

A tensor product of certain two simple modules for finite Chevalley groups

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1 Introduction

Let k be an algebraically closed field of prime characteristic $p > 0$, and let G be a simply connected and simple algebraic k -group which is defined and split over the finite field $\mathbb{F}_p$ (for example, $G = \mathrm{SL}_{l+1}(k)$ for type A_l and $G = \mathrm{Sp}_{2l}(k)$ for type C_l).

The representation theory of G plays an important role to study that of the corresponding finite Chevalley groups $G(p^r)$. Indeed, a simple $kG(p^r)$ -module can be obtained by restricting a simple rational G -module, and a projective indecomposable $kG(p^r)$ -module also can be obtained by restricting a certain rational G -module.

In this article, we give some formulas on a direct sum decomposition of a tensor product of the r -th Steinberg module and a 'small' simple $kG(p^r)$ -module, using the representation theory of G .

2 Preliminaries

We shall use the following standard notation:

- (1) T : maximal split torus of G
- (2) $X = X(T) := \mathrm{Hom}(T, k^\times)$: character group
- (3) Φ : root system relative to the pair (G, T)
- (4) $\Delta := \{\alpha_1, \dots, \alpha_l\}$: set of simple roots for the ordering as in [2]
- (5) Φ^+ : set of positive roots containing Δ
- (6) α_0 : highest short root in Φ^+
- (7) s_α : reflection for $\alpha \in \Phi^+$ in $\mathbb{E} := X \otimes_{\mathbb{Z}} \mathbb{R}$
- (8) $W := N_G(T)/T = \langle s_\alpha | \alpha \in \Delta \rangle$: Weyl group
- (9) w_0 : longest element of W
- (10) $\pi : G \rightarrow G$: graph automorphism (which induces $\pi : X \rightarrow X$)
- (11) $F : G \rightarrow G$: standard Frobenius map relative to $\mathbb{F}_p$ (if $G = \mathrm{SL}_{l+1}(k)$, then $F : (a_{ij}) \mapsto (a_{ij}^p)$)
- (12) $G(p^r) := \{g \in G \mid F^r(\pi(g)) = g\}$: finite Chevalley group (for example, if $G = \mathrm{SL}_{l+1}(k)$, then $G(p^r) = \mathrm{SL}_{l+1}(\mathbb{F}_{p^r})$ if $\pi = id$ and $G(p^r) = \mathrm{SU}_{l+1}(\mathbb{F}_{p^{2r}})$ if $\pi \neq id$)
- (13) $\langle \cdot, \cdot \rangle$: W -invariant inner product on $\mathbb{E} = X \otimes_{\mathbb{Z}} \mathbb{R}$
- (14) $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$: coroot of $\alpha \in \Phi$
- (15) ω_i : fundamental weight with $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ (then $X = \sum_{i=1}^l \mathbb{Z}\omega_i$)

- (16) $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^l \omega_i$
 (17) $X^+ := \sum_{i=1}^l \mathbb{Z}_{\geq 0} \omega_i$: set of dominant weights

In this article, all modules are assumed to be finite-dimensional and even rational for an algebraic group. For a T -module V , set $V_\lambda := \{v \in V \mid tv = \lambda(t)v, \forall t \in T\}$. If $V_\lambda \neq 0$, this space is called the *weight space* in V of *weight* $\lambda \in X$. Then there is a direct sum decomposition $V = \bigoplus_{\lambda \in X} V_\lambda$. Let $V^{[i]}$ be the i -th Frobenius twist for a G -module V . Let $L(\lambda)$ be the simple G -module with highest weight $\lambda \in X^+$. Then $\{L(\lambda) \mid \lambda \in X^+\}$ is a set of all non-isomorphic simple G -modules, where $L(0) \cong k$ is trivial and $L((p^n - 1)\rho) = \text{St}_n$ is called the n -th *Steinberg module*. For $n \in \mathbb{Z}_{>0}$, set $X_n := \{\sum_{i=1}^l c_i \omega_i \in X^+ \mid c_i < p^n, \forall i\}$, whose elements are called p^n -restricted weights. The symbol $\otimes$ denotes a tensor product over k .

Theorem 1 (Steinberg). *Consider $\lambda \in X^+$ and its p -adic expansion $\lambda = \sum_{i=0}^{n-1} p^i \lambda_i$ ($\lambda_i \in X_1$). Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_{n-1})^{[n-1]}$$

as G -modules.

It is well-known that $\{L(\lambda) \mid \lambda \in X_r\}$ is a set of all non-isomorphic simple $kG(p^r)$ -modules, where $L((p^r - 1)\rho) = \text{St}_r$ is the unique simple projective $kG(p^r)$ -module. Let $U_r(\lambda)$ be the projective cover of the simple $kG(p^r)$ -module $L(\lambda)$ ($\lambda \in X_r$).

3 Minuscale weights and (p, a, r) -minuscale weights

Definition 1. $\lambda \in X^+$ is *minuscale* if $\langle \lambda, \alpha_0^\vee \rangle = 1$.

Remark. A minuscale weight is one of the following:

- $\omega_1, \dots, \omega_l$ in type A_l ($l \geq 1$),
- ω_l in type B_l ($l \geq 2$),
- ω_1 in type C_l ($l \geq 3$),
- $\omega_1, \omega_{l-1}, \omega_l$ in type D_l ($l \geq 4$),
- ω_1, ω_6 in type E_6 ,
- ω_7 in type E_7 .

Definition 2. Consider $\lambda \in X_r$ and its p -adic expansion $\lambda = \sum_{i=0}^{r-1} p^i \lambda_i$ ($\lambda_i \in X_1$). Suppose that $a \in \mathbb{Z}_{\geq 0}$ satisfies $a \leq p$. Then

- (i) λ is (p, a, r) -*minuscale* if $\langle \lambda_i, \alpha_0^\vee \rangle \leq a$ for each i .

(ii) λ is (p, r) -minuscule if it is (p, p, r) -minuscule.

Remark. If λ is (p, a, r) -minuscule, then clearly it is (p, r) -minuscule.

Example 1. Let $G = \mathrm{SL}_{l+1}(k)$. Then we have $\alpha_0 = \alpha_1 + \cdots + \alpha_l$ and $\langle \omega_i, \alpha_0^\vee \rangle = 1$ ($1 \leq i \leq l$), and the $(p, a, 1)$ -minuscule weights ($a \leq p$) are

$$\left\{ \sum_{i=1}^l c_i \omega_i \mid 0 \leq c_i \leq p-1, \sum_{i=1}^l c_i \leq a \right\}.$$

4 Formal characters

Let $\mathbb{Z}X$ be the $\mathbb{Z}$ -group algebra of X with basis $\{e(\lambda) \mid \lambda \in X\}$, whose multiplication is defined by $e(\lambda)e(\mu) = e(\lambda + \mu)$ for $\lambda, \mu \in X$.

Definition 3. For a T -module V , define the (*formal*) *character* of V as

$$\mathrm{ch}(V) := \sum_{\lambda \in X} (\dim_k V_\lambda) e(\lambda) \in \mathbb{Z}X.$$

Proposition 1. Let V_1 and V_2 be T -modules. Then the following holds.

- (i) $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$.
- (ii) $\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1) \cdot \mathrm{ch}(V_2)$.
- (iii) If V_1 and V_2 are G -modules, then $\mathrm{ch}(V_1) = \mathrm{ch}(V_2)$ if and only if V_1 and V_2 have the same composition factors with multiplicity.

For $\lambda \in X^+$, set $s(\lambda) := \sum_{\mu \in W\lambda} e(\mu) \in \mathbb{Z}X$.

Proposition 2. If $\lambda \in X^+$ is minuscule, then $\mathrm{ch}(L(\lambda)) = s(\lambda)$.

For $\lambda \in X^+$ and its p -adic expansion $\lambda = \sum_{i=0}^{n-1} p^i \lambda_i$, set

$$s_n(\lambda) := s(\lambda_0) s(\lambda_1)^{[1]} \cdots s(\lambda_{n-1})^{[n-1]},$$

where $s(\mu)^{[i]} = s(p^i \mu)$.

Lemma 1. For a (p, r) -minuscule weight $\mu \in X_r$, the character $\mathrm{ch}(L(\mu))$ can be written uniquely as

$$\mathrm{ch}(L(\mu)) = \sum_{\kappa \in X_r} b_\kappa s_r(\kappa) \quad (b_\kappa \in \mathbb{Z}_{\geq 0}).$$

Example 2. Let $G = \mathrm{SL}_3(k)$, $p = 3$, and

$$X = \{c_1 \omega_1 + c_2 \omega_2 \mid c_1, c_2 \in \mathbb{Z}\} = \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}\}.$$

Take $(6, 1) = (0, 1) + 3(2, 0) \in X_2$. Then

$$\text{ch}(L(0, 1)) = e(0, 1) + e(1, -1) + e(-1, 0) = s(0, 1),$$

$$\begin{aligned} \text{ch}(L(2, 0)) &= e(2, 0) + e(-2, 2) + e(0, -2) \\ &\quad + e(0, 1) + e(1, -1) + e(-1, 0) \\ &= s(2, 0) + s(0, 1), \end{aligned}$$

$$\begin{aligned} \text{ch}(L(6, 1)) &= \text{ch}(L(0, 1) \otimes L(2, 0)^{[1]}) \\ &= \text{ch}(L(0, 1)) \cdot \text{ch}(L(2, 0)^{[1]}) \\ &= s(0, 1) \cdot (s(2, 0)^{[1]} + s(0, 1)^{[1]}) \\ &= s_2(6, 1) + s_2(0, 4). \end{aligned}$$

5 Main results

The following is a result by Anwar in 2011, which gives a direct sum decomposition of a tensor product of the r -th Steinberg module St_r and a simple G -module $L(\lambda)$ for a (p, r) -minuscule weight λ .

Theorem 2 ([1, Theorem 2]). *Suppose that $\lambda \in X_r$ is (p, r) -minuscule, and let $\text{ch}(L(\lambda)) = \sum_{\kappa \in X_r} b_\kappa s_r(\kappa)$ with $b_\kappa \in \mathbb{Z}_{\geq 0}$. Then*

$$\text{St}_r \otimes L(\lambda) \cong \bigoplus_{\kappa \in X_r} b_\kappa T((p^r - 1)\rho + \kappa)$$

as G -modules, where $T(\mu)$ is the indecomposable tilting module with highest weight $\mu \in X^+$.

Now we describe the main results, which are analogous to Theorem 2 for $kG(p^r)$. These are also generalizations of Tsushima's results in [3].

Theorem 3 ([4, Theorem 3.4]). *Suppose that $\lambda \in X_r$ is $(p, p-1, r)$ -minuscule, and let $\text{ch}(L(\lambda)) = \sum_{\kappa \in X_r} b_\kappa s_r(\kappa)$ with $b_\kappa \in \mathbb{Z}_{\geq 0}$. Then*

$$\text{St}_r \otimes L(\lambda) \cong \left(\bigoplus_{\kappa \in X_r} b_\kappa U_r((p^r - 1)\rho + w_0\kappa) \right) \oplus \varepsilon \text{St}_r$$

as $kG(p^r)$ -modules, where

$$\varepsilon = \begin{cases} 1 & \text{if } \lambda = (p^r - 1)\omega \text{ for some minuscule weight } \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4 ([4, Theorem 3.3]). Suppose that $\lambda \in X_r$ is (p, r) -minuscule, and let $\text{ch}(L(\lambda)) = \sum_{\kappa \in X_r} b_\kappa s_r(\kappa)$ with $b_\kappa \in \mathbb{Z}_{\geq 0}$. Moreover, for a p -adic expansion $\lambda = \sum_{i=0}^{r-1} p^i \lambda_i$, suppose that there exists an integer $j \in \{0, 1, \dots, r-1\}$ such that $\langle \lambda_j, \alpha_0^\vee \rangle \leq p-1$ and $\lambda_j \neq (p-1)\omega$ for any minuscule weight ω . Then

$$\text{St}_r \otimes L(\lambda) \cong \bigoplus_{\kappa \in X_r} b_\kappa U_r((p^r - 1)\rho + w_0 \kappa)$$

as $kG(p^r)$ -modules.

Example 3. Consider $G(p^r) = \text{SL}_3(\mathbb{F}_{5^2})$ ($l = 2, p = 5, r = 2$). Then $\Delta = \{\alpha_1, \alpha_2\}$,

$$X = \{c_1 \omega_1 + c_2 \omega_2 \mid c_1, c_2 \in \mathbb{Z}\} = \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}\},$$

$$X^+ = \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}_{\geq 0}\}$$

and

$$X_2 = \{(c_1, c_2) \mid c_1, c_2 \in \{0, 1, \dots, 24\}\}.$$

Since

$$\begin{aligned} \text{ch}(L(3, 11)) &= \text{ch}(L(3, 1) \otimes L(0, 2)^{[1]}) \\ &= (s(3, 1) + s(1, 2) + s(2, 0) + s(0, 1)) \cdot (s(0, 2)^{[1]} + s(1, 0)^{[1]}) \\ &= s_2(3, 11) + s_2(1, 12) + s_2(2, 10) + s_2(0, 11) \\ &\quad + s_2(8, 1) + s_2(6, 2) + s_2(7, 0) + s_2(5, 1), \end{aligned}$$

$$\begin{aligned} \text{ch}(L(15, 11)) &= \text{ch}(L(0, 1) \otimes L(3, 2)^{[1]}) \\ &= s(0, 1) \cdot (s(3, 2)^{[1]} + s(1, 3)^{[1]} + s(4, 0)^{[1]} \\ &\quad + 2s(2, 1)^{[1]} + 2s(0, 2)^{[1]} + 2s(1, 0)^{[1]}) \\ &= s_2(15, 11) + s_2(5, 16) + s_2(20, 1) + 2s_2(10, 6) + 2s_2(0, 11) + 2s_2(5, 1), \end{aligned}$$

and

$$\begin{aligned} \text{ch}(L(24, 0)) &= \text{ch}(L(4, 0) \otimes L(4, 0)^{[1]}) \\ &= (s(4, 0) + s(2, 1) + s(0, 2) + s(1, 0)) \\ &\quad \cdot (s(4, 0)^{[1]} + s(2, 1)^{[1]} + s(0, 2)^{[1]} + s(1, 0)^{[1]}) \\ &= s_2(24, 0) + s_2(22, 1) + s_2(20, 2) + s_2(21, 0) + s_2(14, 5) + s_2(12, 6) \\ &\quad + s_2(10, 7) + s_2(11, 5) + s_2(4, 10) + s_2(2, 11) + s_2(0, 12) + s_2(1, 10) \\ &\quad + s_2(9, 0) + s_2(7, 1) + s_2(5, 2) + s_2(6, 0), \end{aligned}$$

we obtain

$$\begin{aligned} \text{St}_2 \otimes L(3, 11) \cong & U_2(13, 21) \oplus U_2(12, 23) \oplus U_2(14, 22) \oplus U_2(13, 24) \\ & \oplus U_2(23, 16) \oplus U_2(22, 18) \oplus U_2(24, 17) \oplus U_2(23, 19) \end{aligned}$$

and

$$\begin{aligned} \text{St}_2 \otimes L(24, 0) \cong & U_2(24, 0) \oplus U_2(23, 2) \oplus U_2(22, 4) \oplus U_2(24, 3) \oplus U_2(19, 10) \oplus U_2(18, 12) \\ & \oplus U_2(17, 14) \oplus U_2(19, 13) \oplus U_2(14, 20) \oplus U_2(13, 22) \oplus U_2(12, 24) \\ & \oplus U_2(14, 23) \oplus U_2(24, 15) \oplus U_2(23, 17) \oplus U_2(22, 19) \oplus U_2(24, 18) \oplus \text{St}_2 \end{aligned}$$

by Theorem 3, and

$$\begin{aligned} \text{St}_2 \otimes L(15, 11) \cong & U_2(13, 9) \oplus U_2(8, 19) \oplus U_2(23, 4) \oplus 2U_2(18, 14) \\ & \oplus 2U_2(13, 24) \oplus 2U_2(23, 19) \end{aligned}$$

by Theorem 4.

References

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