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$ -Module defined in a neighbourhood of Λ^C and assume that $Y \rightarrow X$ is non microcharacteristic along Λ^C for $\mathfrak{M}$ (cf. Def. 2.2 of [1] and Def. 2.10.3 of [5]). Set $\Sigma = \Lambda \cap \pi^{-1}(Y)$, and denote by Λ_+ a domain of Λ with boundary Σ . Then we can define the microlocal boundary values along Λ to Σ for $\mathcal{B}^2_{\Lambda}$ -solutions to $\mathfrak{M}$. To be precise, there exists the boundary value map

$$bv : R\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}, \Gamma_{\Lambda_+}(\mathcal{B}^2_{\Lambda}))|_{\Sigma} \rightarrow R\mathcal{H}om_{\mathcal{E}_Y}(\mathfrak{M}_Y, \mathcal{B}^2_{\Sigma}),$$

where $\mathcal{B}^2_{\Lambda}$ denotes the sheaf of 2-hyperfunctions on Λ due to Kashiwara (cf. [4], [10]), and $\mathfrak{M}_Y$ denotes the tangential system on Y of $\mathfrak{M}$.

We set

$$\mathcal{E}^2_{\Lambda_+}|_{\tilde{\Lambda}} = \mu\text{hom}(\mathcal{E}_{\Lambda_+}, \mathcal{E}^h_{\tilde{\Lambda}}) \otimes \text{or}_{\Lambda|\tilde{\Lambda}}[\text{codim } \Lambda],$$

where $\mathcal{E}^h_{\tilde{\Lambda}}$ denotes the sheaf of microfunctions with holomorphic parameters on $\tilde{\Lambda}$, and $\text{or}_{\Lambda|\tilde{\Lambda}}$ denotes the relative orientation sheaf on Λ .

Refer to Kashiwara-Schapira [3] for the bifunctor μhom . This complex of $\mathcal{E}_X$ -Modules is a 2-microlocal version of $\mathcal{E}_{\Omega|X}$, the "sheaf" of microfunctions for boundary value problems, which is introduced in Schapira [8]. This allows us to define the "boundary 2-analytic wavefront set" $SS^2_{\Lambda_+}$ for sections of $\Gamma_{\Lambda_+}(\mathcal{B}^2_{\Lambda})$, and we can analyse the boundary value map "bv" 2-microlocally in terms of $SS^2_{\Lambda_+}$. In particular, we can show the reflection of 2-microlocal singularity at the boundary. Refer to [13] for the details.

2. Let Ω be an open subset of M with real analytic boundary $N = \{\varphi=0\}$, Y the complexification of N in X . We denote by ρ the natural

projection $Y \times_X T^*X \longrightarrow T^*Y$. Take a point $y^* \in T^*_N Y$ and a point $x^* \in T^*_M X \times N$ with $\rho(x^*) = y^*$. Let P be a microdifferential operator defined in a neighbourhood of x^* with involutive double characteristics. Precisely we assume that the principal symbol $\sigma(P)$ of P is decomposed by homogeneous holomorphic functions p_1, p_2, q :

$$\sigma(P) = q \cdot p_1^{m_1} \cdot p_2^{m_2}$$

in a neighbourhood of x^* and they satisfy the following conditions:

- (1) p_1 and p_2 are real valued on $T^*_M X$,
- (2) $p_1(x^*) = p_2(x^*) = 0$, $q(x^*) \neq 0$,
- (3) $dp_1 \wedge dp_2 \wedge \omega \neq 0$,
- (4) $\{p_1, p_2\} = 0$ on $\Lambda = \{p_1 = p_2 = 0\}$,
- (5) $\{q, p_i\} \neq 0$ ($i=1, 2$).

In this situation we consider the microlocal boundary value problem

$$(D) \quad \begin{cases} Pu = 0 \text{ at } x^*, \\ (\varphi_X \cdot D_X)^i u|_{\varphi \rightarrow +0} = 0 \text{ at } y^* \quad (0 \leq i < \max(m_1, m_2)). \end{cases}$$

Remark that we take here the boundary value $(\varphi_X \cdot D_X)^i u|_{\varphi \rightarrow +0}$ of u in the microlocal sense from a neighbourhood of x^* .

Then we have the following results.

We denote by $SS_\Omega(u)$ the boundary analytic wavefront set of u (cf. [8] for the definition of the boundary analytic wavefront set).

Theorem 1.---Let Γ be a real bicharacteristic leaf of Λ passing through x^* . For any solution u of the microlocal boundary value problem (D), there exists a subset $\{x^*_s\}$ of $\Gamma \cap \Sigma$ such that

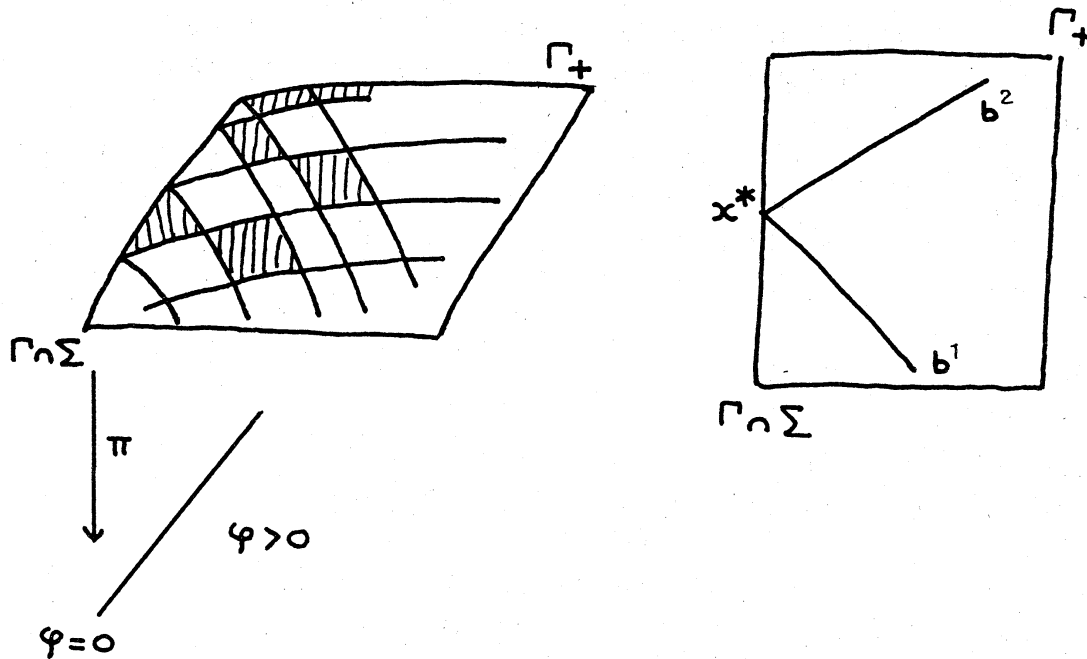
$$SS_\Omega(u) \cap \Gamma = \text{the closure in } \Gamma \text{ of the union of } \{b^i_s; s, i=1, 2\} \text{ and}$$

$$\text{some of connected components of } \Gamma_+ \setminus \bigcup \{b^i_s; s, i=1, 2\},$$

where $\Gamma_+ = \Gamma \times_M \Omega$, and b^i_s denotes the half integral curve of H_{p_i} , the Hamilton vector field of p_i , issued from x^*_s into Γ_+ .

Corollary 2.---For any solution u of the microlocal boundary value problem (D),

$$\begin{aligned} & b^1(x^*) \cup b^2(x^*) \not\subset SS(u|_{\Omega}) \\ \Rightarrow & x^* \notin SS_{\Omega}(u). \end{aligned}$$



Theorem 1 is obtained as an application of the theory of Section 1 (cf. [13]).

Remark.---As for the results in the interior domain for the same operator we refer to Tose [10, 11, 12]. We also refer to Lascar [6] for the similar result as Corollary 2 in the C^∞ category. Note that we assume in Corollary 2 that at least one of the integral curves $b^1(x^*)$, $b^2(x^*)$ is not contained in $SS(u|_{\Omega})$ in a neighbourhood of x^* .

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