

The third order asymptotic admissibility of
estimators in one parameter regular case

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$X_1, X_2, \dots, X_n$ は i. i. d. で共通の確率密度関数 $f(x, \theta_0)$, $\theta_0 \in \Theta \subset \mathbb{R}$ を持つものとする. $f(x, \theta_0)$ は次の条件を満たす.

A 1. Θ は $\mathbb{R}$ の開集合である.

A 2. $f(x, \theta)$ は (x, θ) に関して可測である.

A 3. 各 x に対して, $f(x, \theta)$ は θ に関して 5 回連続偏微分可能で, その微分係数は θ に関して連続である.

A 4. $\{f(x, \theta), \theta \in \Theta\}$ に対応する測度は互いに絶対連続である.

A 5. すべての $\theta \in \Theta$ に対して,

$$E_{\theta} |\log f(X, \theta)| < \infty$$

$$0 < I_{1,1}(\theta) = E_{\theta} \left(\frac{\partial}{\partial \theta} \log f(X, \theta) \right)^2 = -E_{\theta} \left(\frac{\partial^2}{\partial \theta^2} \log f(X, \theta) \right).$$

A 6. 各 $\theta \in \Theta$ に対して, コンパクトな Θ_0 の近傍 Θ_0 および関数 $G(x)$ が存在して, すべての $\theta \in \Theta_0$ に対して

$$\left| \frac{\partial^i}{\partial \theta^i} \log f(X, \theta) \right| \leq G(x), \quad i = 1, 2, 3, 4, 5.$$

$$\sup_{\theta \in \Theta_0} E_{\theta} (G(x))^4 < \infty.$$

A 7. すべての $j (\leq 5)$, $j = i_1 + i_2 + \dots + i_5$, $i_j = 0, 1, 2, \dots, 5$,

に対して,

$$E_{\theta} \left\{ \frac{\partial^{i_1}}{\partial \theta^{i_1}} \log f(x, \theta) \cdot \frac{\partial^{i_2}}{\partial \theta^{i_2}} \log f(x, \theta) \cdot \dots \cdot \frac{\partial^{i_5}}{\partial \theta^{i_5}} \log f(x, \theta) \right\}$$

は $\theta \in \Theta$ に関して, 4回連続微分可能で, その微分係数は θ に関して連続である.

A 8. θ の最尤推定量 $\hat{\theta}$ が一意に定まる.

すべての $\theta_0 \in \Theta$ に対して θ_0 の近傍 Θ_0 が存在して, すべての $\theta \in \Theta_0$ で一様に $o(n^{-1})$ まで Edgeworth 展開可能である θ の推定量 T の全体を π とするとき以下の定理が成り立つ.

定理 推定量 $T \in \pi$ に対して, 関数 $C_0(\theta)$, $C_1(\theta)$, $t(\theta)$, $C_2(\theta, t(\theta))$ が存在して, 統計量

$$\begin{aligned} \bar{\theta} = & \hat{\theta} + \frac{1}{n^{1/2}} C_0(\hat{\theta}) \\ & + \frac{1}{n} \left\{ (C_0(\hat{\theta}) - \frac{1}{2} t(\hat{\theta})) I_{1,1}^{-1}(\hat{\theta}) Z_2(\hat{\theta}) + C_1(\hat{\theta}) \right\} \\ & + \frac{1}{n^{3/2}} \left\{ (C_0(\hat{\theta}) - \frac{1}{2} t(\hat{\theta})) I_{1,1}^{-2}(\hat{\theta}) (K_{2,2}(\hat{\theta}) - I_{1,1}^2(\hat{\theta})) \right. \\ & \quad \left. + C_2(\hat{\theta}, t(\hat{\theta})) \right\} \end{aligned}$$

は次式を満たす:

すべての $x_1, x_2 \geq 0$ に対して,

$$P_{\theta_0}(-x_1 \leq n^{1/2}(\bar{T} - \theta) \leq x_2)$$

$$\leq P_{\theta_0}(-x_1 \leq n^{1/2}(\bar{\theta} - \theta) \leq x_2) + o(n^{-1})$$

ただし

$$Z_2(\theta) = \frac{1}{n^{1/2}} \sum_{i=1}^n \left(\frac{\partial^2}{\partial \theta^2} \log f(x_i, \theta) + I_{1,1}(\theta) \right)$$

$$K_{2,2}(\theta) = E_{\theta} \left(\frac{\partial^2}{\partial \theta^2} \log f(x, \theta) \cdot \frac{\partial^2}{\partial \theta^2} \log f(x, \theta) \right),$$

定理の証明

$$W_n(\theta, t) = \sum_{i=1}^n \log f(x_i, \theta + \frac{t}{n^{1/2}}) - \sum \log f(x_i, \theta) - b(\theta, t, \frac{1}{n^{1/2}})$$

とし, $W_n(\theta, t) = 0$ の解を $\theta^*(t)$ とすれば $C_0(\theta, t)$, $C_1(\theta, t)$, $C_2(\theta, t)$ が存在して, 確率 $1 - o_p(n^{-1})$ で,

$$\begin{aligned} \theta^*(t) = & \theta + \frac{1}{n^{1/2}} \{ I_{1,1}^{-1}(\theta) Z_1(\theta) + C_0(\theta, t) - t \} \\ & + \frac{1}{n} \{ I_{1,1}^{-2}(\theta) Z_1(\theta) Z_2(\theta) + \frac{1}{2} I_{1,1}^{-3}(\theta) J_3(\theta) Z_1^2(\theta) \\ & + (C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-1}(\theta) (Z_2(\theta) - I_{1,1}^{-1}(\theta) J_{2,1}(\theta) Z_1(\theta)) \\ & + C_0'(\theta, t) I_{1,1}^{-1}(\theta) Z_1(\theta) + C_1(\theta, t) \} \\ & + \frac{1}{n^{3/2}} \{ I_{1,1}^{-3}(\theta) Z_1(\theta) Z_2^2(\theta) + \frac{3}{2} I_{1,1}^{-4}(\theta) J_3(\theta) Z_1^2(\theta) Z_2(\theta) \\ & + \frac{1}{2} I_{1,1}^{-3}(\theta) Z_1^2(\theta) Z_3(\theta) + \{ \frac{1}{2} I_{1,1}^{-5}(\theta) J_3^2(\theta) + \frac{1}{6} I_{1,1}^{-4}(\theta) K_4(\theta) \} Z_1^3(\theta) \\ & + \{ -(C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-3}(\theta) (5J_{2,1}(\theta) + J_{1,1,1,1}(\theta)) \\ & + 2C_0'(\theta, t) I_{1,1}^{-2}(\theta) \} Z_1(\theta) Z_2(\theta) + (C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-2}(\theta) Z_1(\theta) Z_3(\theta) \\ & + \{ \frac{1}{2} (C_0(\theta, t) - \frac{t}{2}) I_{1,1}^{-4}(\theta) J_{2,1}(\theta) (-J_3(\theta) + 6J_{2,1}(\theta) + 2J_{1,1,1,1}(\theta)) \} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}(C_0(\theta, t) - \frac{t}{2})I_{1,1}^{-3}(\theta)(2K_{3,1}(\theta) + K_{2,2}(\theta) + K_{2,1,1}(\theta)) \\
& + \frac{1}{2}C_0'(\theta, t)I_{1,1}^{-3}(\theta)(J_3(\theta) - 2J_{2,1}(\theta)) \\
& + \frac{1}{2}C_0''(\theta, t)I_{1,1}^{-2}(\theta)\{Z_1^2(\theta) + (C_0(\theta, t) - \frac{t}{2})I_{1,1}^{-2}(\theta)Z_2^2(\theta) \\
& + \{(C_0(\theta, t) - \frac{t}{2})C_0'(\theta, t) - (C_0(\theta, t) - \frac{t}{2})^2I_{1,1}^{-1}(\theta)J_{2,1}(\theta) + C_1(\theta, t)\} \\
& \times I_{1,1}^{-1}(\theta)(Z_2(\theta) - I_{1,1}^{-1}(\theta)J_{2,1}(\theta)Z_1(\theta)) + C_1'(\theta, t)I_{1,1}^{-1}(\theta)Z_1(\theta) \\
& + \frac{1}{6}I_{1,1}^{-1}(\theta)\{3t(C_0(\theta, t) - t) + 3(C_0(\theta, t) - t)^2 + t^2\} \\
& \times (Z_3(\theta) - I_{1,1}^{-1}(\theta)K_{3,1}(\theta)Z_1(\theta)) + C_2(\theta, t)\}.
\end{aligned}$$

m. l. c. の言葉で言い替へれば,

$$\begin{aligned}
\theta^*(t) = & \hat{\theta} + \frac{1}{n^{1/2}}(C_0(\hat{\theta}, t) - t) \\
& + \frac{1}{n}\left\{(C_0(\hat{\theta}, t) - \frac{t}{2})I_{1,1}^{-1}(\hat{\theta})Z_2(\hat{\theta}) + C_1(\hat{\theta}, t)\right\} \\
& + \frac{1}{n^{3/2}}\left[(C_0(\hat{\theta}, t) - \frac{t}{2})I_{1,1}^{-2}(\hat{\theta})Z_2^2(\hat{\theta}) \right. \\
& \quad \left. + I_{1,1}^{-1}(\hat{\theta})\left\{-(C_0(\hat{\theta}, t) - \frac{t}{2})^2I_{1,1}^{-1}(\hat{\theta})J_{2,1}(\hat{\theta}) \right. \right. \\
& \quad \left. \left. + (C_0(\hat{\theta}, t) - \frac{t}{2})C_0'(\hat{\theta}, t) \right. \right. \\
& \quad \left. \left. + C_1(\hat{\theta}, t)\right\}Z_2(\hat{\theta}) \right. \\
& \quad \left. + \frac{1}{6}\{3t(C_0(\hat{\theta}, t) - t) + 3(C_0(\hat{\theta}, t) - t)^2 + t^2\} \right. \\
& \quad \left. \times I_{1,1}^{-1}(\hat{\theta})Z_3(\hat{\theta}) + C_2(\hat{\theta}, t)\right] + o_p(n^{-1}).
\end{aligned}$$

ところで

$$\begin{aligned}
\theta^{**}(t) = & \hat{\theta} + \frac{1}{n^{1/2}}(C_0(\hat{\theta}, t) - t) \\
& + \frac{1}{n}\left\{(C_0(\hat{\theta}, t) - \frac{t}{2})I_{1,1}^{-1}(\hat{\theta})Z_2(\hat{\theta}) + C_1(\hat{\theta}, t)\right\} \\
& + \frac{1}{n^{3/2}}\left\{(C_0(\hat{\theta}, t) - \frac{t}{2})I_{1,1}^{-2}(\hat{\theta})(K_{2,2}(\hat{\theta}) - I_{1,1}^2(\hat{\theta})) \right. \\
& \quad \left. + C_2(\hat{\theta}, t)\right\}
\end{aligned}$$

なる統計量を導入すれば,

$$\theta^*(t) = \theta^{**}(t) + o_p(n^{-1}) \quad (\text{in law})$$

であることが分る。

今推定量 T が, すべての $\theta \in \Theta$ に対して,

$$P_\theta(T \leq \theta) = f(\theta, \frac{1}{n^{1/2}}) = 1 - P_\theta(T \geq \theta)$$

を満にすしよう。このとき, $\theta^{**}(t)$ が, 任意の $\theta \in \Theta$ に対し,

$$P_{\theta + t/n^{1/2}}(\theta^{**}(t) \leq \theta) \leq f(\theta + \frac{1}{n^{1/2}}, \frac{1}{n^{1/2}}) + o(n^{-1})$$

を満にすならば, 不等式

$$P_\theta(T \leq \theta + \frac{t}{n^{1/2}}) \leq P_\theta(\theta^{**}(t) \leq \theta) + o(n^{-1}) \quad (t > 0)$$

$$P_\theta(T \geq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \geq P_\theta(\theta^{**}(t) \geq \theta) \quad (t < 0)$$

が成り立つ。

従って $\tilde{\theta}(t) = \theta^{**}(t) + \frac{t}{n^{1/2}}$ と置けば, 任意の $\theta \in \Theta$ に対して,

$$P_\theta(T \leq \theta + \frac{t}{n^{1/2}}) \leq P_\theta(\tilde{\theta}(t) \leq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \quad (t > 0)$$

$$P_\theta(T \geq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \geq P_\theta(\tilde{\theta}(t) \geq \theta + \frac{t}{n^{1/2}}) \quad (t < 0)$$

となる。

$T \in \mathcal{F}_1$ に対して, $\theta \in \Theta_0$ における $n^{\frac{1}{2}}(T - \theta)$ の i -th cumulant

$K_i(\theta)$ を

$$K_1(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{3/2}} K_{11}(\theta) + \frac{1}{n} K_{12}(\theta) + o(n^{-1}),$$

$$K_2(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{5/2}} K_{21}(\theta) + \frac{1}{n^{3/2}} K_{22}(\theta) + \frac{1}{n} K_{23}(\theta) + o(n^{-1}),$$

$$K_3(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{7/2}} K_{31}(\theta) + \frac{1}{n^{5/2}} K_{32}(\theta) + \frac{1}{n^{3/2}} K_{33}(\theta) + o(n^{-1}),$$

$$K_4(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{9/2}} K_{41}(\theta) + \frac{1}{n^{7/2}} K_{42}(\theta) + \frac{1}{n^{5/2}} K_{43}(\theta) + \frac{1}{n^{3/2}} K_{44}(\theta) + o(n^{-1}),$$

$$K_i(\theta) = o(n^{-1}) \quad (i \geq 5)$$

て表わそう。 $T \in \mathcal{F}$ であるから、 T に対して定まる $\hat{\theta}(t)$ において

$C_0(\theta, t)$, $C_1(\theta, t)$ は t と無関係であるから、以後 $C_0(\theta, t)$, $C_1(\theta, t)$ に対して、 $C_0(\theta)$, $C_1(\theta)$ を用いる。

ところで推定量 $T \in \mathcal{F}$ が second order admissible であるならば

$$K_{10}(\theta) = C_0(\theta), \quad K_{20}(\theta) = I_{1,1}^{-1}(\theta),$$

$$K_{11}(\theta) = C_1(\theta) - \frac{1}{2} I_{1,1}^{-2}(\theta) (J_{2,1}(\theta) + J_{1,1,1}(\theta)),$$

$$K_{21}(\theta) = 2 I_{1,1}^{-1}(\theta) K_{10}'(\theta),$$

$$K_{31}(\theta) = -I_{1,1}^{-3}(\theta) (3J_{2,1}(\theta) + 2J_{1,1,1}(\theta)),$$

ただし $K_{10}'(\theta) = \frac{d}{d\theta} K_{10}(\theta)$ かつすべての $t \neq 0$ に対して、

$$\begin{aligned} & I_{1,1}^{5/2}(\theta) \{ -A_{31}(\theta) + (K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} t^2 \\ & + I_{1,1}^{5/2}(\theta) [4(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) \\ & + \{ 4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} C_0(\theta)] t \\ & + I_{1,1}^{3/2}(\theta) [12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \\ & - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) I_{1,1}(\theta) C_0(\theta) \\ & + \{ -4A_{31}(\theta) + 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} I_{1,1}(\theta) C_0^2(\theta)] \leq 0 \end{aligned}$$

ただし

$$\begin{aligned} A_{20}(C_0'(\theta), C_1'(\theta), \theta) &= -I_{1,1}^{-3}(\theta) (K_{3,1}(\theta) + 4K_{2,1,1}(\theta) + K_{1,1,1,1}(\theta) + I_{1,1}^{-2}(\theta)) \\ &+ \frac{1}{2} I_{1,1}^{-4}(\theta) (7J_{2,1}^2(\theta) + 14J_{2,1}(\theta) J_{1,1,1}(\theta) + 5J_{1,1,1}^2(\theta) \\ &+ (C_0'(\theta))^2 I_{1,1}^{-1}(\theta) + 2C_1'(\theta) I_{1,1}^{-1}(\theta)), \\ A_{30}(C_0'(\theta), C_0''(\theta), \theta) &= -3C_0'(\theta) I_{1,1}^{-3}(\theta) (3J_{2,1}(\theta) + 2J_{1,1,1}(\theta)) \\ &+ 3C_0''(\theta) I_{1,1}^{-2}(\theta), \\ A_{31}(\theta) &= 3I_{1,1}^{-4}(\theta) \{ I_{1,1}(\theta) (K_{2,2}(\theta) - I_{1,1}^2(\theta)) - J_{2,1}^2(\theta) \}. \end{aligned}$$

$$\begin{aligned} K_{42}^*(\theta) &= -I_{1,1}^{-4}(\theta) (4K_{3,1}(\theta) + 12K_{2,1,1}(\theta) + 3K_{1,1,1,1}(\theta) + 3I_{1,1}^{-2}(\theta)) \\ &+ 12I_{1,1}^{-5}(\theta) (2J_{2,1}(\theta) + J_{1,1,1}(\theta)) (J_{2,1}(\theta) + J_{1,1,1}(\theta)) \end{aligned}$$

上の不等式から

$$K_{42}^*(\theta) - K_{42}(\theta) \leq 0.$$

$$\begin{aligned} & \frac{1}{4} I_{1,1}(\theta) \{ A_{31}(\theta) - (K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} A^2(\theta) \\ & + 12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \\ & \quad - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) I_{1,1}(\theta) C_0(\theta) \\ & \quad + \{ -4A_{31}(\theta) + 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} I_{1,1}(\theta) C_0^2(\theta) \leq 0, \end{aligned}$$

γに代し

$$A(\theta)$$

$$= \frac{4(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) + (4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta)) C_0(\theta)}{-A_{31}(\theta) + (K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta)}$$

$t(\theta) = -\frac{1}{2} A(\theta)$ とおき, $\tilde{\theta}(t(\hat{\theta}))$ とする。このとき,

$$\tilde{\theta}(t(\hat{\theta})) = \tilde{\theta}(t(\theta)) + o_p(n^{-1}) \quad (\text{in law})$$

であって,

$$\begin{aligned} & P_{\theta}(n^{1/2}(T - \theta) \leq x) - P_{\theta}(n^{1/2}(\tilde{\theta}(t(\hat{\theta})) - \theta) \leq x) \\ & = \frac{1}{24n} \phi(I_{1,1}^{1/2}(\theta)(x - C_0(\theta))) I_{1,1}^{5/2}(\theta) x \\ & \quad \times [(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta)(x + \frac{1}{2}A(\theta))^2 \\ & \quad - \frac{1}{4} \{ -A_{31}(\theta) + (K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} A^2(\theta) \\ & \quad + 12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) I_{1,1}^{-1}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) \\ & \quad - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) C_0(\theta) \\ & \quad - \{ 4A_{31}(\theta) - 3(K_{42}^*(\theta) - K_{42}(\theta)) I_{1,1}(\theta) \} C_0^2(\theta)] + o(n^{-1}) \end{aligned}$$

従って,

$$P_{\theta}(n^{1/2}(T - \theta) \leq x) \leq P_{\theta}(n^{1/2}(\tilde{\theta}(t(\hat{\theta})) - \theta) \leq x) + o(n^{-1}) \quad (x \geq 0)$$

$$P_{\theta}(n^{1/2}(T - \theta) < x) + o(n^{-1}) \geq P_{\theta}(n^{1/2}(\tilde{\theta}(t(\hat{\theta})) - \theta) < x) \quad (x \leq 0)$$

ゆえにすべての $x_1, x_2 \geq 0$ に対して,

$$P_{\theta}(-x_1 \leq n^{1/2}(T - \theta) \leq x_2) \leq P_{\theta}(-x_1 \leq n^{1/2}(\tilde{\theta}(t(\hat{\theta})) - \theta) \leq x_2) + o(n^{-1})$$

が成り立つ。