

The mean values of the Dirichlet L -functions

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1 Introduction

First, we recall the definition of the Dirichlet L -function.

Let χ be a Dirichlet character modulo $k \geq 2$ and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the Dirichlet L -function for $\Re s > 1$.

In the present paper, we consider the following mean values of the Dirichlet L -functions for a positive integer r :

$$S_r(m_1, \dots, m_{r+1}) = \left(\frac{2}{\phi(k)} \right)^r \sum_{\chi_1(-1)=(-1)^{m_1}} \cdots \sum_{\chi_r(-1)=(-1)^{m_r}} \left(\prod_{i=1}^r L(m_i, \chi_i) \right) L(m_{r+1}, \overline{\chi_1 \cdots \chi_r}),$$

where $m_1, \dots, m_{r+1} \in \mathbb{N}$ with $m_1 + \cdots + m_{r+1} \equiv 0 \pmod{2}$ and $\chi_1, \dots, \chi_r$ are the Dirichlet character modulo k . The aim of such studies is to express $S_r(m_1, \dots, m_{r+1})$ in terms of the values of the Riemann zeta-function $\zeta(\ell)$ and Jordan's totient functions

$$J_\ell(k) = k^\ell \prod_{p|k} (1 - p^{-\ell}),$$

where ℓ is a positive integer.

In the case $r = 1$, $m_1 = m_2 = 1$ and $k = p$ is a prime, Walum [8] studied $S_1(1, 1)$, and Alkan [1], Qi [7], Louboutin [4] and Zhang [9] independently gave the following explicit formula

$$S_1(1, 1) = \frac{\pi^2}{6k^2} (J_2(k) - 3\phi(k)),$$

in the case $k \geq 2$. Also Louboutin [5] studied $S_1(m, m)$ for a positive integer m and $k > 2$, and Liu and Zhang [3] studied $S_1(m, n)$ for positive integers $m, n \geq 1$ with the same parity (see [3] Theorem 1.1).

Moreover Alkan [2] considered the case $r = 2$, and he gave the explicit formula

$$S_2(1, 1, 2) = \pi^4 \left(\frac{J_4(k)}{90k^4} - \frac{J_2(k)}{18k^4} \right)$$

for $k > 2$.

2 Main theorem

Now we give the main theorem, but first we have to prepare the following some notations which use in the main theorem:

Let $(b_\alpha)_{\alpha=1}^m = (b_1, b_2, \dots, b_m)$ be an m -th row vector and $(c_{ij})_{1 \leq i, j \leq m}$ be an m -th square matrix. Then, for an even positive integer n , we put the $\frac{n}{2}$ -th column vector

$$A_n = \left(\frac{(-1)^i i a_{i,j}}{2^{2i-1} B_{2i}} \right)_{1 \leq i, j \leq \frac{n}{2}}^{-1} \begin{pmatrix} J_2(k) \\ J_4(k) \\ \vdots \\ J_n(k) \end{pmatrix},$$

where B_m is the m -th Bernoulli number and

$$a_{i,j} = \begin{cases} (-1)^i 4^{i-1} & \text{if } j = 1, \\ 2(j-1)(2j-1)a_{i-1,j-1} - 4j^2 a_{i-1,j} & \text{if } 1 < j < i, \\ -(2i-1)! & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases}$$

Theorem 1. Let $k > 2$ be a positive integer and $m_1, \dots, m_{r+1} \in \mathbb{N}$ with $m_1 + \dots + m_{r+1} \equiv 0 \pmod{2}$.

In the case r is odd and $m_1 = m_2 = \dots = m_{r+1} = 1$, we have

$$S_r(1, \dots, 1) = \left(\frac{\pi}{2k} \right)^{r+1} \left\{ \left(\left(\frac{r+1}{2} - \alpha \right) (-1)^{\frac{r+1}{2} - \alpha} \right)_{\alpha=1}^{\frac{r+1}{2}} A_{r+1} + (-1)^{\frac{r+1}{2}} \phi(k) \right\}$$

and the otherwise,

$$S_r(m_1, \dots, m_{r+1}) = \frac{(-1)^{r+1} \pi^{m_1 + \dots + m_{r+1}}}{2^{r+1} k^{m_1 + \dots + m_{r+1}} \{\prod_{i=1}^{r+1} (m_i - 1)!\}} \\ \times \left(\sum_{n=0}^{\lfloor \frac{t}{2} \rfloor} (-1)^{\lfloor \frac{t}{2} \rfloor - n} \binom{\lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor - n} \sum_{\substack{j_1 + \dots + j_{r+1} = \alpha - n \\ 1 \leq j_1 \leq \lfloor \frac{m_1}{2} \rfloor, \dots, 1 \leq j_{r+1} \leq \lfloor \frac{m_{r+1}}{2} \rfloor}} \left\{ \prod_{v=1}^{r+1} f(m_v, j_v) \right\} \right)_{\alpha=1}^{\frac{m_1 + \dots + m_{r+1}}{2}} \\ A_{m_1 + \dots + m_{r+1}},$$

where for $m_v \geq 2$, $f(m_v, j_v) = a_{\lfloor \frac{m_v}{2} \rfloor, j_v} \rho(m_v, j_v)$,

$$\rho(m_v; j_v) = \begin{cases} -2j_v & \text{if } m_v \in 2\mathbb{N} + 1, \\ 1 & \text{otherwise.} \end{cases}$$

and we put

$$t = \# \{m_v \mid m_v \text{ is odd for } 1 \leq v \leq r+1\}.$$

Also, in the case $m_v = 1$, the sum and product corresponding v does not appear.

We note that this result is a generalization of results of [6].

3 Sketch of proof

The proof of Theorem 1 is similar to the proof of Theorem 1.1 and 1.2 of [6]. Therefore we give only the sketch of the proof of Theorem 1 in this paper.

The key of the proof is to use the following result of Louboutin ([5] Proposition 3 (1)):

Let $n \geq 1$ and $k > 2$ be positive integers. Let $\cot^{(n)} x$ denote the n -th derivative of $\cot x$. If χ is a Dirichlet character modulo k and if $\chi(-1) = (-1)^n$ then we have

$$L(n, \chi) = \frac{(-1)^{n-1} \pi^n}{2k^n (n-1)!} \sum_{j=1}^{k-1} \chi(j) \cot^{(n-1)}(\pi j/k). \quad (1)$$

Then, by (1) and

$$\sum_{\chi(-1)=(-1)^n} \chi(j_1) \bar{\chi}(j_2) = \begin{cases} \frac{\phi(k)}{2} & \text{if } j_1 \equiv j_2 \pmod{k}, (j_1, k) = 1, \\ (-1)^n \frac{\phi(k)}{2} & \text{if } j_1 \equiv -j_2 \pmod{k}, (j_1, k) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$S_r(m_1, \dots, m_{r+1}) = \frac{(-1)^{r+1} \pi^{m_1 + \dots + m_{r+1}}}{2k^{m_1 + \dots + m_{r+1}} \left(\prod_{i=1}^{r+1} (m_i - 1)! \right)} \\ \times \sum_{\substack{l=1 \\ (l, k)=1}}^{k-1} \cot^{(m_1-1)}(\pi l/k) \cot^{(m_2-1)}(\pi l/k) \dots \cot^{(m_{r+1}-1)}(\pi l/k).$$

Now, using the following formula:

For any positive integer n , we have

$$\cot^{(2n-1)} x = \sum_{j=1}^n a_{n,j} \sin^{-2j} x, \quad (2)$$

where $a_{i,j}$ is defined in Section 2, we can express $S_r(m_1, \dots, m_{r+1})$ to the terms of $\sum_{j=1}^n a_{n,j} \sin^{-2j}(\pi l/k)$ for $1 \leq j \leq (m_1 + \dots + m_{r+1})/2$.

On the other hand, applying (2) to (1), we have

$$L(2n, \chi_0) = \frac{-\pi^{2n}}{2k^{2n}(2n-1)!} \sum_{j=1}^n a_{n,j} \sum_{\substack{l=1 \\ (l, k)=1}}^{k-1} \sin^{-2j}\left(\frac{\pi l}{k}\right), \quad (3)$$

where χ_0 is a principal Dirichlet character modulo k . Also, by the Euler product expansion, we have

$$L(2n, \chi_0) = \zeta(2n) k^{-2n} J_{2n}(k). \quad (4)$$

Then, by (3) and (4), we can express $\sum_{j=1}^n a_{n,j} \sin^{-2j}(\pi l/k)$ to the terms of J_{2j} for $1 \leq j \leq (m_1 + \dots + m_{r+1})/2$. Therefore we can express $S_r(m_1, \dots, m_{r+1})$ to the terms of J_{2j} for $1 \leq j \leq (m_1 + \dots + m_{r+1})/2$ and we can show Theorem 1.

4 Examples

Lastly we give the following several evaluation formulas for $S_r(m_1, \dots, m_{r+1})$ in the case $r = 2$:

$$S_2(1, 1, 4) = \frac{\pi^6}{12k^6} \left(\frac{4}{315} J_6(k) - \frac{4}{45} J_4(k) + \frac{8}{45} J_2(k) \right),$$

$$S_2(1, 1, 6) = \frac{\pi^8}{240k^8} \left(\frac{8}{315} J_8(k) - \frac{32}{189} J_6(k) + \frac{8}{45} J_4(k) + \frac{32}{63} J_2(k) \right).$$

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