

Molecular predissociation resonances below an energy level crossing

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ABSTRACT. We study the resonances of 2×2 systems of one dimensional Schrödinger operators which are related to the mathematical theory of molecular predissociation. We determine the precise positions of the resonances with real parts below the energy where bonding and anti-bonding potentials intersect transversally. In particular, we find that imaginary parts (widths) of the resonances are exponentially small and that the indices are determined by Agmon distances for the minimum of two potentials.

1. INTRODUCTION

In this paper, we study precise positions of molecular predissociation resonances near the real axis as semiclassical asymptotics. More precisely, we consider resonances whose real parts are apart from a bottom of a well and a crossing of potentials. In particular, we obtain exponentially small imaginary parts (widths) for these resonances.

In the theory of shape resonances of Schrödinger operators it is known that widths of the resonances have exponential bounds as $C_\epsilon e^{-2(1-\epsilon)S/h}$ where ϵ can be taken arbitrarily small, h is the semiclassical parameter and S is the Agmon distance between the bounded and unbounded regions where the potential is below the real part of the resonance (see, e.g., [2, 7, 8]). In one dimensional case, Servat [15] determine the precise positions of the resonances whose real parts are apart from the critical values of potentials using WKB constructions and considering the connection of the solutions and quantization conditions. In particular, it is proved that the exact order of the imaginary parts of the resonances is $he^{-2S/h}$.

When we consider the Schrödinger equation of molecules the study of the equation for electrons and nuclei is reduced to that of semiclassical system of Schrödinger-type operators by the Born–Oppenheimer approximation (see, e.g., [10, 12, 13]). In the Born–Oppenheimer approximation the semiclassical parameter h is the square-root of the ratio of electronic to nuclear mass, and the potentials of the diagonal elements of the matrix of the Schrödinger operators describe electronic energy levels. At sufficiently low energies, since this system is scalar, numerous results from the semiclassical analysis of the Schrödinger operators can be applied. When several electronic levels are involved, states in different electronic levels interact due to the off-diagonal first order differential or pseudodifferential operators.

In Martinez [11], Nakamura [14], Baklouti [1] and Grigis-Martinez [6], they study resonances for potentials that do not intersect and obtain exponential bound on their widths. Klein [9] studies the case of more than two intersecting potentials some of them forming wells to trap nuclei and the others being non-trapping. In this case, it is shown that the widths of the resonances with real parts converging to the bottom of the potential well have the exponential bound as in the case of usual Schrödinger operators with Agmon distance of the minimum of the two potentials. For intersecting two potentials, Grigis-Martinez [5] obtains the full asymptotic of the width of the resonance at the bottom of the well, showing that the exponential bound is optimal.

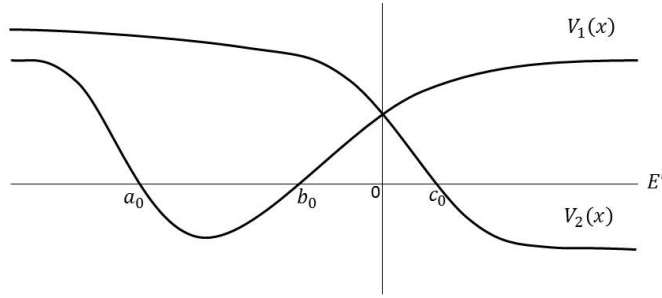


FIGURE 1. The graph of the two potentials on the real axis

In Fujiié-Martinez-Watanabe [3] they consider the resonances with real parts in the distance of order $h^{2/3}$ from a crossing of two potentials in one dimensional space. They construct the solution to the system on the left and right intervals from the crossing and consider the condition that solutions decaying on the left and those outgoing on the right are connected at the crossing. The solutions to the system are constructed as series by successive approximation using the Yafaev's construction for the solution of one dimensional Schrödinger equations (see [16]) and showing that norms of operators including fundamental solutions are small for small h . Under an additional condition of ellipticity on the interaction they obtain the exact order $h^{5/3}$ of the widths of the resonances.

Here, as in [3] we study 2×2 matrix system, the diagonal part of which consists of semiclassical Schrödinger operators, and the off-diagonal parts of first order differential operators. We assume that two potentials V_1 and V_2 cross transversally and that below the crossing V_1 admits a well, while V_2 is non-trapping (see Fig. 1). We study the resonances with imaginary parts $\mathcal{O}(h)$ and with real parts apart from the crossing and the bottom of the well by a constant distance independent of h . The real parts of the resonances behave like eigenvalues obtained from usual Bohr–Sommerfeld quantization condition for the well, that is, the leading terms e_k of the real parts of the resonances satisfy $\int_a^b \sqrt{e_k - V_1(x)} dx = (k + 1/2)\pi h$, where a and b are the endpoints of the well. We also find that the widths of the resonances have exponential bounds as in [9, 5] and under a condition on the interaction and the real part of the resonances they behave exactly like $h^2 e^{-2S/h}$, where S is the Agmon distance for $\min\{V_1, V_2\}$ between endpoints b and c of the region where the potentials are greater than e_k . In our model, the interaction is of the form $h(r_0(x) + hr_1(x)\partial_x)$ and the condition on the interaction is that $r_0(0) + r_1(0)\sqrt{V_1(0) - e_k} \neq 0$, where we assume the position of the crossing is $x = 0$. This result should be compared with [15, Theorem 2.6] for the case of one well. Paying attention to the shape of the graph of $\min\{V_1, V_2\}$ on the real axis our exponential bound is expected.

To prove our main theorem, we construct solutions to the system on four intervals, because if we formally construct solutions on two intervals as in [3] the series in the solutions do not converge. We need to construct four solutions on each of the middle two intervals since the solutions on the left and right intervals are represented as linear combinations of the four solutions. To control the behavior of the series in the solutions, we need four fundamental solutions on each of the middle intervals. We choose the fundamental solutions so that formal leading terms in the solutions will really determine the order of the solution with respect to h . We study the connection of the solutions and calculate the transition matrix at the crossing and change the basis of the solutions in order that some of the elements of the transition matrix will be zero. This makes the calculation of the quantization condition simple and we can find exponentially small terms with the expected index.

The content of this paper is as follows. In Sect. 2 we give the assumptions and state our main result. In Sect. 3 we give some preliminaries. In particular, we introduce solutions to the scalar Schrödinger equations. In Sect. 4 we construct fundamental solutions on four intervals and give estimates on

them. In Sect. 5 we give the solutions to the system on the four intervals. In Sect. 6 we study the connection of the solutions and change the bases of the solutions to make transition matrix suitable for the calculation of the quantization condition. In Sect. 7 the quantization condition is given. Using the quantization condition in Sect. 8 we prove the main theorem.

2. ASSUMPTIONS AND RESULTS

As in [3], we consider a 2×2 Schrödinger operator of the type,

$$(2.1) \quad Pu = Eu, P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix},$$

where $P_j = -h^2\Delta + V_j(x)$, $j = 1, 2$, $W = r_0(x) + hr_1(x)\partial_x$ and W^* is the formal adjoint of W .

We suppose the following conditions on $V_1(x)$, $V_2(x)$ (see Fig. 1) and $r_0(x)$, $r_1(x)$.

Assumption (A1) $V_1(x)$ and $V_2(x)$ are real-valued analytic functions on $\mathbb{R}$ and extend to holomorphic functions in the complex domain,

$$\Gamma = \{x \in \mathbb{C}; |\operatorname{Im} x| < \delta_0 \langle \operatorname{Re} x \rangle\},$$

where $\delta_0 > 0$ is a constant and $\langle t \rangle := (1 + |t|^2)^{1/2}$.

Assumption (A2) For $j = 1, 2$, V_j admits limits as $\operatorname{Re} x \rightarrow \pm\infty$ in Γ and there exists a real number E' such that

$$\begin{aligned} \lim_{\substack{\operatorname{Re} x \rightarrow -\infty \\ x \in \Gamma}} V_1(x) &> E'; & \lim_{\substack{\operatorname{Re} x \rightarrow -\infty \\ x \in \Gamma}} V_2(x) &> E'; \\ \lim_{\substack{\operatorname{Re} x \rightarrow +\infty \\ x \in \Gamma}} V_1(x) &> E'; & \lim_{\substack{\operatorname{Re} x \rightarrow +\infty \\ x \in \Gamma}} V_2(x) &< E'. \end{aligned}$$

Assumption (A3) There exist real numbers $a_0 < b_0 < 0 < c_0$ such that

- $V_1 > E'$ and $V_2 > E'$ on $(-\infty, a_0)$;
- $V_1 < E' < V_2$ on (a_0, b_0) ;
- $E' < V_1 < V_2$ on $(b_0, 0)$;
- $E' < V_2 < V_1$ on $(0, c_0)$;
- $V_2 < E' < V_1$ on $(c_0, +\infty)$,

Moreover, one has

$$V'_1(a_0) < 0, \quad V'_1(b_0) > 0, \quad V'_2(c_0) < 0, \quad V'_1(0) > V'_2(0).$$

Assumption (A4) r_0 and r_1 are bounded analytic functions on Γ , and $r_0(x)$ and $r_1(x)$ are real when x is real.

As in [3] we can define the resonances of P as eigenvalues of the operator P acting on $L^2(\mathbb{R}_\theta) \oplus L^2(\mathbb{R}_\theta)$ where $\mathbb{R}_\theta$ is a complex distortion of $\mathbb{R}$ that coincides with $e^{i\theta}\mathbb{R}$ for $x \gg 1$. We denote by $\operatorname{Res}(P)$ the set of the resonances of P .

For $d > 0$ small enough, we set $I := [E' - d, E' + d]$. We fix C_0 arbitrarily large, and we study the resonances of P lying in the set

$$\mathcal{D}_I := \{E \in \mathbb{C}; \operatorname{Re} E \in I, -C_0 h < \operatorname{Im} E < 0\}.$$

For $E \in \mathcal{D}_I$, $V_1(x) = E$ has only two solutions and we denote the solution with the smaller real part and the larger one by $a = a(E)$ and $b = b(E)$ respectively. We also denote the unique solution of $V_2(x) = E$ for $E \in \mathcal{D}_I$ by $c = c(E)$. For $E \in \mathcal{D}_I$ we define the action,

$$\mathcal{A}(E) := \int_{a(E)}^{b(E)} \sqrt{E - V_1(t)} dt.$$

Then our result is the following.

Theorem 2.1. *Under Assumptions (A1)-(A4), for $h > 0$ small enough, one has,*

$$\text{Res}(P) \cap \mathcal{D}_I = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_I$$

where $E_k(h)$'s are complex numbers that satisfy,

$$(2.2) \quad \begin{aligned} \text{Re } E_k(h) &= e_k(h) + \mathcal{O}(h^{3/2}) \\ \text{Im } E_k(h) &= -\frac{h^2\pi}{4} \mathcal{A}'(e_k(h))^{-1} e^{-2S(e_k)/h} (V_1(0) - e_k(h))^{-1/2} (V_1'(0) - V_2'(0))^{-1} \\ &\quad \cdot (r_0(0) + r_1(0) \sqrt{V_1(0) - e_k(h)})^2 + \mathcal{O}(h^{5/2} e^{-2S(e_k)/h}), \end{aligned}$$

uniformly as $h \rightarrow 0_+$ where

$$\begin{aligned} e_k &= e_k(h) := \mathcal{A}^{-1}\left((k + \frac{1}{2})\pi h\right), \\ S(e_k) &= \int_{b(e_k)}^0 \sqrt{V_1(t) - e_k} dt + \int_0^{c(e_k)} \sqrt{V_2(t) - e_k} dt. \end{aligned}$$

Remark 2.2. *For the first term on the right-hand side of (2.2) to be a semiclassical equivalent of the width of the resonance for sufficiently small h , one needs the additional ellipticity condition of W that $e_k(h)$ satisfies $|r_0(0) + r_1(0) \sqrt{V_1(0) - e_k(h)}| > \rho$, where ρ is a positive constant independent of h .*

3. SOME PRELIMINARIES

As in [3] let $x_\infty > 0$ be a sufficiently large number and $f \in C^\infty(\mathbb{R}_+, \mathbb{R}_+)$ be a function such that $f(x) = x$ for x large enough, $f(x) = 0$ for $x \in [0, x_\infty]$ and

$$f'(t) \geq \delta(t^{-1}f(t) + t^{-3}f(t)^3),$$

for some constant $\delta > 0$. In the sequel we will use the following notation:

$$\begin{aligned} I_b &:= [b, 0] ; \quad I_c := [0, c]; \\ I_L &:= (-\infty, b] ; \quad I_R^\theta := F_\theta([c, +\infty)), \end{aligned}$$

for real b and c where $F_\theta(x) := x + i\theta f(x)$. For complex b and c , we denote by the same notation appropriate curves in the complex plane connecting the end points. Then $\theta > 0$ small enough, $E \in \mathcal{D}_I$ and some positive constant C , we have

$$\text{Im} \int_{x_\infty}^{F_\theta} \sqrt{E - V_2(t)} dt \geq -Ch,$$

where the integral is taken along I_R^θ .

We fix $E' \in I$ and $x_0 \in (a(E'), b(E'))$. Then we can define a function $\xi_1(x; E)$ which depends analytically on $x \in \{x \in \mathbb{C}; |\text{Im}(x - b(E'))| < \delta' \langle \text{Re}(x - b(E')) \rangle, \text{Re } x > x_0\}$ with $\delta' > 0$ small enough and $E \in \mathcal{D}_I$ sufficiently close to E' as follows:

$$\xi_1(x; E) = \left(\frac{3}{2} \int_{b(E)}^x \sqrt{V_1(t) - E} dt \right)^{2/3}.$$

Similarly, we can define,

$$\xi_2(x; E) = \left(\frac{3}{2} \int_{c(E)}^x \sqrt{E - V_2(t)} dt \right)^{2/3},$$

which depends analytically on $E \in \mathcal{D}_I$ and $x \in \{x \in \mathbb{C}; |\text{Im}(x - c(E))| < \delta' \langle \text{Re}(x - c(E)) \rangle\}$ with $\delta' > 0$ small enough.

In the same way as in [3, Appendix A.2] we have solutions to $(P_j - E)u = 0$ for $E \in \mathcal{D}_I$. We denote by $\text{Ai}(x)$ and $\text{Bi}(x)$ the Airy functions.

Proposition 3.1. *Let $E \in \mathcal{D}_I$. Then,*

(i) *For sufficiently small $h > 0$, the equation $(P_1 - E)u = 0$ admits two solutions $u_{1,R}^\pm$ on $\Gamma_1 = \{x \in \mathbb{C}; |\operatorname{Im}(x - b(E))| \leq \delta_1 |\operatorname{Re}(x - b(E))|, \operatorname{Re} x \geq x_0\}$ with sufficiently small $\delta_1 > 0$ such that as $x \rightarrow +\infty$,*

$$u_{1,R}^\pm(x) \sim (1 + \mathcal{O}(h)) \frac{h^{1/6}}{\sqrt{\pi}} (V_1(x) - E)^{-1/4} e^{\pm \int_{b(E)}^x \sqrt{V_1(t) - E} dt/h},$$

uniformly with respect to $h > 0$ small enough, and as $h \rightarrow 0_+$,

$$\begin{aligned} u_{1,R}^- &= 2(\xi_1'(x))^{-1/2} \operatorname{Ai}(h^{-2/3} \xi_1(x)) (1 + \mathcal{O}(h)) \\ &\quad \text{on } \Gamma_1 \cap \{\operatorname{Re} \xi_1(x) \geq 0\}; \\ u_{1,R}^- &= 2(\xi_1'(x))^{-1/2} \operatorname{Ai}(h^{-2/3} \xi_1(x)) + \mathcal{O}(h(1 + h^{-2/3} |\xi_1(x)|)^{-1/4}) \\ &\quad \text{on } \Gamma_1 \cap \{\operatorname{Re} \xi_1(x) \leq 0\}; \\ u_{1,R}^+ &= (\xi_1'(x))^{-1/2} \operatorname{Bi}(h^{-2/3} \xi_1(x)) (1 + \mathcal{O}(h)) \\ &\quad \text{on } \Gamma_1 \cap \{\operatorname{Re} \xi_1(x) \geq 0\}; \\ u_{1,R}^+ &= (\xi_1'(x))^{-1/2} \operatorname{Bi}(h^{-2/3} \xi_1(x)) + \mathcal{O}(h(1 + h^{-2/3} |\xi_1(x)|)^{-1/4}) \\ &\quad \text{on } \Gamma_1 \cap \{\operatorname{Re} \xi_1(x) \leq 0\}. \end{aligned}$$

(ii) *For sufficiently small $h > 0$, there exist two constants $a_2^\pm = 1 + \mathcal{O}(h)$ and the solutions $u_{2,L}^\pm$ to the equation $(P_2 - E)u = 0$ on $\Gamma_2 = \{x \in \mathbb{C}; |\operatorname{Im}(x - c(E))| \leq \delta_2 |\operatorname{Re}(x - c(E))|\}$ with sufficiently small $\delta_2 > 0$ such that,*

$$\begin{aligned} u_{2,L}^\pm(x) &\sim (1 + \mathcal{O}(h)) \frac{h^{1/6}}{\sqrt{\pi}} (V_2(x) - E)^{-1/4} e^{\mp \int_{c(E)}^x \sqrt{V_2(t) - E} dt/h}, \quad (x \rightarrow -\infty); \\ e^{\mp i \frac{\pi}{4}} &\left(\frac{1}{2} a_2^- u_{2,L}^- \pm i a_2^+ u_{2,L}^+ \right) \\ &\sim (1 + \mathcal{O}(h)) \frac{h^{1/6}}{\sqrt{\pi}} (E - V_2(x))^{-1/4} e^{\mp i \int_{c(E)}^x \sqrt{E - V_2(t)} dt/h}, \quad (x \rightarrow +\infty), \end{aligned}$$

uniformly with respect to $h > 0$ small enough, and as $h \rightarrow 0_+$,

$$\begin{aligned} u_{2,L}^- &= 2(\xi_2'(x))^{-1/2} \check{\operatorname{Ai}}(h^{-2/3} \xi_2(x)) + \mathcal{O}(h(1 + h^{-2/3} |\xi_2(x)|)^{-1/4}) \\ &\quad \text{on } \Gamma_2 \cap \{\operatorname{Re} \xi_2(x) \geq 0\}; \\ u_{2,L}^- &= 2(\xi_2'(x))^{-1/2} \check{\operatorname{Ai}}(h^{-2/3} \xi_2(x)) (1 + \mathcal{O}(h)) \\ &\quad \text{on } \Gamma_2 \cap \{\operatorname{Re} \xi_2(x) \leq 0\}; \\ u_{2,L}^+ &= (\xi_2'(x))^{-1/2} \check{\operatorname{Bi}}(h^{-2/3} \xi_2(x)) + \mathcal{O}(h(1 + h^{-2/3} |\xi_2(x)|)^{-1/4}) \\ &\quad \text{on } \Gamma_2 \cap \{\operatorname{Re} \xi_2(x) \geq 0\}; \\ u_{2,L}^+ &= (\xi_2'(x))^{-1/2} \check{\operatorname{Bi}}(h^{-2/3} \xi_2(x)) (1 + \mathcal{O}(h)) \\ &\quad \text{on } \Gamma_2 \cap \{\operatorname{Re} \xi_2(x) \leq 0\}, \end{aligned}$$

where $\check{\operatorname{Ai}}(x) = \operatorname{Ai}(-x)$ and $\check{\operatorname{Bi}}(x) = \operatorname{Bi}(-x)$.

Remark 3.2. *Similarly, we have two solutions $u_{1,L}^\pm$ on $\tilde{\Gamma}_1 = \{x \in \mathbb{C}; |\operatorname{Im}(x - a(E))| \leq \tilde{\delta}_1 |\operatorname{Re}(x - a(E))|, \operatorname{Re} x \leq x_0\}$ with sufficiently small $\tilde{\delta}_1 > 0$ with the asymptotic behavior,*

$$u_{1,L}^\pm(x) \sim (1 + \mathcal{O}(h)) \frac{h^{1/6}}{\sqrt{\pi}} (V_1(x) - E)^{-1/4} e^{\mp \int_{a(E)}^x \sqrt{V_1(t) - E} dt/h}, \quad (x \rightarrow -\infty).$$

As in [3, Proposition 5.1] we have,

Proposition 3.3. *Let $u_{1,R}^\pm$ and $u_{1,L}^\pm$ as above. Then one has*

$$u_{1,L}^\pm = a_\pm u_{1,R}^\mp + b_\pm u_{1,R}^\pm$$

with,

$$\begin{aligned} a_- &= \sin \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h) \quad ; \quad b_- = 2 \cos \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h) \\ a_+ &= \frac{1}{2} \cos \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h) \quad ; \quad b_+ = -\sin \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h), \end{aligned}$$

as $h \rightarrow 0_+$.

Setting

$$u_{2,R}^\pm := e^{\mp i \frac{\pi}{4}} \left(\frac{1}{2} a_2^- u_{2,L}^\mp \pm i a_2^+ u_{2,L}^\pm \right),$$

by Proposition 3.1 we have,

$$u_{2,R}^\pm \sim (1 + \mathcal{O}(h)) \frac{h^{1/6}}{\sqrt{\pi}} (E - V_2(x))^{-1/4} e^{\mp i \int_c^x \sqrt{E - V_2(t)} dt/h}, \quad (x \rightarrow +\infty).$$

The Wronskians $\mathcal{W}[u_{j,L}^-, u_{j,L}^+]$ and $\mathcal{W}[u_{j,R}^-, u_{j,R}^+]$ are independent of the variable x and satisfies

$$\begin{aligned} \mathcal{W}[u_{j,L}^-, u_{j,L}^+] &= -\frac{2}{\pi h^{2/3}} (1 + \mathcal{O}(h)), \\ \mathcal{W}[u_{1,R}^-, u_{1,R}^+] &= \frac{2}{\pi h^{2/3}} (1 + \mathcal{O}(h)), \\ \mathcal{W}[u_{2,R}^-, u_{2,R}^+] &= -\frac{2i}{\pi h^{2/3}} (1 + \mathcal{O}(h)). \end{aligned} \tag{3.1}$$

To see the growth and decay from the points b and c , we define the solutions $u_{j,b}^\pm$ on I_L , $u_{j,c}^\pm$ on I_R^θ , $v_{j,b}^\pm$ on I_b and $v_{j,c}^\pm$ on I_c as follows:

$$\begin{aligned} u_{1,b}^\pm &:= u_{1,L}^\pm, \quad u_{2,b}^\pm := e^{-S_2/h} u_{2,L}^\pm, \quad u_{2,b}^\mp := e^{S_2/h} u_{2,L}^\mp, \\ u_{2,c}^\pm &:= u_{2,R}^\pm, \quad u_{1,c}^\pm := e^{-S_1/h} u_{1,R}^\pm, \quad u_{1,c}^\mp := e^{S_1/h} u_{1,R}^\mp, \\ v_{1,b}^\pm &:= u_{1,R}^\pm, \quad v_{2,b}^\pm := e^{S_2/h} u_{2,L}^\pm, \quad v_{2,b}^\mp := e^{-S_2/h} u_{2,L}^\mp, \\ v_{2,c}^\pm &:= u_{2,L}^\pm, \quad v_{1,c}^\pm := e^{S_1/h} u_{1,R}^\pm, \quad v_{1,c}^\mp := e^{-S_1/h} u_{1,R}^\mp, \end{aligned}$$

where $S_1 := \int_b^c \sqrt{V_1(t) - E} dt$, $S_2 := \int_b^c \sqrt{V_2(t) - E} dt$.

4. FUNDAMENTAL SOLUTIONS

In this section we introduce fundamental solutions used to construct solutions to the system.

4.1. Fundamental solutions on I_b .

We define fundamental solutions

$$K_{1,b}, \quad K'_{1,b}, \quad K''_{1,b} : C(I_b) \rightarrow C^2(I_b),$$

of $P_1 - E$ and

$$K_{2,b} : C(I_b) \rightarrow C^2(I_b),$$

of $P_2 - E$ by setting for $v \in C(I_b)$,

$$\begin{aligned} K_{1,b}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{1,b}^-, v_{1,b}^+]} \left(v_{1,b}^-(x) \int_b^x v_{1,b}^+(t) v(t) dt + v_{1,b}^+(x) \int_x^0 v_{1,b}^-(t) v(t) dt \right), \\ K'_{1,b}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{1,b}^-, v_{1,b}^+]} \left(v_{1,b}^-(x) \int_b^x v_{1,b}^+(t) v(t) dt - v_{1,b}^+(x) \int_b^x v_{1,b}^-(t) v(t) dt \right), \\ K''_{1,b}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{1,b}^-, v_{1,b}^+]} \left(-v_{1,b}^-(x) \int_x^0 v_{1,b}^+(t) v(t) dt + v_{1,b}^+(x) \int_x^0 v_{1,b}^-(t) v(t) dt \right), \\ K_{2,b}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{2,b}^-, v_{2,b}^+]} \left(v_{2,b}^-(x) \int_b^x v_{2,b}^+(t) v(t) dt + v_{2,b}^+(x) \int_x^0 v_{2,b}^-(t) v(t) dt \right), \end{aligned}$$

where $\mathcal{W}[v_{1,b}^-, v_{1,b}^+]$ is the Wronskian of $v_{1,b}^-$ and $v_{1,b}^+$ and so on. Then $K_{1,b}$, $K'_{1,b}$ and $K''_{1,b}$ satisfy

$$(P_1 - E)K_{1,b} = \mathbf{1}, \quad (P_1 - E)K'_{1,b} = \mathbf{1}, \quad (P_1 - E)K''_{1,b} = \mathbf{1},$$

and $K_{2,b}$ satisfies

$$(P_2 - E)K_{2,b} = \mathbf{1}.$$

For an interval I and sufficiently small $h_0 > 0$, let $v(x, h) \in C(I)$ be a family of functions satisfying $v(x, h) \neq 0$ for $x \in I$, $0 < h < h_0$. For $h \in (0, h_0]$ we define $C(v, h)$ as the set of continuous functions u on I equipped with the norm $\|u\|_{C(v, h)} := \sup_{x \in I} |u(x)| |v(x, h)|^{-1}$. In the following, we consider families of functions $u(x, h) \in C(v, h)$, $h \in (0, h_0]$ and families of operators $A(h) \in \mathcal{L}(C(v, h))$, $h \in (0, h_0]$.

Note that $v_{j,b}^\pm \neq 0$ and $v_{j,c}^\pm \neq 0$ on I_b and I_c respectively for sufficiently small h . In view of the construction of solutions to the system, we prove,

Lemma 4.1. *As h goes to 0_+ one has,*

$$(4.1) \quad \|hK_{1,b}W\|_{\mathcal{L}(C(v_{1,b}^\pm, h))} = \mathcal{O}(1),$$

$$(4.2) \quad \|hK_{2,b}W^*\|_{\mathcal{L}(C(v_{1,b}^\pm, h))} = \mathcal{O}(h^{1/2}),$$

$$(4.3) \quad \|hK'_{1,b}W\|_{\mathcal{L}(C(v_{2,b}^\pm, h))} = \mathcal{O}(h^{1/2}),$$

$$(4.4) \quad \|hK''_{1,b}W\|_{\mathcal{L}(C(v_{2,b}^\pm, h))} = \mathcal{O}(h^{1/2}),$$

$$(4.5) \quad \|hK_{2,b}W^*\|_{\mathcal{L}(C(v_{2,b}^\pm, h))} = \mathcal{O}(1).$$

Moreover, there exist complex numbers $\alpha_{j,b}, \beta_{1,b}^\pm, \beta_{2,b}$ depending on v such that,

$$(4.6) \quad \begin{pmatrix} hK_{1,b}Wv(b) \\ h\partial_x(K_{1,b}Wv)(b) \end{pmatrix} = \alpha_{1,b} \begin{pmatrix} v_{1,b}^+(b) \\ \partial_x v_{1,b}^+(b) \end{pmatrix},$$

$$(4.7) \quad \begin{pmatrix} hK_{2,b}W^*v(b) \\ h\partial_x(K_{2,b}W^*v)(b) \end{pmatrix} = \alpha_{2,b} \begin{pmatrix} v_{2,b}^+(b) \\ \partial_x v_{2,b}^+(b) \end{pmatrix},$$

$$(4.8) \quad \begin{pmatrix} hK'_{1,b}Wv(b) \\ h\partial_x(K'_{1,b}Wv)(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$(4.9) \quad \begin{pmatrix} hK''_{1,b}Wv(b) \\ h\partial_x(K''_{1,b}Wv)(b) \end{pmatrix} = \beta_{1,b}^- \begin{pmatrix} v_{1,b}^-(b) \\ \partial_x v_{1,b}^-(b) \end{pmatrix} + \beta_{1,b}^+ \begin{pmatrix} v_{1,b}^+(b) \\ \partial_x v_{1,b}^+(b) \end{pmatrix},$$

$$(4.10) \quad \begin{pmatrix} hK_{2,b}W^*v(b) \\ h\partial_x(K_{2,b}W^*v)(b) \end{pmatrix} = \beta_{2,b} \begin{pmatrix} v_{2,b}^+(b) \\ \partial_x v_{2,b}^+(b) \end{pmatrix},$$

and

$$|\alpha_{1,b}| \leq C\|v\|_{C(v_{1,b}^\pm, h)}, \quad |\alpha_{2,b}| \leq Ch^{5/6}\|v\|_{C(v_{1,b}^\pm, h)}, \quad |\beta_{1,b}^\pm| \leq Ch^{5/6}\|v\|_{C(v_{2,b}^\pm, h)}, \quad |\beta_{2,b}| \leq C\|v\|_{C(v_{2,b}^\pm, h)}.$$

Proof. (i) First, we shall prove (4.1) and (4.6). We set,

$$\begin{aligned} \tilde{v}_{1,b}^\pm &:= \max\{|v_{1,b}^\pm|, |h\partial_x v_{1,b}^\pm|\}, \\ \mathcal{V}_{1,b}(x, t) &:= \tilde{v}_{1,b}^-(x)\tilde{v}_{1,b}^+(t)\mathbf{1}_{\{t < x\}} + \tilde{v}_{1,b}^+(x)\tilde{v}_{1,b}^-(t)\mathbf{1}_{\{t > x\}}. \end{aligned}$$

By an integration by parts we have,

$$|hK_{1,b}Wv(x)| = \mathcal{O}(h^{-1/3}) \left(\int_b^0 \mathcal{V}_{1,b}(x, t)|v(t)|dt + h\mathcal{V}_{1,b}(x, 0)|v(0)| + h\mathcal{V}_{1,b}(x, b)|v(b)| \right).$$

Using the asymptotics of $v_{1,b}^\pm$ and $h\partial_x v_{1,b}^\pm$ on I_b and fixing some constant $C_1 > 0$ sufficiently large we obtain,

- If $b \leq x, t \leq b + C_1h^{2/3}$, then,

$$\mathcal{V}_{1,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(1).$$

- If $b \leq x \leq b + C_1h^{2/3} \leq t \leq 0$, then,

$$\mathcal{V}_{1,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/3}) \frac{e^{-\operatorname{Re} \int_b^t (V_1 - E)^{1/2}/h \pm \operatorname{Re} \int_b^t (V_1 - E)^{1/2}/h}}{|V_1(t) - E|^{1/2}}.$$

- If $b \leq t \leq b + C_1h^{2/3} \leq x \leq 0$, then,

$$\mathcal{V}_{1,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(1)e^{-\operatorname{Re} \int_b^x (V_1 - E)^{1/2}/h \mp \operatorname{Re} \int_b^x (V_1 - E)^{1/2}/h}.$$

- If $b + C_1h^{2/3} \leq x, t \leq 0$, then,

$$\mathcal{V}_{1,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/3}) \frac{e^{-|\operatorname{Re} \int_t^x (V_1 - E)^{1/2}/h \pm \operatorname{Re} \int_x^t (V_1 - E)^{1/2}/h}}{|V_1(t) - E|^{1/2}}.$$

Hence, observing $|V_1(t) - E|^{1/4} \geq C|t - b| \geq Ch^{1/6}$ for $b + C_1h^{2/3} \leq t \leq 0$, we have

$$\mathcal{V}_{1,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(1) \text{ as } h \rightarrow 0_+,$$

uniformly with respect to $b \leq x, t \leq 0$. Thus we obtain

$$\|\mathcal{V}_{1,b}(x, 0)|v(0)|\|_{C(v_{1,b}^\pm, h)}, \quad \|\mathcal{V}_{1,b}(x, b)|v(b)|\|_{C(v_{1,b}^\pm, h)} \leq C\|v\|_{C(v_{1,b}^\pm, h)}.$$

Moreover, regardless of $b \leq x \leq b + C_1h^{2/3}$ or $b + C_1h^{2/3} < x \leq 0$, we have

$$\begin{aligned} \int_b^0 \mathcal{V}_{1,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1}dt &= \mathcal{O}(1) \int_b^{b+C_1h^{2/3}} dt + \mathcal{O}(h^{1/3}) \int_{b+C_1h^{2/3}}^0 (t-b)^{-1/2}dt \\ &= \mathcal{O}(h^{1/3}). \end{aligned}$$

Hence we obtain

$$\left\| \int_b^0 \mathcal{V}_{1,b}(x, t)|v(t)|dt \right\|_{C(v_{1,b}^\pm, h)} \leq Ch^{1/3}\|v\|_{C(v_{1,b}^\pm, h)},$$

which completes the proof of (4.1). Observing

$$\begin{aligned} hK_{1,b}Wv(b) &= \frac{1}{h^2\mathcal{W}[v_{1,b}^-, v_{1,b}^+]} v_{1,b}^+(b) \int_b^0 v_{1,b}^-(t)v(t)dt, \\ h\partial_x K_{1,b}Wv(b) &= \frac{1}{h^2\mathcal{W}[v_{1,b}^-, v_{1,b}^+]} \partial_x v_{1,b}^+(b) \int_b^0 v_{1,b}^-(t)v(t)dt, \end{aligned}$$

we estimate $|(hK_{1,b}Wv(b))(v_{1,b}^+(b))^{-1}|$ and $|(h\partial_x K_{1,b}Wv(b))(\partial_x v_{1,b}^+(b))^{-1}|$ by the similar calculation as above and obtain (4.6).

(ii) We shall prove (4.2) and (4.7). We set,

$$\begin{aligned} \tilde{v}_{2,b}^\pm &:= \max\{|v_{2,b}^\pm|, |h\partial_x v_{2,b}^\pm|\}, \\ \mathcal{V}_{2,b}(x, t) &:= \tilde{v}_{2,b}^-(x)\tilde{v}_{2,b}^+(t)\mathbf{1}_{\{t < x\}} + \tilde{v}_{2,b}^+(x)\tilde{v}_{2,b}^-(t)\mathbf{1}_{\{t > x\}}. \end{aligned}$$

By an integration by parts we have,

$$(4.11) \quad |hK_{2,b}W^*v(x)| = \mathcal{O}(h^{-1/3}) \left(\int_b^0 \mathcal{V}_{2,b}(x, t)|v(t)|dt + h\mathcal{V}_{2,b}(x, 0)|v(0)| + h\mathcal{V}_{2,b}(x, b)|v(b)| \right).$$

Using the asymptotics of $v_{j,b}^\pm$ and $h\partial_x v_{2,b}^\pm$ on I_b and fixing some constant $C_1 > 0$ sufficiently large we obtain,

- If $b \leq x, t \leq b + C_1 h^{2/3}$, then,

$$\mathcal{V}_{2,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/3})e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}/h|}.$$

- If $b \leq x \leq b + C_1 h^{2/3} \leq t \leq 0$, then,

$$\mathcal{V}_{2,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/2}) \frac{e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}/h \pm \operatorname{Re} \int_b^t (V_1 - E)^{1/2}/h|}}{|V_1(t) - E|^{1/4}}.$$

- If $b \leq t \leq b + C_1 h^{2/3} \leq x \leq 0$, then,

$$\mathcal{V}_{2,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/6})e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}/h \mp \operatorname{Re} \int_b^x (V_1 - E)^{1/2}/h|}.$$

- If $b + C_1 h^{2/3} \leq x, t \leq 0$, then,

$$\mathcal{V}_{2,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/3}) \frac{e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}/h \pm \operatorname{Re} \int_x^t (V_1 - E)^{1/2}/h|}}{|V_1(t) - E|^{1/4}}.$$

Hence, observing that $\int_b^{b+C_1 h^{2/3}} (V_1(t) - E)^{1/2} dt = \mathcal{O}(h)$, we have

$$\mathcal{V}_{2,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/6}),$$

uniformly with respect to $b \leq x, t \leq 0$. Thus we have

$$\|\mathcal{V}_{2,b}(x, 0)|v(0)|\|_{C(v_{1,b}^\pm, h)}, \|\mathcal{V}_{2,b}(x, b)|v(b)|\|_{C(v_{1,b}^\pm, h)} \leq Ch^{1/6}\|v\|_{C(v_{1,b}^\pm, h)}.$$

When $b \leq x \leq b + C_1 h^{2/3}$, there exist constants $\delta > 0$ small enough and $\alpha > 0$ such that

$$\int_b^0 \mathcal{V}_{2,b}(x, t)|v_{1,b}^\pm(t)||v_{1,b}^\pm(x)|^{-1} dt = \mathcal{O}(h^{1/3}) \int_b^{b+\delta} e^{-\alpha|x-t|/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h^{4/3}).$$

On the other hand, if $b + C_1 h^{2/3} < x \leq -\delta'$ with $\delta' > 0$ small enough there exist a constant $\alpha > 0$ such that

$$\int_b^0 \mathcal{V}_{2,b}(x, t) |v_{1,b}^\pm(t)| |v_{1,b}^\pm(x)|^{-1} dt = \mathcal{O}(h^{1/6}) \int_b^{-\delta'/2} e^{-\alpha|x-t|/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h^{7/6}).$$

Finally, when $-\delta' < x \leq 0$ there exist constants $\alpha, \beta > 0$ such that

$$\int_b^0 \mathcal{V}_{2,b}(x, t) |v_{1,b}^\pm(t)| |v_{1,b}^\pm(x)|^{-1} dt = \mathcal{O}(h^{1/3}) \int_{-2\delta'}^0 e^{-\beta|x^2-t^2|/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h^{5/6}).$$

Here we used $\int_{-2\delta'}^0 e^{-\beta|x^2-t^2|/h} dt = \mathcal{O}(h^{1/2})$. To prove this estimate we write

$$\int_{-2\delta'}^0 e^{-\beta|x^2-t^2|/h} dt = \int_{-2\delta'}^x e^{-\beta|x^2-t^2|/h} dt + \int_x^0 e^{-\beta|x^2-t^2|/h} dt.$$

The first term is estimated as follows;

$$\int_{-2\delta'}^x e^{-\beta|x^2-t^2|/h} dt = \int_{-2\delta'-x}^0 e^{-\beta|x^2-(t+x)^2|/h} dt \leq \int_{-2\delta'-x}^0 e^{-\beta t^2/h} dt = \mathcal{O}(h^{1/2}),$$

and the second term is estimated as follows;

$$\begin{aligned} \int_x^0 e^{-\beta|x^2-t^2|/h} dt &= \int_0^{-x} e^{-\beta|x^2-(t+x)^2|/h} dt = \int_0^{-x} e^{-\beta|2(-x)t-t^2|/h} dt \leq \int_0^{-x} e^{-\beta|(-x)t|/h} dt \\ &\leq \int_0^{-x} e^{-\beta t^2/h} dt = \mathcal{O}(h^{1/2}), \end{aligned}$$

where we used that $(-x)t \geq t^2$ for $0 \leq t \leq -x$ in the inequalities. Hence, we obtain

$$\left\| \int_b^0 \mathcal{V}_{2,b}(x, t) |v(t)| dt \right\|_{C(v_{1,b}^\pm, h)} \leq C h^{5/6} \|v\|_{C(v_{1,b}^\pm, h)},$$

which completes the proof of (4.2). Noting that there exists a constant $C > 0$ such that $|v_{1,b}^\pm(b)| \leq C$ and $C^{-1} h^{1/6} \leq |v_{2,b}^\pm(b)|$, we estimate $|(hK_{2,b}W^*v(b))(v_{2,b}^\pm(b))^{-1}|$ and $|(h\partial_x K_{2,b}W * v(b))(\partial_x v_{2,b}^\pm(b))^{-1}|$ by the similar calculation as above and obtain (4.7).

(iii) We shall prove (4.3) and (4.8). We set,

$$\mathcal{V}'_{1,b}(x, t) := \tilde{v}_{1,b}^-(x) \tilde{v}_{1,b}^+(t) + \tilde{v}_{1,b}^+(x) \tilde{v}_{1,b}^-(t).$$

By an integration by parts we have,

$$|hK'_{1,b}Wv(x)| = \mathcal{O}(h^{-1/3}) \left(\int_b^x \mathcal{V}'_{1,b}(x, t) |v(t)| dt + h\mathcal{V}''_{1,b}(x, x) |v(x)| + h\mathcal{V}'_{1,b}(x, b) |v(b)| \right).$$

We obtain,

- If $b \leq t \leq x \leq b + C_1 h^{2/3}$, then,

$$\mathcal{V}'_{1,b}(x, t) |v_{2,b}^+(t)| |v_{2,b}^+(x)|^{-1} = \mathcal{O}(1) e^{-\operatorname{Re} \int_t^x (V_2 - E)^{1/2}/h}.$$

- If $b \leq t \leq b + C_1 h^{2/3} \leq x \leq 0$, then,

$$\mathcal{V}'_{1,b}(x, t) |v_{2,b}^+(t)| |v_{2,b}^+(x)|^{-1} = \mathcal{O}(h^{1/6}) \frac{e^{-\operatorname{Re} \int_t^x (V_2 - E)^{1/2}/h + \operatorname{Re} \int_b^x (V_1 - E)^{1/2}/h}}{|V_1(x) - E|^{1/4}}.$$

- If $b + C_1 h^{2/3} \leq t \leq x \leq 0$, then,

$$\mathcal{V}'_{1,b}(x, t) |v_{2,b}^+(t)| |v_{2,b}^+(x)|^{-1} = \mathcal{O}(h^{1/3}) \frac{e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}|/h + \operatorname{Re} \int_t^x (V_1 - E)^{1/2}/h}}{|V_1(t) - E|^{1/4} |V_1(x) - E|^{1/4}}.$$

Hence, we have $\mathcal{V}'_{1,b}(x, t) |v_{2,b}^+(t)| |v_{2,b}^+(x)|^{-1} = \mathcal{O}(1)$ as $h \rightarrow 0_+$. Thus we have

$$\|\mathcal{V}'_{1,b}(x, b) |v(b)|\|_{C(v_{2,b}^+, h)}, \|\mathcal{V}'_{1,b}(x, x) |v(x)|\|_{C(v_{2,b}^-, h)} \leq C \|v\|_{C(v_{2,b}^+, h)}.$$

When $b \leq x \leq b + C_1 h^{2/3}$ or $b + C_1 h^{2/3} < x \leq -\delta'$ with $\delta' > 0$ small enough, there exist a constant $\alpha > 0$ such that

$$\int_b^x \mathcal{V}'_{1,b}(x, t) |v_{2,b}^+(t)| |v_{2,b}^+(x)|^{-1} dt \leq \mathcal{O}(1) \int_b^x e^{-\alpha(t-x)/h} dt = \mathcal{O}(h).$$

When $-\delta' < x \leq 0$ there exist constants $\alpha, \beta > 0$ such that

$$\int_b^x \mathcal{V}'_{1,b}(x, t) |v_{2,b}^+(t)| |v_{2,b}^+(x)|^{-1} dt = \mathcal{O}(h^{1/3}) \int_{-2\delta'}^x e^{-\beta|x^2-t^2|/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h^{5/6}).$$

Hence, we obtain

$$\left\| \int_b^x \mathcal{V}'_{1,b}(x, t) |v(t)| dt \right\|_{C(v_{2,b}^+, h)} \leq C h^{5/6} \|v\|_{C(v_{2,b}^+, h)},$$

which completes the proof of (4.3). The equation (4.8) is obvious from the definition of $K'_{1,b}$.

(iv) We shall prove (4.4) and (4.9). We set,

$$\mathcal{V}''_{1,b}(x, t) = \mathcal{V}'_{1,b}(x, t).$$

By an integration by parts we have,

$$|h K'_{1,b} W v(x)| = \mathcal{O}(h^{-1/3}) \left(\int_x^0 \mathcal{V}''_{1,b}(x, t) |v(t)| dt + h \mathcal{V}''_{1,b}(x, 0) |v(0)| + h \mathcal{V}''_{1,b}(x, x) |v(x)| \right).$$

We obtain,

- If $b \leq x \leq t \leq b + C_1 h^{2/3}$, then,

$$\mathcal{V}''_{1,b}(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} = \mathcal{O}(1) e^{-\operatorname{Re} \int_x^t (V_2 - E)^{1/2}/h}.$$

- If $b \leq x \leq b + C_1 h^{2/3} \leq t \leq 0$, then,

$$\mathcal{V}''_{1,b}(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} = \mathcal{O}(h^{1/6}) \frac{e^{-\operatorname{Re} \int_x^t (V_2 - E)^{1/2}/h + \operatorname{Re} \int_b^t (V_1 - E)^{1/2}/h}}{|V_1(t) - E|^{1/4}}.$$

- If $b + C_1 h^{2/3} \leq x \leq t \leq 0$, then,

$$\mathcal{V}''_{1,b}(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} = \mathcal{O}(h^{1/3}) \frac{e^{-|\operatorname{Re} \int_x^t (V_2 - E)^{1/2}|/h + \operatorname{Re} \int_x^t (V_1 - E)^{1/2}/h}}{|V_1(t) - E|^{1/4} |V_1(x) - E|^{1/4}}.$$

Hence, we have $\mathcal{V}''_{1,b}(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} = \mathcal{O}(1)$ as $h \rightarrow 0_+$. Thus we have

$$\|\mathcal{V}''_{1,b}(x, 0) |v(0)|\|_{C(v_{2,b}^-, h)}, \|\mathcal{V}''_{1,b}(x, x) |v(x)|\|_{C(v_{2,b}^-, h)} \leq C \|v\|_{C(v_{2,b}^-, h)}.$$

When $b \leq x \leq b + C_1 h^{2/3}$, there exist constants $\delta > 0$ small enough and $\alpha > 0$ such that

$$\int_x^0 \mathcal{V}''_{1,b}(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} dt = \mathcal{O}(1) \int_x^{b+\delta} e^{-\alpha(t-x)/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h).$$

On the other hand, if $b + C_1 h^{2/3} < x \leq -\delta'$ with $\delta' > 0$ small enough, there exist a constant α such that

$$\int_x^0 \mathcal{V}_{1,b}''(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} dt = \mathcal{O}(1) \int_x^{-\delta'/2} e^{-\alpha(t-x)/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h).$$

Finally, when $-\delta' < x \leq 0$ there exist constants $\alpha, \beta > 0$ such that

$$\int_x^0 \mathcal{V}_{1,b}''(x, t) |v_{2,b}^-(t)| |v_{2,b}^-(x)|^{-1} dt = \mathcal{O}(h^{1/3}) \int_x^0 e^{-\beta|x^2-t^2|/h} dt + \mathcal{O}(e^{-\alpha/h}) = \mathcal{O}(h^{5/6}).$$

Hence, we obtain

$$\left\| \int_b^0 \mathcal{V}_{1,b}''(x, t) |v(t)| dt \right\|_{C(v_{2,b}^-, h)} \leq Ch^{5/6} \|v\|_{C(v_{2,b}^-, h)},$$

which completes the proof of (4.4). Noting that there exists a constant $C > 0$ such that $C^{-1} \leq |v_{1,b}^\pm(b)|$ and $|v_{2,b}^-(b)| \leq Ch^{1/6}$, we obtain (4.9) by the similar calculation as above.

(v) Finally, we shall prove (4.5) and (4.10). We shall estimate the terms in (4.11). We obtain for any $b \leq x, t \leq 0$,

$$\mathcal{V}_{2,b}(x, t) |v_{2,b}^\pm(t)| |v_{2,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/3}) e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}|/h \pm \operatorname{Re} \int_x^t (V_2 - E)^{1/2}/h}.$$

Hence, we have $\mathcal{V}_{2,b}(x, t) |v_{2,b}^\pm(t)| |v_{2,b}^\pm(x)|^{-1} = \mathcal{O}(h^{1/3})$ as $h \rightarrow 0_+$. Thus we have

$$\|\mathcal{V}_{2,b}(x, b) |v(b)|\|_{C(v_{2,b}^\pm, h)}, \|\mathcal{V}_{2,b}(x, x) |v(x)|\|_{C(v_{2,b}^\pm, h)} \leq C \|v\|_{C(v_{2,b}^\pm, h)}.$$

We have for any $b \leq x \leq 0$,

$$\int_b^0 \mathcal{V}_{2,b}(x, t) |v_{2,b}^\pm(t)| |v_{2,b}^\pm(x)|^{-1} dt = \mathcal{O}(h^{1/3}) \int_b^0 dt = \mathcal{O}(h^{1/3}).$$

Noting that there exists a constant $C > 0$ such that $C^{-1} h^{1/6} \leq v_{2,b}^\pm(b) \leq Ch^{1/6}$, we obtain (4.10) by the similar calculation as above. $\square$

4.2. Fundamental solutions on I_c .

We define the fundamental solutions of $P_j - E$ on I_c as,

$$\begin{aligned} K_{1,c}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{1,c}^+, v_{1,c}^-]} \left(v_{1,c}^+(x) \int_0^x v_{1,c}^-(t) v(t) dt + v_{1,c}^-(x) \int_x^c v_{1,c}^+(t) v(t) dt \right), \\ K_{2,c}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{2,c}^+, v_{2,c}^-]} \left(v_{2,c}^+(x) \int_0^x v_{2,c}^-(t) v(t) dt + v_{2,c}^-(x) \int_x^c v_{2,c}^+(t) v(t) dt \right), \\ K'_{2,c}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{2,c}^+, v_{2,c}^-]} \left(-v_{2,c}^+(x) \int_x^c v_{2,c}^-(t) v(t) dt + v_{2,c}^-(x) \int_x^c v_{2,c}^+(t) v(t) dt \right), \\ K''_{2,c}[v](x) &:= \frac{1}{h^2 \mathcal{W}[v_{2,c}^+, v_{2,c}^-]} \left(v_{2,c}^+(x) \int_0^x v_{2,c}^-(t) v(t) dt - v_{2,c}^-(x) \int_0^x v_{2,c}^+(t) v(t) dt \right), \end{aligned}$$

for $v \in C(I_c)$.

Then, one can prove exactly as for Lemma 4.1 that we have,

Lemma 4.2. *As h goes to 0_+ one has,*

$$\begin{aligned} \|hK_{2,c}W^*\|_{\mathcal{L}(C(v_{2,c}^\pm, h))} &= \mathcal{O}(1), \\ \|hK_{1,c}W\|_{\mathcal{L}(C(v_{2,c}^\pm, h))} &= \mathcal{O}(h^{1/2}), \\ \|hK'_{2,c}W^*\|_{\mathcal{L}(C(v_{1,c}^+, h))} &= \mathcal{O}(h^{1/2}), \\ \|hK''_{2,c}W^*\|_{\mathcal{L}(C(v_{1,c}^-, h))} &= \mathcal{O}(h^{1/2}), \\ \|hK_{1,c}W\|_{\mathcal{L}(C(v_{1,c}^\pm, h))} &= \mathcal{O}(1). \end{aligned}$$

Moreover, there exist complex numbers $\alpha_{j,c}, \beta_{1,c}, \beta_{2,c}^\pm$ depending on v such that,

$$\begin{aligned} \begin{pmatrix} hK_{2,c}W^*v(c) \\ h\partial_x(K_{2,c}W^*v)(c) \end{pmatrix} &= \alpha_{2,c} \begin{pmatrix} v_{2,c}^+(c) \\ \partial_x v_{2,c}^+(c) \end{pmatrix}, \\ \begin{pmatrix} hK_{1,c}Wv(c) \\ h\partial_x(K_{1,c}Wv)(c) \end{pmatrix} &= \alpha_{1,c} \begin{pmatrix} v_{1,c}^+(c) \\ \partial_x v_{1,c}^+(c) \end{pmatrix}, \\ \begin{pmatrix} hK'_{2,c}W^*v(c) \\ h\partial_x(K'_{2,c}W^*v)(c) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} hK''_{2,c}W^*v(c) \\ h\partial_x(K''_{2,c}W^*v)(c) \end{pmatrix} &= \beta_{2,c}^- \begin{pmatrix} v_{2,c}^-(c) \\ \partial_x v_{2,c}^-(c) \end{pmatrix} + \beta_{2,c}^+ \begin{pmatrix} v_{2,c}^+(c) \\ \partial_x v_{2,c}^+(c) \end{pmatrix}, \\ \begin{pmatrix} hK_{1,c}Wv(c) \\ h\partial_x(K_{1,c}Wv)(c) \end{pmatrix} &= \beta_{1,c} \begin{pmatrix} v_{1,c}^+(c) \\ \partial_x v_{1,c}^+(c) \end{pmatrix}, \end{aligned}$$

and

$$|\alpha_{2,c}| \leq C\|v\|_{C(v_{2,c}^\pm, h)}, \quad |\alpha_{1,c}| \leq Ch^{5/6}\|v\|_{C(v_{2,c}^\pm, h)}, \quad |\beta_{2,c}^\pm| \leq Ch^{5/6}\|v\|_{C(v_{1,c}^\pm, h)}, \quad |\beta_{1,c}| \leq C\|v\|_{C(v_{1,c}^\pm, h)}.$$

4.3. Fundamental solutions on I_L and I_R^θ .

For any $k \in \mathbb{N}$ we set

$$\begin{aligned} C_b^0(I_L) &:= \{u : I_L \rightarrow \mathbb{C} \text{ of class } C^k; \sum_{0 \leq j \leq k} \sup_{x \in I_L} |u^{(j)}(x)| \leq +\infty\}, \\ C_b^0(I_R^\theta) &:= \{u : I_R^\theta \rightarrow \mathbb{C} \text{ of class } C^k; \sum_{0 \leq j \leq k} \sup_{x \in I_R^\theta} |u^{(j)}(x)| \leq +\infty\}, \end{aligned}$$

and $\|u\|_{C_b^0(I_L)} := \sum_{0 \leq j \leq k} \sup_{x \in I_L} |u^{(j)}(x)|$, $\|u\|_{C_b^0(I_L)} := \sum_{0 \leq j \leq k} \sup_{x \in I_R^\theta} |u^{(j)}(x)|$. We also define the fundamental solutions $K_{j,L} : C_b^0(I_L) \rightarrow C_b^2(I_L)$ and $K_{j,R} : C_b^0(I_R^\theta) \rightarrow C_b^2(I_R^\theta)$ of $P_j - E$ on I_L and I_R^θ as,

$$K_{j,L}[v](x) := \frac{1}{h^2 \mathcal{W}[u_{j,b}^+, u_{j,b}^-]} \left(u_{j,b}^+(x) \int_{-\infty}^x u_{j,b}^-(t) v(t) dt + u_{j,b}^-(x) \int_x^b u_{j,b}^+(t) v(t) dt \right),$$

for $v \in C_b^0(I_L)$ and

$$K_{j,R}[v](x) := \frac{1}{h^2 \mathcal{W}[u_{j,c}^-, u_{j,c}^+]} \left(u_{j,c}^-(x) \int_c^x u_{j,c}^+(t) v(t) dt + u_{j,c}^+(x) \int_x^{+\infty} u_{j,c}^-(t) v(t) dt \right),$$

for $v \in C_b^0(I_R^\theta)$ respectively. Then one has the following lemma.

Lemma 4.3. *As h goes to 0_+ one has,*

$$(4.12) \quad \|hK_{2,L}W^*\|_{\mathcal{L}(C_b^0(I_L))} = \mathcal{O}(h),$$

$$(4.13) \quad \|hK_{1,L}W\|_{\mathcal{L}(C_b^0(I_L))} = \mathcal{O}(h^{-1/6}),$$

Moreover, there exist complex numbers $\eta_b, \theta_b, \theta'_b$ such that,

$$(4.14) \quad \begin{pmatrix} hK_{2,L}W^*u(b) \\ h\partial_x(K_{2,L}W^*u)(b) \end{pmatrix} = \eta_b \begin{pmatrix} u_{2,b}^+(b) \\ \partial_x u_{2,b}^+(b) \end{pmatrix},$$

$$(4.15) \quad \begin{pmatrix} hK_{1,L}Wu(b) \\ h\partial_x(K_{1,L}Wu)(b) \end{pmatrix} = \theta_b \begin{pmatrix} u_{1,b}^+(b) \\ \partial_x u_{1,b}^+(b) \end{pmatrix},$$

$$(4.16) \quad \begin{pmatrix} hK_{1,L}Wu_{2,b}^-(b) \\ h\partial_x(K_{1,L}Wu_{2,b}^-)(b) \end{pmatrix} = \theta'_b \begin{pmatrix} u_{1,b}^+(b) \\ \partial_x u_{1,b}^+(b) \end{pmatrix},$$

and

$$|\eta_b| \leq Ch^{5/6}\|u\|_{C_b^0(I_L)}, \quad |\theta_b| \leq Ch^{-1/6}\|u\|_{C_b^0(I_L)}, \quad |\theta'_b| \leq Ch^{5/6}.$$

Proof. We set,

$$\begin{aligned} \tilde{u}_{1,b}^\pm &:= \max\{|u_{1,b}^\pm|, |h\partial_x u_{1,b}^\pm|\}, \\ \tilde{u}_{2,b}^\pm &:= \max\{|u_{2,b}^\pm|, |h\partial_x u_{2,b}^\pm|\}, \\ \mathcal{U}_{1,L}(x, t) &:= \tilde{u}_{1,b}^+(x)\tilde{u}_{1,b}^-(t)\mathbf{1}_{\{t < x\}} + \tilde{u}_{1,b}^-(x)\tilde{u}_{1,b}^+(t)\mathbf{1}_{\{t > x\}}, \\ \mathcal{U}_{2,L}(x, t) &:= \tilde{u}_{2,b}^+(x)\tilde{u}_{2,b}^-(t)\mathbf{1}_{\{t < x\}} + \tilde{u}_{2,b}^-(x)\tilde{u}_{2,b}^+(t)\mathbf{1}_{\{t > x\}}. \end{aligned}$$

By an integration by parts we have,

$$|hK_{1,L}Wv(x)| = \mathcal{O}(h^{-1/3}) \left(\int_{-\infty}^b \mathcal{U}_{1,L}(x, t)|v(t)|dt + h\mathcal{U}_{1,L}(x, b)|v(b)| \right),$$

and

$$|hK_{2,L}W^*v(x)| = \mathcal{O}(h^{-1/3}) \left(\int_{-\infty}^b \mathcal{U}_{2,L}(x, t)|v(t)|dt + h\mathcal{U}_{2,L}(x, b)|v(b)| \right).$$

For any $x, t \leq b$ we have

$$\mathcal{U}_{2,L}(x, t) = \mathcal{O}(h^{1/3})e^{-|\operatorname{Re} \int_t^x (V_2 - E)^{1/2}|/h}.$$

Hence we have $\mathcal{U}_{2,L}(x, b) = \mathcal{O}(h^{1/3})$ and there exists $\alpha > 0$ such that,

$$\int_{-\infty}^b \mathcal{U}_{2,L}(x, t)dt = \mathcal{O}(h^{1/3}) \int_{-\infty}^b e^{-\alpha|x-t|/h}dt = \mathcal{O}(h^{4/3}),$$

which proves (4.12). By the similar calculation we obtain (4.14).

Next we estimate $\mathcal{U}_{1,L}$. For any $\delta > 0$ small enough, there exists $\alpha > 0$ such that,

- If $|t - a| \leq C_1 h^{2/3}$ or $b - C_1 h^{2/3} \leq t \leq b$, then for any $-\infty \leq x \leq b$,

$$\mathcal{U}_{1,L}(x, t) = \mathcal{O}(1).$$

- If $a + C_1 h^{2/3} \leq t \leq b - C_1 h^{2/3}$, then for any $-\infty \leq x \leq b$

$$\mathcal{U}_{1,L}(x, t) = \mathcal{O}(h^{1/6})|t - b|^{-1/4}.$$

- If $a - 2\delta \leq t \leq a - C_1 h^{2/3}$, then for any $-\infty \leq x \leq b$,

$$\mathcal{U}_{1,L}(x, t) = \mathcal{O}(h^{1/6})|t - a|^{-1/4}.$$

- If $t \leq a - 2\delta$ and $x \leq a - \delta$ then,

$$\mathcal{U}_{1,L}(x, t) = \mathcal{O}(h^{1/3})e^{-\alpha|t-x|/h}.$$

- If $t \leq a - 2\delta$ and $a - \delta \leq x \leq b$ then,

$$\mathcal{U}_{1,L}(x, t) = \mathcal{O}(h^{1/6})e^{-\alpha|t-a+\delta|/h}.$$

Hence we have $\mathcal{U}_{1,L}(x, b) = \mathcal{O}(1)$. Moreover, when $x \leq a - \delta$ we have

$$\begin{aligned} \int_{-\infty}^b \mathcal{U}_{1,L}(x, t)dt &= \mathcal{O}(h^{1/3}) \int_{-\infty}^{a-2\delta} e^{-\alpha|t-x|/h} dt + \mathcal{O}(1) \int_{a-C_1h^{2/3}}^{a+C_1h^{2/3}} dt \\ &\quad + \mathcal{O}(1) \int_{b-C_1h^{2/3}}^b dt + \mathcal{O}(h^{1/6}) \int_{a+C_1h^{2/3}}^{b-C_1h^{2/3}} |t-b|^{-1/4} dt \\ &\quad + \mathcal{O}(h^{1/6}) \int_{a-2\delta}^{a-C_1h^{2/3}} |t-a|^{-1/4} dt = \mathcal{O}(h^{1/6}). \end{aligned}$$

When $a - \delta \leq x \leq b$ we have

$$\begin{aligned} \int_{-\infty}^b \mathcal{U}_{1,L}(x, t)dt &= \mathcal{O}(h^{1/6}) \int_{-\infty}^{a-2\delta} e^{-\alpha|t-a+\delta|/h} dt + \mathcal{O}(1) \int_{a-C_1h^{2/3}}^{a+C_1h^{2/3}} dt \\ &\quad + \mathcal{O}(1) \int_{b-C_1h^{2/3}}^b dt + \mathcal{O}(h^{1/6}) \int_{a+C_1h^{2/3}}^{b-C_1h^{2/3}} |t-b|^{-1/4} dt \\ &\quad + \mathcal{O}(h^{1/6}) \int_{a-2\delta}^{a-C_1h^{2/3}} |t-a|^{-1/4} dt = \mathcal{O}(h^{1/6}), \end{aligned}$$

which completes the proof of (4.13). By the similar calculation we obtain (4.15).

Finally, we shall prove (4.16). By an integration by parts we have

$$|hK_{1,L}Wu_{2,b}^-(b)| = \mathcal{O}(h^{-1/3}) \left(\int_{-\infty}^b \tilde{u}_{1,b}^-(t) \tilde{u}_{2,b}^-(t) dt + h \tilde{u}_{1,b}^-(b) \tilde{u}_{2,b}^-(b) \right) \tilde{u}_{1,b}^+(b),$$

Since $\tilde{u}_{2,b}^-(b) \leq Ch^{1/6}$ we have $\tilde{u}_{1,b}^-(b) \tilde{u}_{2,b}^-(b) \leq Ch^{1/6}$. As for the integral we have

$$\int_{-\infty}^b \tilde{u}_{1,b}^-(t) \tilde{u}_{2,b}^-(t) dt = \mathcal{O}(h^{1/6}) \int_{-\infty}^b e^{-\alpha|t-b|/h} dt = \mathcal{O}(h^{7/6}),$$

which proves the estimate for the first element of (4.16). The estimate for the derivative follows from the similar calculation. $\square$

In the same way as for Lemma 4.3 we have,

Lemma 4.4. *As h goes to 0_+ one has,*

$$\|hK_{1,R}W\|_{\mathcal{L}(C_b^0(I_R^\theta))} = \mathcal{O}(h),$$

$$\|hK_{2,R}W^*\|_{\mathcal{L}(C_b^0(I_R^\theta))} = \mathcal{O}(h^{-1/6}),$$

Moreover, there exist complex numbers $\eta_c, \theta_c, \theta'_c$ such that,

$$\begin{aligned} \begin{pmatrix} hK_{1,R}Wu(c) \\ h\partial_x(K_{1,R}Wu)(c) \end{pmatrix} &= \eta_c \begin{pmatrix} u_{1,c}^+(c) \\ \partial_x u_{1,c}^+(c) \end{pmatrix}, \\ \begin{pmatrix} hK_{2,R}W^*u(c) \\ h\partial_x(K_{2,R}W^*u)(c) \end{pmatrix} &= \theta_c \begin{pmatrix} u_{2,c}^+(c) \\ \partial_x u_{2,c}^+(c) \end{pmatrix}, \\ \begin{pmatrix} hK_{2,R}W^*u_{1,c}^-(c) \\ h\partial_x(K_{2,R}W^*u_{1,c}^-)(c) \end{pmatrix} &= \theta'_c \begin{pmatrix} u_{2,c}^+(c) \\ \partial_x u_{2,c}^+(c) \end{pmatrix}, \end{aligned}$$

and

$$|\eta_c| \leq Ch^{5/6} \|u\|_{C_b^0(I_R^\theta)}, \quad |\theta_c| \leq Ch^{-1/6} \|u\|_{C_b^0(I_R^\theta)}, \quad |\theta'_c| \leq Ch^{5/6}.$$

5. SOLUTIONS TO THE SYSTEM

In this section, we construct solutions to (2.1) on each interval. By Lemma 4.1, the operators $M_b := h^2 K_{2,b} W^* K_{1,b} W$, $M'_b := h^2 K_{2,b} W^* K'_{1,b} W$ and $M''_b := h^2 K_{2,b} W^* K''_{1,b} W$ are $\mathcal{O}(h^{1/2})$ when acting on $C(v_{1,b}^\pm, h)$, $C(v_{2,b}^+, h)$ and $C(v_{2,b}^-, h)$ respectively. Therefore, we can define

$$\begin{aligned} w_{1,b}^\pm &:= \begin{pmatrix} v_{1,b}^\pm + hK_{1,b}W \sum_{j \geq 0} M_b^j (hK_{2,b}W^* v_{1,b}^\pm) \\ - \sum_{j \geq 0} M_b^j (hK_{2,b}W^* v_{1,b}^\pm) \end{pmatrix}, \\ w_{2,b}^+ &:= \begin{pmatrix} -hK'_{1,b}W \sum_{j \geq 0} (M'_b)^j v_{2,b}^+ \\ \sum_{j \geq 0} (M'_b)^j v_{2,b}^+ \end{pmatrix}, \\ w_{2,b}^- &:= \begin{pmatrix} -hK''_{1,b}W \sum_{j \geq 0} (M''_b)^j v_{2,b}^- \\ \sum_{j \geq 0} (M''_b)^j v_{2,b}^- \end{pmatrix}. \end{aligned}$$

These are solutions to (2.1) on I_b and we have,

$$w_{1,b}^\pm \rightarrow \begin{pmatrix} v_{1,b}^\pm \\ 0 \end{pmatrix}, \quad w_{2,b}^\pm \rightarrow \begin{pmatrix} 0 \\ v_{2,b}^\pm \end{pmatrix},$$

as $h \rightarrow 0_+$.

In the same way, the operators $M_c := h^2 K_{1,c} W K_{2,c} W^*$, $M'_c := h^2 K_{1,c} W K'_{2,c} W^*$ and $M''_c := h^2 K_{1,c} W K''_{2,c} W^*$ are $\mathcal{O}(h^{1/2})$ when acting on $C(v_{2,c}^\pm, h)$, $C(v_{1,c}^+, h)$ and $C(v_{1,c}^-, h)$ respectively. Therefore, we can define

$$\begin{aligned} w_{1,c}^+ &:= \begin{pmatrix} \sum_{j \geq 0} (M'_c)^j v_{1,c}^+ \\ -hK'_{2,c}W^* \sum_{j \geq 0} (M'_c)^j v_{1,c}^+ \end{pmatrix}, \\ w_{1,c}^- &:= \begin{pmatrix} \sum_{j \geq 0} (M''_c)^j v_{1,c}^- \\ -hK''_{2,c}W^* \sum_{j \geq 0} (M''_c)^j v_{1,c}^- \end{pmatrix}, \\ w_{2,c}^\pm &:= \begin{pmatrix} - \sum_{j \geq 0} M_c^j (hK_{1,c}W v_{2,c}^\pm) \\ v_{2,c}^\pm + hK_{2,c}W \sum_{j \geq 0} M_c^j (hK_{1,c}W v_{2,c}^\pm) \end{pmatrix}. \end{aligned}$$

These are solutions to (2.1) on I_c and we have,

$$w_{1,c}^\pm \rightarrow \begin{pmatrix} v_{1,c}^\pm \\ 0 \end{pmatrix}, \quad w_{2,c}^\pm \rightarrow \begin{pmatrix} 0 \\ v_{2,c}^\pm \end{pmatrix},$$

as $h \rightarrow 0_+$.

The following lemmas are consequences of Lemma 4.1, 4.2 and the definitions of fundamental solutions. The estimates for the derivatives follow observing that if v is any of the functions $v_{j,b}^\pm$, $v_{j,c}^\pm$, $j = 1, 2$, the order of $\partial v(0)$ is that of $v(0)$ times h^{-1} .

Lemma 5.1. *There exist complex numbers $\gamma_{1,b}^\pm$, $\gamma_{2,b}^\pm$, $\zeta_{2,b}^\pm$ and $\delta_{j,b}^\pm$, $j = 1, 2$ such that,*

$$\begin{aligned} w_{1,b}^\pm(0) &= \begin{pmatrix} v_{1,b}^\pm(0) + \gamma_{1,b}^\pm v_{1,b}^\mp(0) \\ \delta_{1,b}^\pm v_{2,b}^\mp(0) \end{pmatrix} = \begin{pmatrix} v_{1,b}^\pm(0) \\ 0 \end{pmatrix} + \mathcal{O}(h^{1/2}|v_{1,b}^\pm(0)|), \\ \partial w_{1,b}^\pm(0) &= \begin{pmatrix} \partial v_{1,b}^\pm(0) + \gamma_{1,b}^\pm \partial v_{1,b}^\mp(0) \\ \delta_{1,b}^\pm \partial v_{2,b}^\mp(0) \end{pmatrix} = \begin{pmatrix} \partial v_{1,b}^\pm(0) \\ 0 \end{pmatrix} + \mathcal{O}(h^{1/2}|\partial v_{1,b}^\pm(0)|), \\ w_{2,b}^+(0) &= \begin{pmatrix} \gamma_{2,b}^+ v_{1,b}^+(0) + \zeta_{2,b}^+ v_{1,b}^-(0) \\ v_{2,b}^+(0) + \delta_{2,b}^+ v_{2,b}^-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ v_{2,b}^+(0) \end{pmatrix} + \mathcal{O}(h^{1/2}|v_{2,b}^+(0)|), \\ \partial w_{2,b}^+(0) &= \begin{pmatrix} \gamma_{2,b}^+ \partial v_{1,b}^+(0) + \zeta_{2,b}^+ \partial v_{1,b}^-(0) \\ \partial v_{2,b}^+(0) + \delta_{2,b}^+ \partial v_{2,b}^-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \partial v_{2,b}^+(0) \end{pmatrix} + \mathcal{O}(h^{1/2}|\partial v_{2,b}^+(0)|), \\ w_{2,b}^-(0) &= \begin{pmatrix} 0 \\ v_{2,b}^-(0) + \delta_{2,b}^- v_{2,b}^-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ v_{2,b}^-(0) \end{pmatrix} + \mathcal{O}(h^{1/2}|v_{2,b}^-(0)|), \\ \partial w_{2,b}^-(0) &= \begin{pmatrix} 0 \\ \partial v_{2,b}^-(0) + \delta_{2,b}^- \partial v_{2,b}^-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \partial v_{2,b}^-(0) \end{pmatrix} + \mathcal{O}(h^{1/2}|\partial v_{2,b}^-(0)|). \end{aligned}$$

Lemma 5.2. *There exist complex numbers $\gamma_{1,c}^\pm$, $\gamma_{2,c}^\pm$, $\zeta_{1,c}^\pm$ and $\delta_{j,c}^\pm$, $j = 1, 2$ such that,*

$$\begin{aligned} w_{2,c}^\pm(0) &= \begin{pmatrix} \delta_{2,c}^\pm v_{1,c}^\mp(0) \\ v_{2,c}^\pm(0) + \gamma_{2,c}^\pm v_{2,c}^\mp(0) \end{pmatrix} = \begin{pmatrix} 0 \\ v_{2,c}^\pm(0) \end{pmatrix} + \mathcal{O}(h^{1/2}|v_{2,c}^\pm(0)|), \\ \partial w_{2,c}^\pm(0) &= \begin{pmatrix} \delta_{2,c}^\pm \partial v_{1,c}^\mp(0) \\ \partial v_{2,c}^\pm(0) + \gamma_{2,c}^\pm \partial v_{2,c}^\mp(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \partial v_{2,c}^\pm(0) \end{pmatrix} + \mathcal{O}(h^{1/2}|\partial v_{2,c}^\pm(0)|), \\ w_{1,c}^+(0) &= \begin{pmatrix} v_{1,c}^+(0) + \delta_{1,c}^+ v_{1,c}^-(0) \\ \gamma_{1,c}^+ v_{2,c}^+(0) + \zeta_{1,c}^+ v_{2,c}^-(0) \end{pmatrix} = \begin{pmatrix} v_{1,c}^+(0) \\ 0 \end{pmatrix} + \mathcal{O}(h^{1/2}|v_{1,c}^+(0)|), \\ \partial w_{1,c}^+(0) &= \begin{pmatrix} \partial v_{1,c}^+(0) + \delta_{1,c}^+ \partial v_{1,c}^-(0) \\ \gamma_{1,c}^+ \partial v_{2,c}^+(0) + \zeta_{1,c}^+ \partial v_{2,c}^-(0) \end{pmatrix} = \begin{pmatrix} \partial v_{1,c}^+(0) \\ 0 \end{pmatrix} + \mathcal{O}(h^{1/2}|\partial v_{1,c}^+(0)|), \\ w_{1,c}^-(0) &= \begin{pmatrix} v_{1,c}^-(0) + \delta_{1,c}^- v_{1,c}^-(0) \\ 0 \end{pmatrix} = \begin{pmatrix} v_{1,c}^-(0) \\ 0 \end{pmatrix} + \mathcal{O}(h^{1/2}|v_{1,c}^-(0)|), \\ \partial w_{1,c}^-(0) &= \begin{pmatrix} \partial v_{1,c}^-(0) + \delta_{1,c}^- \partial v_{1,c}^-(0) \\ 0 \end{pmatrix} = \begin{pmatrix} \partial v_{1,c}^-(0) \\ 0 \end{pmatrix} + \mathcal{O}(h^{1/2}|\partial v_{1,c}^-(0)|). \end{aligned}$$

We also define the solutions on I_L and I_R^θ . By Lemma 4.3 the operator $M_L := h^2 K_{1,L} W K_{2,L} W^*$ and $M_R := h^2 K_{2,R} W^* K_{1,R} W$ are $\mathcal{O}(h^{2/3})$ when acting on $C_b^0(I_L)$ and $C_b^0(I_R^\theta)$ respectively. Thus we can define,

$$\begin{aligned} w_{1,L} &:= \begin{pmatrix} \sum_{j \geq 0} M_L^j u_{1,b}^- \\ -h K_{2,L} W^* \sum_{j \geq 0} M_L^j u_{1,b}^- \end{pmatrix}, \\ w_{2,L} &:= \begin{pmatrix} -\sum_{j \geq 0} M_L^j (h K_{1,L} W u_{2,b}^-) \\ u_{2,b}^- + h K_{2,L} W^* \sum_{j \geq 0} M_L^j (h K_{1,L} W u_{2,b}^-) \end{pmatrix}, \end{aligned}$$

on I_L and

$$\begin{aligned} w_{1,R} &:= \begin{pmatrix} u_{1,c}^- + hK_{1,R}W \sum_{j \geq 0} M_R^j(hK_{2,R}W^*u_{1,c}^-) \\ - \sum_{j \geq 0} M_R^j(hK_{2,R}W^*u_{1,c}^-) \end{pmatrix}, \\ w_{2,R} &:= \begin{pmatrix} -hK_{1,R}W \sum_{j \geq 0} M_R^j u_{2,c}^- \\ \sum_{j \geq 0} M_R^j u_{2,c}^- \end{pmatrix}, \end{aligned}$$

on I_R^θ .

For these solutions we have the following proposition corresponding to [3, Proposition 4.1].

Proposition 5.3. *The solutions $w_{j,L}$ and $w_{j,R}$, ($j = 1, 2$) satisfy,*

$$w_{j,L} \in L^2(I_L) \oplus L^2(I_L) ; w_{j,R} \in L^2(I_R^\theta) \oplus L^2(I_R^\theta).$$

6. CONNECTION OF THE SOLUTIONS

In this section we investigate the connection of the basis $w_{j,b}^\pm$ and $w_{j,c}^\pm$ and that of $w_{j,L}$ (resp., $w_{j,R}$) and $w_{j,b}^\pm$ (resp., $w_{j,c}^\pm$). We define the 4×4 transition matrix T as follows:

$$\begin{pmatrix} w_{1,b}^+ \\ w_{1,b}^- \\ w_{2,b}^+ \\ w_{2,b}^- \end{pmatrix} = T \begin{pmatrix} w_{1,c}^+ \\ w_{1,c}^- \\ w_{2,c}^+ \\ w_{2,c}^- \end{pmatrix}.$$

From now on, we set,

$$\begin{aligned} A_1 &:= \int_b^0 \sqrt{V_1(t) - E} dt/h, \quad B_1 = \int_0^c \sqrt{V_1(t) - E} dt/h, \\ A_2 &:= \int_0^c \sqrt{V_2(t) - E} dt/h, \quad B_2 = \int_b^0 \sqrt{V_2(t) - E} dt/h. \end{aligned}$$

Lemma 6.1. *One has*

$$\begin{aligned} T &= \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(h^{1/2}e^{A_1-B_1}) & \mathcal{O}(e^{A_1+B_1}) & \mathcal{O}(h^{1/2}e^{A_1-A_2}) & \mathcal{O}(he^{A_1+A_2}) \\ \mathcal{O}(e^{-A_1-B_1}) & \mathcal{O}(h^{1/2}e^{-A_1+B_1}) & \mathcal{O}(h^{1/2}e^{-A_1-A_2}) & \mathcal{O}(h^{1/2}e^{-A_1+A_2}) \\ \mathcal{O}(h^{1/2}e^{B_2-B_1}) & \mathcal{O}(h^{1/2}e^{B_2+B_1}) & \mathcal{O}(h^{1/2}e^{B_2-A_2}) & \mathcal{O}(e^{B_2+A_2}) \\ 0 & \mathcal{O}(h^{1/2}e^{-B_2+B_1}) & \mathcal{O}(e^{-B_2-A_2}) & \mathcal{O}(h^{1/2}e^{-B_2+A_2}) \end{pmatrix}. \end{aligned}$$

Moreover, one has,

$$(6.1) \quad t_{12} = (1 + \mathcal{O}(h^{1/2}))e^{A_1+B_1},$$

$$(6.2) \quad t_{34} = (1 + \mathcal{O}(h^{1/2}))e^{A_2+B_2},$$

$$(6.3) \quad \begin{aligned} t_{23} &= -h^{1/2}\sqrt{\pi}e^{-A_1-A_2}(V_1(0) - E)^{-1/4}(V_1'(0) - V_2'(0))^{-1/2} \\ &\quad \cdot (r_0(0) + r_1(0)\sqrt{V_1(0) - E}) + \mathcal{O}(he^{-A_1-A_2}), \end{aligned}$$

$$(6.4) \quad \begin{aligned} t_{32} &= h^{1/2}\sqrt{\pi}e^{B_1+B_2}(V_1(0) - E)^{-1/4}(V_1'(0) - V_2'(0))^{-1/2} \\ &\quad \cdot (r_0(0) + r_1(0)\sqrt{V_1(0) - E}) + \mathcal{O}(he^{B_1+B_2}). \end{aligned}$$

Proof. First, we consider t_{12} . By Cramer's formula we see that

$$(6.5) \quad t_{12} = \frac{\mathcal{W}[w_{1,c}^+, w_{1,b}^+, w_{2,c}^+, w_{2,c}^-]}{\mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{2,c}^+, w_{2,c}^-]}.$$

For the calculation of Wronskians, we will use the following notation: If w is any of the vectors of functions $w_{j,b}^\pm, w_{j,c}^\pm$, $j = 1, 2$ and written as

$$w(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix},$$

we set,

$$\mathbf{w}(x) := \begin{pmatrix} w_1(x) \\ \partial w_1(x) \\ w_2(x) \\ \partial w_2(x) \end{pmatrix}.$$

To estimate the remainder terms in $\mathbf{w}$ we notice

$$(6.6) \quad \begin{aligned} v_{1,b}^\pm(0) &= \mathcal{O}(h^{1/6}e^{\pm A_1}), \quad \partial v_{1,b}^\pm(0) = \mathcal{O}(h^{-5/6}e^{\pm A_1}), \\ v_{2,b}^\pm(0) &= \mathcal{O}(h^{1/6}e^{\pm B_2}), \quad \partial v_{2,b}^\pm(0) = \mathcal{O}(h^{-5/6}e^{\pm B_2}), \\ v_{1,c}^\pm(0) &= \mathcal{O}(h^{1/6}e^{\pm B_1}), \quad \partial v_{1,c}^\pm(0) = \mathcal{O}(h^{-5/6}e^{\pm B_1}), \\ v_{2,c}^\pm(0) &= \mathcal{O}(h^{1/6}e^{\pm A_2}), \quad \partial v_{2,c}^\pm(0) = \mathcal{O}(h^{-5/6}e^{\pm A_2}). \end{aligned}$$

By Lemma 5.1, Lemma 5.2 and (6.6) we have

$$(6.7) \quad \begin{aligned} \mathbf{w}_{1,b}^\pm(0) &= \begin{pmatrix} v_{1,b}^\pm(0) \\ \partial v_{1,b}^\pm(0) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}e^{\pm A_1}) \\ \mathcal{O}(h^{-1/3}e^{\pm A_1}) \\ \mathcal{O}(h^{2/3}e^{\pm A_1}) \\ \mathcal{O}(h^{-1/3}e^{\pm A_1}) \end{pmatrix}, \\ \mathbf{w}_{2,b}^\pm(0) &= \begin{pmatrix} 0 \\ 0 \\ v_{2,b}^\pm(0) \\ \partial v_{2,b}^\pm(0) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}e^{\pm B_2}) \\ \mathcal{O}(h^{-1/3}e^{\pm B_2}) \\ \mathcal{O}(h^{2/3}e^{\pm B_2}) \\ \mathcal{O}(h^{-1/3}e^{\pm B_2}) \end{pmatrix}, \\ \mathbf{w}_{1,c}^\pm(0) &= \begin{pmatrix} v_{1,c}^\pm(0) \\ \partial v_{1,c}^\pm(0) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}e^{\pm B_1}) \\ \mathcal{O}(h^{-1/3}e^{\pm B_1}) \\ \mathcal{O}(h^{2/3}e^{\pm B_1}) \\ \mathcal{O}(h^{-1/3}e^{\pm B_1}) \end{pmatrix}, \\ \mathbf{w}_{2,c}^\pm(0) &= \begin{pmatrix} 0 \\ 0 \\ v_{2,c}^\pm(0) \\ \partial v_{2,c}^\pm(0) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}e^{\pm A_2}) \\ \mathcal{O}(h^{-1/3}e^{\pm A_2}) \\ \mathcal{O}(h^{2/3}e^{\pm A_2}) \\ \mathcal{O}(h^{-1/3}e^{\pm A_2}) \end{pmatrix}. \end{aligned}$$

The Wronskian $\mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{2,c}^+, w_{2,c}^-]$ is written as the determinant

$$\mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{2,c}^+, w_{2,c}^-] = \det(\mathbf{w}_{1,c}^+(0), \mathbf{w}_{1,c}^-(0), \mathbf{w}_{2,c}^+(0), \mathbf{w}_{2,c}^-(0)).$$

We calculate the determinant using the multilinearity with respect to columns. We regard each column in the determinant as the sum of vectors whose upper or lower two elements are 0, and expand the determinant into determinants whose columns are such vectors. Then the order of the upper and

lower two elements in the remainder terms in (6.7) are those of the upper or lower two elements of the leading terms multiplied by $h^{1/2}$. Thus by (3.1) and the definitions of $v_{j,b}^\pm$ and $v_{j,c}^\pm$, we have

$$(6.8) \quad \begin{aligned} \mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{2,c}^+, w_{2,c}^-] &= \mathcal{W}[v_{1,c}^+, v_{1,c}^-] \mathcal{W}[v_{2,c}^+, v_{2,c}^-] + \mathcal{O}(h^{-5/6}) \\ &= \frac{4}{\pi^2 h^{4/3}} + \mathcal{O}(h^{-5/6}), \end{aligned}$$

$$(6.9) \quad \begin{aligned} \mathcal{W}[w_{1,c}^+, w_{1,b}^+, w_{2,c}^+, w_{2,c}^-] &= \mathcal{W}[v_{1,c}^+, v_{1,b}^+] \mathcal{W}[v_{2,c}^+, v_{2,c}^-] + \mathcal{O}(h^{-5/6} e^{A_1+B_1}) \\ &= \frac{4}{\pi^2 h^{4/3}} e^{A_1+B_1} + \mathcal{O}(h^{-5/6} e^{A_1+B_1}). \end{aligned}$$

By (6.5), (6.8) and (6.9) we obtain (6.1). By the similar calculation we obtain (6.2).

Next we study t_{23} and t_{32} . For the calculation of higher order terms we notice

$$(6.10) \quad \begin{aligned} \mathbf{w}_{1,b}^- &= \begin{pmatrix} v_{1,b}^-(0) \\ \partial v_{1,b}^-(0) \\ -hK_{2,b}W^*v_{1,b}^-(0) \\ -h\partial(K_{2,b}W^*v_{1,b}^-(0)) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}e^{-A_1}) \\ \mathcal{O}(h^{-1/3}e^{-A_1}) \\ \mathcal{O}(h^{7/6}e^{-A_1}) \\ \mathcal{O}(h^{1/6}e^{-A_1}) \end{pmatrix}, \\ \mathbf{w}_{2,b}^+ &= \begin{pmatrix} -hK'_{1,b}Wv_{2,b}^+(0) \\ -h\partial(K'_{1,b}Wv_{2,b}^+(0)) \\ v_{2,b}^+(0) \\ \partial v_{2,b}^+(0) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{7/6}e^{B_2}) \\ \mathcal{O}(h^{1/6}e^{B_2}) \\ \mathcal{O}(h^{2/3}e^{B_2}) \\ \mathcal{O}(h^{-1/3}e^{B_2}) \end{pmatrix}, \\ \mathbf{w}_{1,c}^+ &= \begin{pmatrix} v_{1,c}^+(0) \\ \partial v_{1,c}^+(0) \\ -hK'_{2,c}W^*v_{1,c}^+(0) \\ -h\partial(K'_{2,c}W^*v_{1,c}^+(0)) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}e^{B_1}) \\ \mathcal{O}(h^{-1/3}e^{B_1}) \\ \mathcal{O}(h^{7/6}e^{B_1}) \\ \mathcal{O}(h^{1/6}e^{B_1}) \end{pmatrix}, \\ \mathbf{w}_{2,c}^- &= \begin{pmatrix} -hK_{1,c}Wv_{2,c}^-(0) \\ -h\partial(K_{1,c}Wv_{2,c}^-(0)) \\ v_{2,c}^-(0) \\ \partial v_{2,c}^-(0) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{7/6}e^{-A_2}) \\ \mathcal{O}(h^{1/6}e^{-A_2}) \\ \mathcal{O}(h^{2/3}e^{-A_2}) \\ \mathcal{O}(h^{-1/3}e^{-A_2}) \end{pmatrix}. \end{aligned}$$

Since $\mathcal{W}[v_{1,c}^+, v_{1,b}^-] = 0$, we have

$$(6.11) \quad \begin{aligned} \mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{1,b}^-, w_{2,c}^-] &= \mathcal{W}[v_{1,c}^+, v_{1,c}^-] \mathcal{W}[-hK_{2,b}W^*v_{1,b}^-, v_{2,c}^-] \\ &\quad + \mathcal{W}[v_{1,c}^-, v_{1,b}^-] \mathcal{W}[-hK'_{2,c}W^*v_{1,c}^+, v_{2,c}^-] \\ &\quad + \mathcal{O}(h^{-1/3}e^{-A_1-A_2}) \\ &= -h^{-1} \mathcal{W}[v_{1,c}^+, v_{1,c}^-] e^{-A_2-B_2} \int_b^0 v_{2,b}^-(t) W^* v_{1,b}^-(t) dt \\ &\quad + h^{-1} \mathcal{W}[v_{1,c}^-, v_{1,b}^-] \int_0^c v_{2,c}^-(t) W^* v_{1,c}^+(t) dt \\ &\quad + \mathcal{O}(h^{-1/3}e^{-A_1-A_2}) \\ &= h^{-1} \mathcal{W}[v_{1,c}^-, v_{1,c}^+] \int_b^c u_{2,L}^-(t) W^* u_{1,R}^-(t) dt \\ &\quad + \mathcal{O}(h^{-1/3}e^{-A_1-A_2}). \end{aligned}$$

As for the derivative of $u_{1,R}^-$, for x near 0 we have

$$\partial u_{1,R}^-(x) = (1 + \mathcal{O}(h)) \frac{1}{h^{5/6}\sqrt{\pi}} (V_1(x) - E)^{3/4} e^{-\int_b^x \sqrt{V_1(t) - E} dt/h}.$$

We apply the stationary phase theorem to $\int_b^c u_{2,L}^-(t) W^* u_{1,R}^-(t) dt$. Estimating the derivatives of $u_{2,L}^-(t) W^* u_{1,R}^-(t) e^{\int_b^t \sqrt{V_1(r) - E} dr/h + \int_t^c \sqrt{V_2(r) - E} dr/h}$ near 0 by Cauchy's integral formula, we have

$$(6.12) \quad \begin{aligned} \int_b^c u_{2,L}^-(t) W^* u_{1,R}^-(t) dt &= \frac{2h^{5/6}}{\sqrt{\pi}} (V_1(0) - E)^{-1/4} (V_1'(0) - V_2'(0))^{-1/2} \\ &\quad \cdot (r_0(0) + r_1(0) \sqrt{V_1(0) - E}) e^{-A_1 - A_2} \\ &\quad + \mathcal{O}(h^{4/3} e^{-A_1 - A_2}). \end{aligned}$$

Here we used $V_1(0) = V_2(0)$. From (6.11), (6.12) and that

$$t_{23} = \frac{\mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{1,b}^-, w_{2,c}^-]}{\mathcal{W}[w_{1,c}^+, w_{1,c}^-, w_{2,c}^+, w_{2,c}^-]},$$

(6.3) follows. In the similar way we can obtain (6.4).

As for t_{41} , since by Lemmas 5.1 and 5.2 upper two elements of $\mathbf{w}_{2,b}^-$, $\mathbf{w}_{1,c}^-$ and $\mathbf{w}_{2,c}^\pm$ can be written as

$$C \begin{pmatrix} v_{1,c}^- \\ \partial v_{1,c}^- \end{pmatrix},$$

we have $t_{41} = 0$. As for t_{14} , since we can see by Lemmas 5.1 and 5.2 that we need to choose remainder terms from at least two columns of the determinant $\det(\mathbf{w}_{1,c}^+, \mathbf{w}_{1,c}^-, \mathbf{w}_{2,c}^+, \mathbf{w}_{1,b}^-)$, we obtain $t_{14} = \mathcal{O}(he^{A_1 + A_2})$. The estimates for the other terms can be obtained by the similar way as above. $\square$

To make some of the elements of the transition matrix 0, we change the basis on I_b and I_c .

Lemma 6.2. *There exist complex numbers $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j$, $j = 1, 2$ such that if we set,*

$$\begin{aligned} \tilde{w}_{1,b}^+ &= w_{1,b}^+, \quad \tilde{w}_{2,b}^+ = w_{2,b}^+, \\ \tilde{w}_{1,b}^- &= w_{1,b}^- - \tilde{a}_1 w_{1,b}^+ - \tilde{a}_2 w_{2,b}^+, \\ \tilde{w}_{2,b}^- &= w_{2,b}^- - \tilde{b}_1 w_{1,b}^+ - \tilde{b}_2 w_{2,b}^+, \\ \tilde{w}_{1,c}^+ &= w_{1,c}^+, \quad \tilde{w}_{2,c}^+ = w_{2,c}^+, \\ \tilde{w}_{1,c}^- &= w_{1,c}^- + \tilde{c}_1 w_{1,c}^+ + \tilde{c}_2 w_{2,c}^+, \\ \tilde{w}_{2,c}^- &= w_{2,c}^- + \tilde{d}_1 w_{1,c}^+ + \tilde{d}_2 w_{2,c}^+, \end{aligned}$$

the matrix $\tilde{T}$ defined by

$$\begin{pmatrix} \tilde{w}_{1,b}^+ \\ \tilde{w}_{1,b}^- \\ \tilde{w}_{2,b}^+ \\ \tilde{w}_{2,b}^- \end{pmatrix} = \tilde{T} \begin{pmatrix} \tilde{w}_{1,c}^+ \\ \tilde{w}_{1,c}^- \\ \tilde{w}_{2,c}^+ \\ \tilde{w}_{2,c}^- \end{pmatrix},$$

has the following form;

$$\tilde{T} = \begin{pmatrix} 0 & \tilde{t}_{12} & 0 & \tilde{t}_{14} \\ \tilde{t}_{21} & 0 & \tilde{t}_{23} & 0 \\ 0 & \tilde{t}_{32} & 0 & \tilde{t}_{34} \\ \tilde{t}_{41} & 0 & \tilde{t}_{43} & 0 \end{pmatrix}.$$

with $\tilde{t}_{23}$ and $\tilde{t}_{32}$ having the same asymptotics as t_{23} and t_{32} respectively. Moreover, we have the following estimates.

$$\begin{aligned}\tilde{a}_1 &= \mathcal{O}(h^{1/2}e^{-2A_1}), \quad \tilde{a}_2 = \mathcal{O}(h^{1/2}e^{-A_1-B_2}), \\ \tilde{b}_1 &= \mathcal{O}(h^{1/2}e^{-A_1-B_2}), \quad \tilde{b}_2 = \mathcal{O}(h^{1/2}e^{-2B_2}), \\ \tilde{c}_1 &= \mathcal{O}(h^{1/2}e^{-2B_1}), \quad \tilde{c}_2 = \mathcal{O}(h^{1/2}e^{-B_1-A_2}), \\ \tilde{d}_1 &= \mathcal{O}(h^{1/2}e^{-B_1-A_2}), \quad \tilde{d}_2 = \mathcal{O}(h^{1/2}e^{-2A_2}).\end{aligned}$$

Proof. We define $\tilde{a}_j$, $\tilde{b}_j$, $\tilde{c}_j$ and $\tilde{d}_j$ by

$$\begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 \\ \tilde{b}_1 & \tilde{b}_2 \end{pmatrix} \begin{pmatrix} t_{12} & t_{14} \\ t_{32} & t_{34} \end{pmatrix} = \begin{pmatrix} t_{22} & t_{24} \\ t_{42} & t_{44} \end{pmatrix},$$

$$\begin{pmatrix} t_{12} & t_{14} \\ t_{32} & t_{34} \end{pmatrix} \begin{pmatrix} \tilde{c}_1 & \tilde{c}_2 \\ \tilde{d}_1 & \tilde{d}_2 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{pmatrix}.$$

Then, it is easy to see that $\tilde{T}$ has the form as in the lemma, and by Lemma 6.1 we have

$$\begin{aligned}\begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 \\ \tilde{b}_1 & \tilde{b}_2 \end{pmatrix} &= (t_{12}t_{34} - t_{14}t_{32})^{-1} \begin{pmatrix} t_{22} & t_{24} \\ t_{42} & t_{44} \end{pmatrix} \begin{pmatrix} t_{34} & -t_{14} \\ -t_{32} & t_{12} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(h^{1/2}e^{-2A_1}) & \mathcal{O}(h^{1/2}e^{-A_1-B_2}) \\ \mathcal{O}(h^{1/2}e^{-A_1-B_2}) & \mathcal{O}(h^{1/2}e^{-2B_2}) \end{pmatrix}, \\ \begin{pmatrix} \tilde{c}_1 & \tilde{c}_2 \\ \tilde{d}_1 & \tilde{d}_2 \end{pmatrix} &= (t_{12}t_{34} - t_{14}t_{32})^{-1} \begin{pmatrix} t_{34} & -t_{14} \\ -t_{32} & t_{12} \end{pmatrix} \begin{pmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(h^{1/2}e^{-2B_1}) & \mathcal{O}(h^{1/2}e^{-B_1-A_2}) \\ \mathcal{O}(h^{1/2}e^{-B_1-A_2}) & \mathcal{O}(h^{1/2}e^{-2A_2}) \end{pmatrix}.\end{aligned}$$

□

Note that $\tilde{w}_{j,b}^\pm$ and $\tilde{w}_{j,c}^\pm$ have the same asymptotics as $w_{j,b}^\pm$ and $w_{j,c}^\pm$ with respect to h respectively. We next consider the connection of solutions at b and c .

Lemma 6.3. Set $a_j^\pm, b_j^\pm, c_j^\pm, d_j^\pm$ as follows:

$$\begin{aligned}(6.13) \quad w_{1,L} &= D_L^{-1}(a_1^+ \tilde{w}_{1,b}^+ + a_1^- \tilde{w}_{1,b}^- + a_2^+ \tilde{w}_{2,b}^+ + a_2^- \tilde{w}_{2,b}^-), \\ w_{2,L} &= D_L^{-1}(b_1^+ \tilde{w}_{1,b}^+ + b_1^- \tilde{w}_{1,b}^- + b_2^+ \tilde{w}_{2,b}^+ + b_2^- \tilde{w}_{2,b}^-), \\ w_{1,R} &= D_R^{-1}(c_1^+ \tilde{w}_{1,c}^+ + c_1^- \tilde{w}_{1,c}^- + c_2^+ \tilde{w}_{2,c}^+ + c_2^- \tilde{w}_{2,c}^-), \\ w_{2,R} &= D_R^{-1}(d_1^+ \tilde{w}_{1,c}^+ + d_1^- \tilde{w}_{1,c}^- + d_2^+ \tilde{w}_{2,c}^+ + d_2^- \tilde{w}_{2,c}^-),\end{aligned}$$

where D_L and D_R are Wronskians

$$\begin{aligned}D_L &= \mathcal{W}[\tilde{w}_{1,b}^+, \tilde{w}_{1,b}^-, \tilde{w}_{2,b}^+, \tilde{w}_{2,b}^-], \\ D_R &= \mathcal{W}[\tilde{w}_{1,c}^+, \tilde{w}_{1,c}^-, \tilde{w}_{2,c}^+, \tilde{w}_{2,c}^-].\end{aligned}$$

Then we have

$$(6.14) \quad a_1^+ = \frac{8}{\pi^2 h^{4/3}} \cos \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h^{-5/6}),$$

$$(6.15) \quad a_1^- = \frac{4}{\pi^2 h^{4/3}} \sin \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h^{-5/6}),$$

$$(6.16) \quad a_2^\pm = \mathcal{O}(h^{-1/2})$$

$$(6.17) \quad b_1^\pm = \mathcal{O}(h^{-1/2})$$

$$(6.18) \quad b_2^+ = \frac{4}{\pi^2 h^{4/3}} + \mathcal{O}(h^{-5/6}),$$

$$(6.19) \quad b_2^- = \mathcal{O}(h^{-1/2}),$$

$$(6.20) \quad c_1^+ = \frac{4}{\pi^2 h^{4/3}} + \mathcal{O}(h^{-5/6}),$$

$$(6.21) \quad c_1^- = \mathcal{O}(h^{-1/2}),$$

$$(6.22) \quad c_2^\pm = \mathcal{O}(h^{-1/2}),$$

$$(6.23) \quad d_1^\pm = \mathcal{O}(h^{-1/2}),$$

$$(6.24) \quad d_2^+ = -\frac{4e^{i\frac{\pi}{4}}i}{\pi^2 h^{4/3}} + \mathcal{O}(h^{-5/6}),$$

$$(6.25) \quad d_2^- = \frac{2e^{i\frac{\pi}{4}}}{\pi^2 h^{4/3}} + \mathcal{O}(h^{-5/6}).$$

Proof. We start with a_1^+ . By the Cramer's formula we have

$$a_1^+ = \mathcal{W}[w_{1,L}, \tilde{w}_{1,b}^-, \tilde{w}_{2,b}^+, \tilde{w}_{2,b}^-].$$

By Lemma 6.2 there exist $\alpha > 0$ such that

$$\mathcal{W}[w_{1,L}, \tilde{w}_{1,b}^-, \tilde{w}_{2,b}^+, \tilde{w}_{2,b}^-] = \mathcal{W}[w_{1,L}, w_{1,b}^-, w_{2,b}^+, w_{2,b}^-] + \mathcal{O}(e^{-\alpha/h}).$$

We use the notation $\mathbf{w}$ as in the proof of Lemma 6.1. The Wronskian in the right-hand side is written as the determinant;

$$\mathcal{W}[w_{1,L}, w_{1,b}^-, w_{2,b}^+, w_{2,b}^-] = \det(\mathbf{w}_{1,L}(b), \mathbf{w}_{1,b}^-(b), \mathbf{w}_{2,b}^+(b), \mathbf{w}_{2,b}^-(b)).$$

By Lemma 4.3 we have

$$(6.26) \quad \mathbf{w}_{1,L}(b) = \begin{pmatrix} u_{1,b}^-(b) \\ \partial u_{1,b}^-(b) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{2/3}) \\ \mathcal{O}(1) \\ \mathcal{O}(h) \\ \mathcal{O}(1) \end{pmatrix}.$$

By Lemma 4.1 we also have

$$\begin{aligned}
 \mathbf{w}_{1,b}^- &= \begin{pmatrix} v_{1,b}^-(b) \\ \partial v_{1,b}^-(b) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{1/2}) \\ \mathcal{O}(h^{-1/6}) \\ \mathcal{O}(h) \\ \mathcal{O}(1) \end{pmatrix}, \\
 \mathbf{w}_{2,b}^+ &= \begin{pmatrix} 0 \\ 0 \\ v_{2,b}^+(b) \\ \partial v_{2,b}^+(b) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathcal{O}(h^{2/3}) \\ \mathcal{O}(h^{-1/3}) \end{pmatrix}, \\
 \mathbf{w}_{2,b}^- &= \begin{pmatrix} 0 \\ 0 \\ v_{2,b}^-(b) \\ \partial v_{2,b}^-(b) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(h^{5/6}) \\ \mathcal{O}(h^{1/6}) \\ \mathcal{O}(h^{2/3}) \\ \mathcal{O}(h^{-1/3}) \end{pmatrix}.
 \end{aligned}
 \tag{6.27}$$

We calculate the determinant using the multilinearity with respect to columns as in the proof of Lemma 6.1. Then the order of the upper and lower two elements in the remainder terms in (6.26) and (6.27) are those of the leading terms multiplied by $h^{1/2}$. Thus we obtain

$$\mathcal{W}[w_{1,L}, w_{1,b}^-, w_{2,b}^+, w_{2,b}^-] = \mathcal{W}[u_{1,b}^-, v_{1,b}^-] \mathcal{W}[v_{2,b}^+, v_{2,b}^-] + \mathcal{O}(h^{-5/6}).$$

From the definition we have $u_{1,b}^- = u_{1,L}^-$, $v_{1,b}^- = u_{1,R}^-$. Hence by Proposition 3.3 we obtain

$$\mathcal{W}[u_{1,b}^-, v_{1,b}^-] = b_- \mathcal{W}[u_{1,R}^+, u_{1,R}^-] = -\frac{4}{\pi h^{2/3}} \cos \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h^{1/3}).$$

Since $v_{2,b}^+ = e^{S_2/h} u_{2,L}^-$ and $v_{2,b}^- = e^{-S_2/h} u_{2,L}^+$, we have

$$\mathcal{W}[v_{2,b}^+, v_{2,b}^-] = \mathcal{W}[u_{2,L}^-, u_{2,L}^+] = -\frac{2}{\pi h^{2/3}} (1 + \mathcal{O}(h))$$

which completes the proof of (6.14). The proof of (6.15) is similar.

The estimates (6.16) and (6.17) follow from the calculation of the determinant as above, Lemma 4.1 and Lemma 4.3. We can prove (6.18) by the similar calculation as in the proof of (6.14). The estimate (6.19) is obtained using $\mathcal{W}[v_{2,b}^+, u_{2,b}^-] = 0$. The estimates (6.20)-(6.25) are obtained in the same way as (6.14)-(6.19). $\square$

7. QUANTISATION CONDITION

Proposition 7.1. *$E \in \mathcal{D}_I$ is a resonance of P if and only if,*

$$(7.1) \quad \cos \frac{\mathcal{A}(E)}{h} + f(E, h) = hF(E, h).$$

where $f(E, h)$ and $F(E, h)$ are analytic for $E \in \mathcal{D}_I$, $f(E, h)$ is real for real E and

$$\begin{aligned}
 f(E, h) &= \mathcal{O}(h^{1/2}), \\
 F(E, h) &= -\frac{\pi}{4i} \left(\sin \frac{\mathcal{A}(E)}{h} \right) e^{-2A_1 - 2A_2} (V_1(0) - E)^{-1/2} \\
 &\quad \cdot (V_1'(0) - V_2'(0))^{-1} (r_0(0) + r_1(0) \sqrt{V_1(0) - E})^2 + \mathcal{O}(h^{1/2} e^{-2A_1 - 2A_2}).
 \end{aligned}$$

Proof. As in [3], E is a resonance if and only if $w_{1,L}, w_{2,L}, w_{1,R}$ and $w_{2,R}$ are linearly dependent, that is

$$(7.2) \quad \mathcal{W}_0(E) = \mathcal{W}[w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}] = 0.$$

We substitute the right-hand side of (6.13) for $w_{j,L}, w_{j,R}$ in (7.2) and develop the Wronskian as a sum of terms of the form $C(h)\mathcal{W}[w_1, w_2, w_3, w_4]$ where $C(h)$ is a constant, w_1, w_2 and w_3, w_4 are chosen from $w_{j,b}^\pm$, ($j = 1, 2$) and $w_{j,c}^\pm$, ($j = 1, 2$) respectively. If only one of the w_j , ($j = 1, \dots, 4$) is chosen from $w_{j,b}^-$ or $w_{j,c}^-$, then by the form of $\tilde{T}$ in Lemma 6.2 we can see that $\mathcal{W}[w_1, w_2, w_3, w_4] = 0$. Thus by Lemma 6.3 we have

$$(7.3) \quad \begin{aligned} D_L^2 D_R^2 \mathcal{W}_0(E) = & (a_1^+ b_2^+ - b_1^+ a_2^+) (c_1^+ d_2^+ - d_1^+ c_2^+) \mathcal{W}[\tilde{w}_{1,b}^+, \tilde{w}_{2,b}^+, \tilde{w}_{1,c}^+, \tilde{w}_{2,c}^+] \\ & + (a_1^- b_2^+ - b_1^- a_2^+) (c_1^+ d_2^- - d_1^+ c_2^-) \mathcal{W}[\tilde{w}_{1,b}^-, \tilde{w}_{2,b}^+, \tilde{w}_{1,c}^+, \tilde{w}_{2,c}^+] \\ & + \mathcal{O}(e^{\max\{A_1 - B_1 - A_2 + B_2, -A_1 + B_1 + A_2 - B_2\}}). \end{aligned}$$

By Lemma 6.2 we can easily see that

$$\begin{aligned} \mathcal{W}[\tilde{w}_{1,b}^-, \tilde{w}_{2,b}^+, \tilde{w}_{1,c}^+, \tilde{w}_{2,c}^-] &= \tilde{t}_{23} \tilde{t}_{32} \mathcal{W}[\tilde{w}_{2,c}^+, \tilde{w}_{1,c}^-, \tilde{w}_{1,c}^+, \tilde{w}_{2,c}^-] \\ &= \tilde{t}_{23} \tilde{t}_{32} \mathcal{W}[w_{2,c}^+, w_{1,c}^-, w_{1,c}^+, w_{2,c}^-], \end{aligned}$$

and by Lemma 6.3 we have

$$(7.4) \quad \begin{aligned} a_1^+ b_2^+ - b_1^+ a_2^+ &= \frac{32}{\pi^4 h^{8/3}} \left(\cos \frac{\mathcal{A}(E)}{h} + f(E, h) \right), \\ c_1^+ d_2^+ - d_1^+ c_2^+ &= -\frac{16e^{i\frac{\pi}{4}} i}{\pi^4 h^{8/3}} (1 + \mathcal{O}(h^{1/2})), \\ a_1^- b_2^+ - b_1^- a_2^+ &= \frac{16}{\pi^4 h^{8/3}} \left(\sin \frac{\mathcal{A}(E)}{h} + \mathcal{O}(h^{1/2}) \right), \\ c_1^+ d_2^- - d_1^+ c_2^- &= \frac{8e^{i\frac{\pi}{4}}}{\pi^4 h^{8/3}} (1 + \mathcal{O}(h^{1/2})), \end{aligned}$$

where $|f(E, h)| = \mathcal{O}(h^{1/2})$ uniformly with respect to $E \in \mathcal{D}_I$. From the construction of $u_{j,L}^\pm$ and $u_{1,R}^\pm$ we can easily see that $f(E, h)$ is real for real E . We can also see by the similar calculation as in the proof of Lemma 6.1,

$$(7.5) \quad \begin{aligned} \mathcal{W}[\tilde{w}_{1,b}^+, \tilde{w}_{2,b}^+, \tilde{w}_{1,c}^+, \tilde{w}_{2,c}^+] &= -\frac{4}{\pi^2 h^{4/3}} e^{A_1 + A_2 + B_1 + B_2} (1 + \mathcal{O}(h^{1/2})), \\ \mathcal{W}[w_{2,c}^+, w_{1,c}^-, w_{1,c}^+, w_{2,c}^-] &= -\frac{4}{\pi^2 h^{4/3}} (1 + \mathcal{O}(h^{1/2})). \end{aligned}$$

By (7.3), (7.4), (7.5) and asymptotics of t_{23} and t_{32} , we have

$$\begin{aligned} D_L^2 D_R^2 \mathcal{W}_0(E) &= \frac{2^{11} e^{i\frac{\pi}{4}} i}{\pi^{10} h^{20/3}} \left(\cos \frac{\mathcal{A}(E)}{h} + f(E, h) \right) e^{A_1 + A_2 + B_1 + B_2} (1 + \mathcal{O}(h^{1/2})) \\ &\quad + \frac{2^9 e^{i\frac{\pi}{4}}}{\pi^9 h^{17/3}} \left(\sin \frac{\mathcal{A}(E)}{h} \right) e^{-A_1 - A_2 + B_1 + B_2} (V_1(0) - E)^{-1/2} \\ &\quad \cdot (V_1'(0) - V_2'(0))^{-1} (r_0(0) + r_1(0) \sqrt{V_1(0) - E})^2 \\ &\quad + \mathcal{O}(h^{-31/6} e^{-A_1 - A_2 + B_1 + B_2}). \end{aligned}$$

Therefore, $\mathcal{W}_0(E) = 0$ is equivalent to

$$(7.6) \quad \begin{aligned} \cos \frac{\mathcal{A}(E)}{h} + f(E, h) &= -\frac{h\pi}{4i} \left(\sin \frac{\mathcal{A}(E)}{h} \right) e^{-2A_1 - 2A_2} (V_1(0) - E)^{-1/2} \\ &\quad \cdot (V_1'(0) - V_2'(0))^{-1} (r_0(0) + r_1(0) \sqrt{V_1(0) - E})^2 \\ &\quad + \mathcal{O}(h^{3/2} e^{-2A_1 - 2A_2}). \end{aligned}$$

□

8. COMPLETION OF THE PROOF OF THEOREM 2.1

In order to solve (7.1), we first observe that the roots of $\cos(\mathcal{A}(E)/h) = 0$ are given by $E = e_k(h)$ with,

$$e_k(h) := \mathcal{A}^{-1}\left((k + \frac{1}{2})\pi h\right) \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

For sufficiently small $d' > 0$ we set $\tilde{I} = I + [-d', d']$. Then since there exist constants $m, M > 0$ such that for $E \in \tilde{I}$ we have $m \leq \mathcal{A}'(E) \leq M$, the estimate $|e_k(h) - e_l(h)| \geq \pi h|k - l|/M$ holds. Since $\mathcal{A}(E)$ is holomorphic in $\mathcal{D}_I$ (see, e.g., Fujiié-Ramond [4]), by the Cauchy's integral formula we have for fixed $C' > 0$ and $z \in \mathbb{C}$ such that $|z - e_k| < hC'/2$

$$(8.1) \quad \begin{aligned} \cos \frac{\mathcal{A}(z)}{h} &= -h^{-1} \mathcal{A}'(e_k)(z - e_k) \\ &+ \frac{1}{2\pi i} \oint_{|\zeta - e_k| = C'h} \frac{1}{\zeta - z} \left(\frac{z - e_k}{\zeta - e_k} \right)^2 \cos \frac{\mathcal{A}(\zeta)}{h} d\zeta. \end{aligned}$$

Since $\mathcal{A}'(e_k) > m$ for $e_k \in \tilde{I}$, $f(E, h) = \mathcal{O}(h^{1/2})$ and

$$\left| \frac{1}{2\pi i} \oint_{|\zeta - e_k| = C'h} \frac{1}{\zeta - z} \left(\frac{z - e_k}{\zeta - e_k} \right)^2 \cos \frac{\mathcal{A}(\zeta)}{h} d\zeta \right| \leq Ch^{-2}|z - e_k|^2,$$

by the Rouché's theorem we can see that for sufficiently large $C'_0 > 0$ and sufficiently small $h > 0$, $\cos \mathcal{A}(z)/h + f(z, h) = 0$ has a unique solution $\tilde{e}_k(h)$ in $B(e_k; C'_0 h^{3/2})$ for $e_k \in \tilde{I}$ and conversely, all the roots in $\mathcal{D}_I$ are of this type. Since $f(E, h)$ is real for $E \in \mathbb{R}$ and by (8.1) there exists a number $C > 0$ such that $[-CC'_0 h^{1/2}, CC'_0 h^{1/2}] \subset \{\cos(\mathcal{A}(E)/h); -C'_0 h^{3/2} < E - e_k < C'_0 h^{3/2}\}$, we can see that $\tilde{e}_k(h)$ is real.

Estimating the remainder terms by the Cauchy's integral formula as above, the left-hand side of (7.1) is written as

$$(8.2) \quad \begin{aligned} \cos \frac{\mathcal{A}(z)}{h} + f(z, h) &= -h^{-1}(\mathcal{A}'(\tilde{e}_k(h)) + \mathcal{O}(h^{1/2})) \sin \frac{\mathcal{A}(\tilde{e}_k(h))}{h} (z - \tilde{e}_k(h)) \\ &+ \mathcal{O}(h^{-2}|z - \tilde{e}_k(h)|^2). \end{aligned}$$

Thus again by the Rouché's theorem we can see that for sufficiently large $C''_0 > 0$ and sufficiently small $h > 0$, $\mathcal{W}_0(E) = 0$ has a unique solution $E_k(h)$ in

$$B(\tilde{e}_k; C''_0 h^2 e^{-2A_1 - 2A_2}),$$

and all the roots in $\mathcal{D}_I$ are of this type. In the same way as above we also have

$$\sin \frac{\mathcal{A}(E_k(h))}{h} = \sin \frac{\mathcal{A}(\tilde{e}_k(h))}{h} + \mathcal{O}(h e^{-2A_1 - 2A_2}).$$

Hence substituting $E_k(h)$ into (7.1) for z we can see

$$\begin{aligned} E_k(h) - \tilde{e}_k(h) &= -\frac{h^2 \pi i}{4} \mathcal{A}'(e_k(h))^{-1} e^{-2A_1 - 2A_2} (V_1(0) - E_k(h))^{-1/2} (V_1'(0) - V_2'(0))^{-1} \\ &\quad \cdot (r_0(0) + r_1(0) \sqrt{V_1(0) - E_k(h)})^2 + \mathcal{O}(h^{5/2}) e^{-2A_1 - 2A_2}, \end{aligned}$$

from which Theorem 2.1 follows.

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