

Combinatorial Analysis on Soliton Cellular Automata

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Chapter 1

Introduction

In 1990, Takahashi and Satsuma introduced the *box-ball system (BBS)*, a cellular automaton with solitonic behavior [32]. It has since been studied from various perspectives, including ultradiscretization of soliton equations [23, 35], crystal basis [14], the inverse scattering method [28, 37], and the probability method [1, 2, 3]. In particular, the time evolution equation of the BBS is derived by applying a limiting procedure known as ultradiscretization to the discrete Korteweg–de Vries (KdV) equation [35], a representative example of discrete integrable systems. This process endows the BBS with properties such as an infinite number of conserved quantities and soliton solutions. In 1999, Nagai et al. revealed that the ultradiscrete Toda molecule equation provides an alternative representation of the time evolution of the BBS [23]. Furthermore, the time evolution of the BBS is described by a Mealy automaton [30].

In 2006, Kuniba et al. found the time evolution of the BBS can be linearized by associating states of the BBS with rigged configurations [14]. Consequently, the initial value problem of the BBS can be solved. This correspondence, known as the Kerov–Kirillov–Reshetikhin (KKR) bijection, was originally introduced in the study of solvable lattice models [11, 15]. The linearization method by the KKR bijection can also be applied to the extensions, such as the BBS with a box capacity larger than one, finite carrier capacity, or multicolor balls [9]. Proving the linearization property of the BBS required Kashiwara’s crystal basis theory [7, 14]. Mada et al. developed a combinatorial approach for the initial value problem based on “10-elimination” [19]. The relationship between these two approaches, rigged configurations and 10-eliminations, is discussed in [16]. In 2018, Kakei et al. proved the linearization property by an elementary approach using “arc lines” and “01/10-eliminations”. This approach can be easily extended to the BBS with finite carrier capacity [13].

In recent years, various extensions of the BBS have been proposed [8, 17, 18]; however, the relationship between some of these extensions and algebraic structures, such as the quantum groups underlying the KKR bijection, remains unexplored. It is expected that linearizing these extensions using elementary methods such as arc lines could elucidate these relationships.

In this thesis, we study the three types of generalized BBS: box capacity extension de-

scribed by the ultradiscrete KdV equation, the relativistic BBS described by the ultradiscrete relativistic Toda equation, and the soliton Mealy automata. We show the integrability of the generalized BBS by linearizing them using a combinatorial approach.

The following section briefly reviews the original box–ball system and its linearization method.

1.1 Box–ball system

In this section, let us review the original box–ball system and some descriptions of its time evolution: ultradiscrete KdV equation, ultradiscrete Toda molecule equation, and Mealy automaton.

1.1.1 Ultradiscrete Korteweg–de Vries equation

The state of the original BBS consists of a row of infinitely many boxes and finitely many balls, where each box can hold at most one ball (that is, box capacity is one) [27]. In this chapter, when we refer to ‘BBS’, we mean the original BBS with a box capacity of one. The time evolution of the state is defined as follows:

- (i) Move the leftmost ball to the nearest empty box on the right,
- (ii) Repeat step (i) until all the balls have been moved once.

By mapping the empty box to ‘0’ and the box containing a ball to ‘1’, the state of the BBS is represented by the semi-infinite sequence $\boldsymbol{\eta} = (\eta_j)_{j=0}^{\infty}$, $\eta_j \in \{0, 1\}$ as

$$\eta_j = \begin{cases} 0, & \text{if the } j\text{th box is empty,} \\ 1, & \text{if the } j\text{th box contains a ball.} \end{cases} \quad (1.1)$$

By repeating the time evolution, a series of BBS states is obtained (see Fig 1.1).

In this thesis, we suppose the number of balls is finite. Assuming that there are finitely many balls and a row of boxes continues infinitely to the left, the inverse time evolution of the BBS is uniquely defined. Specifically, the inverse time evolution can be obtained by reversing the left and right sides of the row of boxes and performing the above time evolution. On the other hand, the BBS with infinitely many balls has been studied from the viewpoint of the probability method [1, 2, 3].

The behavior of the balls in the BBS has soliton property. So here, we define solitary waves and soliton automata. A more general definition is given in Section 4.2.2. Let S be the alphabet set and $\boldsymbol{\eta} = (\eta_j)_{j=0}^{\infty} \in S^{\mathbb{Z}_{\geq 0}}$ be a state of the automaton. Let T be the time evolution on $S^{\mathbb{Z}_{\geq 0}}$. A string ν on S of a finite length is called the *solitary wave*, if there exists a pair of $(m, l) \in \mathbb{Z}_{>0}^2$ such that $T^m(\nu 0^\infty) = 0^l \nu 0^\infty$, where 0^l means the string of length l as $\underbrace{0 \cdots 0}_l$. This pair of nonnegative integers (m, l) determines the (average) speed $\text{sp}(\nu) = l/m$. An automaton

is called the *soliton automaton*, if for all initial state η , there is some $t \in \mathbb{Z}_{>0}$ such that for some $n \in \mathbb{Z}_{>0}$ and solitary waves $v_1, v_2, \dots, v_n$,

$$T^t(\eta) = 0^{k_1} v_1 0^{k_2} v_2 0^{k_3} \dots 0^{k_n} v_n 0^\infty, \quad (1.2)$$

$$\text{sp}(v_1) < \text{sp}(v_2) < \dots < \text{sp}(v_n), \quad (1.3)$$

where $k_1, k_2, \dots, k_n$ are sufficiently large. Each solitary wave v_j can be divided into smaller solitary waves whose speeds are equal to $\text{sp}(v_j)$. These solitary waves obtained through this division are called *solitons*. In the soliton automaton, solitons satisfy the following properties:

1. A soliton moves at a constant speed when not interacting, and
2. The sizes (the number of balls) and speeds of solitons are unchanged before and after the interactions.

In the original BBS, the alphabet set $S = \{0, 1\}$, and the block of l contiguous balls ('1's) forms a soliton of speed l .

```

t = 0 : 011110000110001000000000000000000000...
t = 1 : 000001111001100100000000000000000000...
t = 2 : 000000000110011011100000000000000000...
t = 3 : 000000000001100100011110000000000000...
t = 4 : 000000000000011010000001111000000000...
t = 5 : 000000000000000101100000000111100000...

```

Fig 1.1: Time evolution of the BBS. There are three solitons of sizes four, two, and one.

Because it is necessary for the discussion in Chapter 2 and Chapter 4, we introduce another definition of the time evolution of the BBS. Consider a carrier that transports balls from left to right [31]. The carrier moves according to the following rules:

- (i) The carrier starts from the leftmost site with no balls and runs to the right.
- (ii) When the carrier passes in front of the j th box,
 - If the box contains a ball, the carrier picks up the ball, and
 - If the box is empty and the carrier holds at least one ball, the carrier drops off the ball to the box.
- (iii) The carrier stops when all the balls have been transported to another box.

If we assume that the carrier loads and unloads balls in a first-in, first-out manner, the movement of balls aligns with the initial definition.

Then, the time evolution $T : \eta^t \mapsto \eta^{t+1}$ of the BBS from time t to $t + 1$ is rewritten as the following piecewise linear equation:

$$\eta_j^{t+1} = \min \left(L - \eta_j^t, \sum_{i=0}^{j-1} (\eta_i^t - \eta_i^{t+1}) \right), \quad (1.4)$$

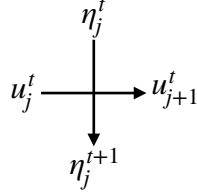
where $L = 1$. The equation (1.4) is called the *ultradiscrete Korteweg–de Vries (uKdV) equation*. The uKdV equation is obtained by ultradiscretization of the discrete Korteweg–de Vries equation [29, 35]. The N -soliton solution of (1.4) is given by Mada et al. [20]. By introducing

auxiliary variables $u_j^t = \sum_{i=0}^{j-1} (\eta_i^t - \eta_i^{t+1})$, the equation (1.4) can be rewritten as

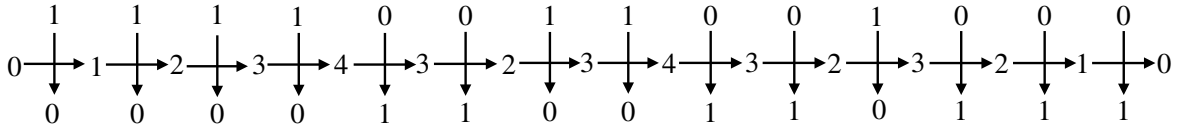
$$\eta_j^{t+1} = \min(L - \eta_j^t, u_j^t), \quad (1.5)$$

$$u_{j+1}^t = u_j^t + \eta_j^t - \eta_j^{t+1}, \quad (1.6)$$

with $u_0^t = 0$. The first term of min on the right-hand side of (1.5), $L - \eta_j^t$, represents the remaining capacity of the j th box and the second term u_j^t represents the number of balls that the carrier holds just before passing the j th box. The equation (1.6) means that the total number of balls is conserved in the time evolution as $u_j^t + \eta_j^t = \eta_j^{t+1} + u_{j+1}^t$. The following diagram shows the changes in the number of balls on the carrier and the number of balls in the box.



Example 1.1.1. The time evolution for $\eta^t = 11110011001000$. The state at $t + 1$ becomes $\eta^{t+1} = 00001100110111$.



By choosing a general natural number as L in the equation (1.5), we can consider the extended BBS with a box capacity of L , discussed in Chapter 2. It is also possible to consider the BBS with different box capacities for each box, which is derived from the ultradiscretization of the nonautonomous discrete Kadomtsev–Petviashvili (KP) equation [34].

1.1.2 Ultradiscrete Toda equation

In 1999, Nagai et al. showed that the time evolution of the BBS can also be expressed in terms of the *ultradiscrete Toda molecule (uToda) equation* [23]:

$$Q_j^{t+1} = \min(E_j^t, D_j^{t+1}), \quad (1.7)$$

$$E_j^{t+1} = E_j^t + Q_{j+1}^t - Q_j^{t+1}, \quad (1.8)$$

$$D_j^{t+1} = D_{j-1}^{t+1} + Q_j^t - Q_{j-1}^{t+1}, \quad (1.9)$$

with initial values $Q_0^{t+1} = D_0^{t+1} = 0$. Let us set the relation between the uToda variables and uKdV variables η^t . Let $N \in \mathbb{Z}_{>0}$ be the number of blocks of contiguous balls and suppose $E_N^t = +\infty$. For $i = 0, 1, \dots, 2N + 1$, we set r_i^t as follows:

$$\begin{aligned} r_0^t &= 0, \\ r_{2i-1}^t &= r_{2i-2}^t + E_{i-1}^t \\ &= \sum_{j=0}^{i-1} E_j^t + \sum_{j=1}^{i-1} Q_j^t \quad (i = 1, 2, \dots, N+1), \\ r_{2i}^t &= r_{2i-1}^t + Q_i^t \\ &= \sum_{j=0}^{i-1} E_j^t + \sum_{j=1}^i Q_j^t \quad (i = 1, 2, \dots, N). \end{aligned}$$

We set the sequence $\boldsymbol{\eta}^t = (\eta_j^t)_{j=0}^\infty$ from $(r_i^t)_{i=0}^{2N+1}$ as

$$\eta_j^t = \begin{cases} 0, & r_{2i}^t \leq j < r_{2i+1}^t \text{ for some } i = 0, 1, \dots, N, \\ 1, & r_{2i-1}^t \leq j < r_{2i}^t \text{ for some } i = 1, 2, \dots, N. \end{cases}$$

Here, E_j^t represents the number of empty boxes between the j th and $(j + 1)$ th blocks of balls and Q_j^t represents the length of the j th block of contiguous balls at time t . D_j^{t+1} in the uToda equation (1.7)–(1.9) represents the number of balls that the carrier holds just before passing the r_{2j}^t th box. Fig 1.2 shows the values of Q_j^t and E_j^t for the same example in Fig 1.1.

	E_0	Q_1	E_1	Q_2	E_2	Q_3	E_3
$t = 0$:	0	1	1	1	0	0	0
$t = 1$:	0	0	0	0	0	1	1
$t = 2$:	0	0	0	0	0	0	0
$t = 3$:	0	0	0	0	0	0	0
$t = 4$:	0	0	0	0	0	0	0
$t = 5$:	0	0	0	0	0	0	0

Fig 1.2: Q_j^t and E_j^t variables.

From the viewpoint of ultradiscretization of integrable discrete systems, various extensions of the BBS can be obtained [8, 17, 18, 39], and the relativistic BBS discussed in Chapter 3 is one of them.

1.1.3 Mealy automaton

The time evolution of the BBS is also described by a Mealy automaton with infinitely many states [30]. First, we recall the definition of Mealy automaton.

Let Q and S be a non-empty set of states and a non-empty set of letters called *alphabet*, respectively. We introduce a *transition function* φ and an *exit function* ψ :

$$\varphi : Q \times S \rightarrow Q; (q, s) \mapsto \varphi(q, s), \quad (1.10)$$

$$\psi : Q \times S \rightarrow S; (q, s) \mapsto \psi(q, s). \quad (1.11)$$

We also introduce the mappings $\varphi_s : Q \rightarrow Q; q \mapsto \varphi(q, s)$ for $s \in S$ and the mapping $\psi_q : S \rightarrow S; s \mapsto \psi(q, s)$ for $q \in Q$. An automaton $\mathcal{A}$ is defined by a quadruple (Q, S, φ, ψ) . The action of $\mathcal{A}$ on the product of the set of states and the set of words with finite length n is defined by

$$(\varphi, \psi) : Q \times S^n \rightarrow Q \times S^n; (q; s_1 s_2 \cdots s_n) \mapsto (\tilde{q}; \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_n), \quad (1.12)$$

where $q_1 = q$, $q_{j+1} = \varphi(q_j, s_j)$, $\tilde{s}_j = \psi(q_j, s_j)$ for $j = 1, 2, \dots, n$ and $\tilde{q} = q_{n+1}$. The automaton with initial state $q' \in Q$ is denoted by $\mathcal{A}_{q'} = (Q, S, \varphi, \psi; q')$ which acts on S^n as

$$\psi_{q'} : S^n \rightarrow S^n; s_1 \cdots s_n \mapsto \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_n. \quad (1.13)$$

A Mealy automaton $\mathcal{A} = (Q, S, \varphi, \psi)$ is illustrated as a directed graph, with each state q corresponding to a vertex. When $\varphi(q, s) = q'$ and $\psi(q, s) = s'$, an arrow is drawn from the vertex q to q' , with $s|s'$ labeled next to the arrow (see Fig 1.3).

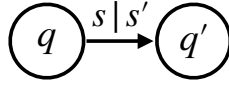


Fig 1.3: A directed graph corresponding to $\varphi(q, s) = q'$ and $\psi(q, s) = s'$.

Now, let us explain the Mealy automaton that describes the BBS. Consider the Mealy automaton shown in Fig 1.4, with the state set $Q = \{q_0, q_1, q_2, \dots\}$, the initial state q_0 and the alphabet $S = \{0, 1\}$. Suppose the input is the binary sequence η^t that denotes the state of the BBS. Then, the outputs sequence becomes $\eta^{t+1} = T(\eta^t)$. The state q_k of the Mealy automaton corresponds to the state of the BBS in which the carrier holds k balls. If the number of states of this automaton is truncated to a finite number, the BBS with a finite carrier capacity proposed in [31] is obtained, and the linearization method of the time evolution of this automaton is also discussed in [13].

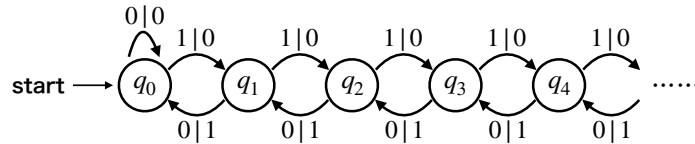
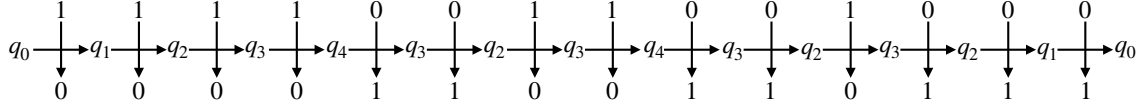


Fig 1.4: The Mealy automaton describing the BBS time evolution.

Example 1.1.2. The time evolution for $\eta' = 11110011001000$ (same as Example 1.1.1). This diagram matches the diagram in Example 1.1.1.



1.2 Linearization of the box-ball system

One of the reasons for the integrability of the BBS is that its time evolution can be linearized. Kuniba et al. showed that the time evolution of the BBS can be linearized by the *Kerov–Kirillov–Reshetikhin (KKR) bijection* [14], which was originally introduced for the analysis of solvable lattice models [11, 15]. In this section, we review the KKR bijection for the BBS [9] and show that the KKR bijection can be interpreted by using arc lines [13, 16, 19, 40].

1.2.1 Kerov–Kirillov–Reshetikhin bijection

In general, the KKR bijection can be applied to the extended BBS with a box capacity larger than one or multicolor balls [14], but here we review the case corresponding to the original BBS with a box capacity of one and only one type of ball. First, let us review the definition of rigged configuration. Consider the data of the form

$$(\mu^{(0)}, (\mu^{(1)}, J^{(1)})), \quad (1.14)$$

where $\mu^{(a)} = (\mu_1^{(a)}, \mu_2^{(a)}, \dots, \mu_{l_a}^{(a)}) \in (\mathbb{Z}_{>0})^{l_a}$ ($a = 0, 1$) and $J^{(1)} = (J_1^{(1)}, J_2^{(1)}, \dots, J_{l_1}^{(1)}) \in (\mathbb{Z}_{\geq 0})^{l_1}$. $\mu^{(1)}$ is a Young diagram (i.e. $\mu_1^{(1)} \geq \mu_2^{(1)} \geq \dots \geq \mu_{l_1}^{(1)} > 0$). $\mu_i^{(a)}$ is the length of the i th row and l_a is the number of rows in $\mu^{(a)}$. Here, we set $\mu^{(0)} = \underbrace{(1, 1, \dots, 1)}_{l_0} = (1^{l_0})$. Since the length

of each row of $\mu^{(0)}$ corresponds to the capacity of each box in considering the KKR bijection to the extended BBS, $\mu^{(0)}$ is not necessarily a Young diagram in general. Define the *vacancy number* $p_j^{(1)}$ corresponding to the row of length j in $\mu^{(1)}$ as

$$q_j^{(a)} = \sum_{i=1}^{l_a} \min(j, \mu_i^{(a)}), \quad \text{for } a = 0, 1, \quad (1.15)$$

$$p_j^{(1)} = q_j^{(0)} - 2q_j^{(1)}. \quad (1.16)$$

$q_j^{(a)}$ denotes the number of boxes up to the j th column from the left in $\mu^{(a)}$, and therefore $q_j^{(0)} = l_0$ for $j \in \mathbb{Z}_{>0}$. A pair $(\mu^{(0)}, \mu^{(1)})$ is called *configuration* if all $p_j^{(1)}$ are nonnegative. The data (1.14) is called a *rigged configuration* if the following conditions are satisfied:

- For all $j \in \mathbb{Z}_{\geq 0}$,

$$0 \leq J_{i_j+m_j-1}^{(1)} \leq \dots \leq J_{i_j+1}^{(1)} \leq J_{i_j}^{(1)} \leq p_j^{(1)}, \quad (1.17)$$

where $\{k \mid \mu_k^{(1)} = j\} = \{i_j, i_j + 1, \dots, i_j + m_j - 1\}$.

Nonnegative integers $J_i^{(1)}$ are called *riggings*, and the differences $p_j^{(1)} - J_i^{(1)} (\geq 0)$ are called *corrigings*. If the rigging associated with a row satisfies $J_i^{(1)} = p_j^{(1)}$ (that is, the corriging is equal to zero), we call that row a *singular string*. By convention, the empty row in the Young diagram is always considered singular. Here is an example of a rigged configuration in Fig 1.5.

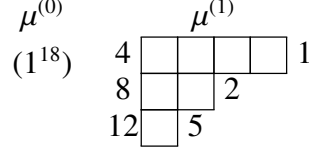


Fig 1.5: A rigged configuration.

The vacancy numbers $p_j^{(1)}$ are written on the left of the Young diagram $\mu^{(1)}$ and the riggings $J_i^{(1)}$ are written on the right. Define the *weight* of the rigged configuration as

$$w = (w_0, w_1) = (|\mu^{(0)}| - |\mu^{(1)}|, |\mu^{(1)}|), \quad (1.18)$$

where $|\mu^{(a)}|$ denotes the total number of boxes in $\mu^{(a)}$ as $|\mu^{(a)}| = \mu_1^{(a)} + \mu_2^{(a)} + \dots + \mu_{l_a}^{(a)}$. From the nonnegativity of the vacancy numbers, we see $w_0 \geq w_1$. For a partition w , let $\text{RC}(w)$ denote the set of rigged configurations with the weight w .

Next, let us define the highest path. Consider the binary sequence $\boldsymbol{\eta} = (\eta_j)_{j=0}^{l-1} \in \{0, 1\}^l$. $\boldsymbol{\eta}$ is called a *highest path* if the following conditions are satisfied:

- For all $0 \leq j \leq l-1$,

$$\#\{k \in \{0, 1, \dots, j\} \mid \eta_k = 0\} \geq \#\{k \in \{0, 1, \dots, j\} \mid \eta_k = 1\}. \quad (1.19)$$

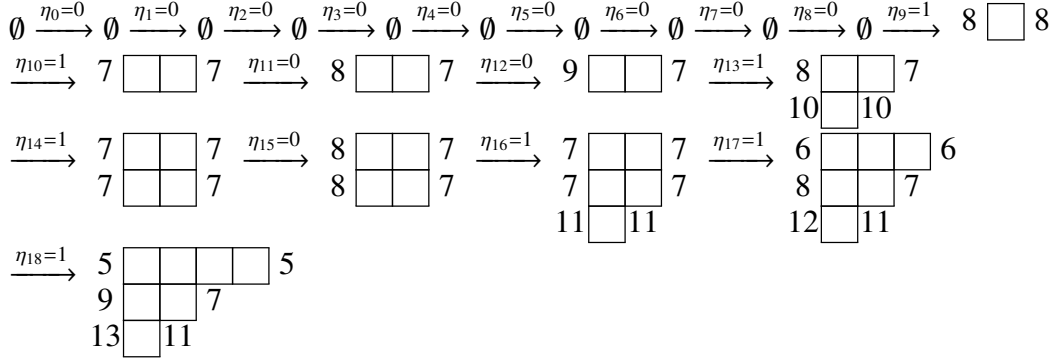
The *weight* $w = (w_0, w_1)$ of the highest path $\boldsymbol{\eta}$ is defined as

$$w_0 = \#\{k \in \{0, 1, \dots, l-1\} \mid \eta_k = 0\}, w_1 = \#\{k \in \{0, 1, \dots, l-1\} \mid \eta_k = 1\}. \quad (1.20)$$

The condition (1.19) for $j = l-1$ shows $w_0 \geq w_1$. For a partition w , let $\mathcal{P}(w)$ denote the set of highest paths with the weight w .

There is a bijection between $\text{RC}(w)$ and $\mathcal{P}(w)$. For $\boldsymbol{\eta} = (\eta_j)_{j=0}^{l-1} \in \mathcal{P}(w)$, the rigged configuration associated with $\boldsymbol{\eta}$ is defined recursively as follows. Assume the rigged configuration associated with $\hat{\boldsymbol{\eta}} = (\eta_j)_{j=0}^{l-2}$ is $(\hat{\mu}^{(0)} = (1^{l-1}), (\hat{\mu}^{(1)}, \hat{J}^{(1)}))$. If $\eta_{l-1} = 0$, $(\mu^{(1)}, J^{(1)}) = (\hat{\mu}^{(1)}, \hat{J}^{(1)})$. If $\eta_{l-1} = 1$, we add a box to the longest singular string of $(\hat{\mu}^{(1)}, \hat{J}^{(1)})$ and make that string singular by updating the rigging attaching that row to the new vacancy number. Since $(\mu^{(1)}, J^{(1)})$ does not change when $\eta_{l-1} = 0$, we can consider the correspondence between the BBS state $\boldsymbol{\eta}$ with finitely many balls and $(\mu^{(1)}, J^{(1)})$.

Example 1.2.1. For $\eta = 0000000001100110111000 \cdots$ (same as the sequence at $t = 2$ in Fig 1.1), the corresponding rigged configuration is constructed as follows:



In this case, the rigged configuration is $(\mu^{(1)}, J^{(1)}) = ((4, 2, 1), (5, 7, 11))$.

Let us recall the important proposition. The length of each row $\mu_j^{(1)}$ is the size (or speed) of each soliton and the rigging corresponding to each row $J_j^{(1)}$ represents the position of each soliton.

Proposition 1.2.2. (Proposition 2.6. in [14]) Define the time evolution T on a rigged configuration $(\tilde{\mu}^{(1)}, \tilde{J}^{(1)}) = T(\mu^{(1)}, J^{(1)})$ as

$$\tilde{\mu}^{(1)} = \mu^{(1)}, \quad (1.21)$$

$$\tilde{J}_k^{(1)} = J_k^{(1)} + \mu_k^{(1)}. \quad (1.22)$$

Then, the following commutative diagram holds.

$$\begin{array}{ccc}
\eta^t & \xleftrightarrow{\text{KKR bijection}} & (\mu^{(1)}, J^{(1)}) \\
T \downarrow & & T \downarrow \\
\eta^{t+1} & \xleftrightarrow{\text{KKR bijection}} & (\tilde{\mu}^{(1)}, \tilde{J}^{(1)})
\end{array}$$

This proposition shows that the Young diagram $\mu^{(1)}$ remains unchanged and the riggings increase by the length of the corresponding row in the time evolution of the BBS. That is, the time evolution of the BBS is linearized, allowing us to solve the initial value problem for the BBS. This is an inverse scattering method for the BBS.

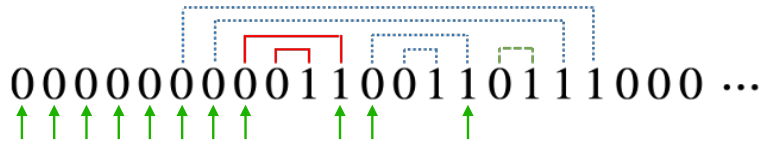
1.2.2 Arc lines for the box-ball system

The construction procedure of the KKR bijection can be interpreted using arc lines that connect balls ('1's) to empty boxes ('0's). Arc lines are typically drawn in order of increasing 'depth' (as explained later), but here, to clarify the correspondence with the KKR bijection,

we define the arc lines to be drawn sequentially from left to right. The 01-arc lines for the highest path $\boldsymbol{\eta} = (\eta_j)_{j=0}^{l-1}$ are drawn recursively. Assume the arc lines corresponding to $\hat{\boldsymbol{\eta}} = (\eta_j)_{j=0}^{l-2}$ have already been drawn. If $\eta_{l-1} = 0$, no action is taken. If $\eta_{l-1} = 1$, draw an arc line that connects the ball in the $(l-1)$ th box to the nearest empty box on the left that has not yet been connected by an arc line. When adjacent ‘0’ and ‘1’ are connected, the corresponding arc line is called an arc line of *depth 1*. Additionally, if the maximum depth of the arc lines below an arc line is i , then the depth of that arc line is defined as $(i+1)$. An arc line of depth i is considered part of the same soliton as the rightmost arc line of depth $(i-1)$ that lies below it. Let λ_j be the number of arc lines of depth j for $j \in \mathbb{Z}_{>0}$. Swapping ‘0’s and ‘1’s connected by 01-arc lines gives the reverse time evolution of the BBS. If we assume that the carrier loads and unloads balls in a last-in, first-out manner, the movement of the balls corresponds to the arc lines.

In 1996, Torii et al. proved $(\lambda_1, \lambda_2, \dots)$ are conserved quantities in the time evolution of the BBS [36]. The conjugate of the Young diagram $(\lambda_1, \lambda_2, \dots)$, the Young diagram obtained by reflecting $(\lambda_1, \lambda_2, \dots)$ along its main diagonal, is $\mu^{(1)} = (\mu_1^{(1)}, \mu_2^{(1)}, \dots)$ in the corresponding rigged configuration. Furthermore, for a soliton of size n in a finite-length path $\boldsymbol{\eta} = (\eta_j)_{j=0}^{l-1}$, the rigging (or corigging) corresponding to that soliton is equal to the number of characters (‘0’ or ‘1’) located to the left (or right) of that soliton, ignoring the characters connected by arc lines of up to depth n [13, 16].

Example 1.2.3. For $\boldsymbol{\eta} = 0000000001100110111000 \dots$ in Fig 1.6 (as in Example 1.2.1). There are three solitons of sizes two (shown by solid red lines), four (dotted blue lines), and one (dashed green lines). The rigging $J_3^{(1)} = 11$ corresponding to the soliton of length one counts the number of letters that the arrows point to.



speed one, and a soliton sequence, which evolves according to the time evolution of the BBS with box capacity one.

In Chapter 3, we consider the soliton system derived from interpreting the ultradiscrete relativistic Toda equation as a cellular automaton. We introduce arc lines based on new rules that are distinct from the conventional ones and derive invariants corresponding to the sizes of the solitons.

In Chapter 4, we show that there are only three types of soliton Mealy automata with three states and two letters, and we discuss their integrability. The first one is the well-known BBS with a carrier capacity of two. The second one follows the rule ‘Skip to the box two spaces ahead’, for which we provided a simple linearization method. The third one follows the rule ‘Jump on to the second nearest vacant box’. We show the time evolution of this system is equivalent to the ultradiscrete Lotka–Volterra equation. It is known that the ultradiscrete Lotka–Volterra equation can be transformed into the ultradiscrete KdV equation, which describes the time evolution of the original BBS, through a variable transformation.

In Chapter 5, we give conclusions.

Chapter 2

Linearization of the box–ball system with a box capacity of L for a sequence of integer values

In this chapter, we construct a bijection between the state of the box–ball system with a box capacity of L and a pair of two sequences. During the time evolution, one of the sequences moves at speed one, and the other follows the rules of the box–ball system with a box capacity of one, which can be linearized by the KKR bijection. Our method can be applied to a state with dependent variables taking a negative value or a value greater than the box capacity.

2.1 Time evolution

In this section, we consider the BBS with a box capacity of $L \in \mathbb{Z}_{>0}$ (BBS(L)). First, we introduce the BBS(L) for $L + 1$ values. The set of the BBS(L) states is denoted by $\mathcal{S}_L$:

$$\mathcal{S}_L = \left\{ \boldsymbol{\eta} = (\eta_j)_{j=0}^{\infty} \in \{0, 1, \dots, L\}^{\mathbb{Z}_{\geq 0}} \left| \sum_{j=0}^{\infty} \eta_j < \infty, \eta_0 = 0, \sum_{j=0}^i (L - \eta_j) \geq \sum_{j=0}^{i+1} \eta_j (i = 0, 1, \dots) \right. \right\}. \quad (2.1)$$

The condition $\sum_{j=0}^i (L - \eta_j) \geq \sum_{j=0}^{i+1} \eta_j$ corresponds to the condition for ‘highest path’ in the case of the BBS(1). The same rules can define the time evolution $T_L : \mathcal{S}_L \rightarrow \mathcal{S}_L; \boldsymbol{\eta}' \mapsto \boldsymbol{\eta}^{t+1} = T_L(\boldsymbol{\eta}')$ of the BBS(L) as for the BBS(1) in Chapter 1, but the carrier loads and unloads balls simultaneously [31]. A carrier that transports balls from left to right moves according to the following rules:

- (i) The carrier starts from the leftmost site with no balls, and it runs to the right.
- (ii) When the carrier passes in front of the j th box, it performs the following two operations simultaneously:

- if that box contains at least one ball, the carrier picks the ball(s) up,
- if that box is not full and the carrier has at least one ball, the carrier drops off as many balls as possible into that box.

(iii) The carrier stops when all the balls have been transported to another box.

By repeating this procedure, the time evolution series of the BBS(L) states can be obtained (see Example 2.1.1).

The time evolution T_L can be rewritten as a piecewise linear equation:

$$u_0^t = 0, \quad (2.2)$$

$$\eta_j^{t+1} = \min(L - \eta_j^t, u_j^t), \quad (2.3)$$

$$\begin{aligned} u_{j+1}^t &= \eta_j^t + u_j^t - \eta_j^{t+1} \\ &= \eta_j^t + \max(0, \eta_j^t + u_j^t - L), \end{aligned} \quad (2.4)$$

where η_j^t is the number of balls in the j th box at time t , and u_j^t is the number of balls in the carrier just before passing the j th box at time t .

Example 2.1.1. Box capacity $L = 3$. There are three solitons of sizes seven, three, and one in the following series of time evolution of the BBS(3).

```

t = 0 : 00023200021010000000000000...
t = 1 : 00000133002101000000000000...
t = 2 : 00000000331210100000000000...
t = 3 : 00000000002123210000000000...
t = 4 : 00000000000210123200000000...
t = 5 : 00000000000021010133000000...
t = 6 : 0000000000000210100033100...

```

In the argument in the next section, we allow the state value η_j^t to take a negative value or a value greater than the box capacity L . For a state of arbitrary integer values, let M be the nonnegative integer which corresponds to the maximal excess below 0 or above L , as $M = \max_{j \in \mathbb{Z}_{\geq 0}} \max(-\eta_j^t, \eta_j^t - L)$. One can find that M is invariant under the time evolution equations (2.2)–(2.4). Define the set of states as

$$\mathcal{S}_{L,M} = \left\{ \boldsymbol{\eta} = (\eta_j)_{j=0}^{\infty} \in \{-M, -M+1, \dots, L+M\}^{\mathbb{Z}_{\geq 0}} \left| \sum_{j=0}^{\infty} |\eta_j| < \infty, \eta_0 \leq 0, \right. \right. \\ \left. \left. \max_{j \in \mathbb{Z}_{\geq 0}} \max(-\eta_j^t, \eta_j^t - L) = M, \sum_{j=0}^i (L - \eta_j) \geq \sum_{j=0}^{i+1} \eta_j \ (i = 0, 1, \dots) \right\}. \quad (2.5)$$

We set variables $\tilde{\eta}_j^t = \eta_j^t + M$, $\tilde{u}_j^t = u_j^t + M$, and $\tilde{L} = L + 2M$ [4]. Then, the time evolution equations (2.2)–(2.4) become

$$\tilde{u}_0^t = M, \quad (2.6)$$

$$\tilde{\eta}_j^{t+1} = \min(\tilde{L} - \tilde{\eta}_j^t, \tilde{u}_j^t), \quad (2.7)$$

$$\begin{aligned} \tilde{u}_{j+1}^t &= \tilde{\eta}_j^t + \tilde{u}_j^t - \tilde{\eta}_j^{t+1} \\ &= \tilde{\eta}_j^t + \max(0, \tilde{\eta}_j^t + \tilde{u}_j^t - \tilde{L}). \end{aligned} \quad (2.8)$$

Since equations (2.7) and (2.8) are equivalent to equations (2.3) and (2.4) respectively, we can regard this system as the BBS($\tilde{L}$). Note that the carrier starts with M balls and stops when the carrier and every box to the right of the carrier have M balls. Let T_L^M denote the time evolution defined by equations (2.3) and (2.4) with the initial value of the carrier $u_0 = M$.

The time evolution of the BBS(L) can be expressed in terms of the BBS(1) by considering a transformation between a binary sequence and an $L + 1$ value sequence [33]. We define two binary sequences $\mathbf{r}_I$ and $\mathbf{l}_{I'}$ from a state $\boldsymbol{\eta}^t \in \mathcal{S}_{L,M}$. Let $J(\boldsymbol{\eta}^t)$ be a set of indices j , where $\eta_j^t + u_j^t \geq L$, as

$$\begin{aligned} J(\boldsymbol{\eta}^t) &= \{j \in \mathbb{Z}_{\geq 0} \mid \eta_j^t + u_j^t \geq L\} \\ &= \{j \in \mathbb{Z}_{\geq 0} \mid \tilde{\eta}_j^t + \tilde{u}_j^t \geq \tilde{L}\}. \end{aligned} \quad (2.9)$$

For a subset $I = \{i_0, i_1, \dots, i_m\} \subset J(\boldsymbol{\eta}^t)$, let $r_I(\tilde{\eta}_j^t)$ be the binary sequence of length $\tilde{L}$ as

$$r_I(\tilde{\eta}_j^t) = \begin{cases} 1^{\tilde{\eta}_j^t} 0^{\tilde{L} - \tilde{\eta}_j^t}, & j \in I, \\ 0^{\tilde{L} - \tilde{\eta}_j^t} 1^{\tilde{\eta}_j^t}, & j \notin I, \end{cases} \quad (2.10)$$

and let $\mathbf{r}_I(\boldsymbol{\eta}^t)$ be the concatenation of $r_I(\tilde{\eta}_j^t)$ as

$$\mathbf{r}_I(\boldsymbol{\eta}^t) = r_I(\tilde{\eta}_0^t) \cdot r_I(\tilde{\eta}_1^t) \cdot r_I(\tilde{\eta}_2^t) \cdots. \quad (2.11)$$

Similarly, let $J'(\boldsymbol{\eta}^t)$ be a set of indices j , where $\eta_j^t + u_{j+1}^{t-1} \geq L$, as

$$\begin{aligned} J'(\boldsymbol{\eta}^t) &= \{j \in \mathbb{Z}_{\geq 0} \mid \eta_j^t + u_{j+1}^{t-1} \geq L\} \\ &= \{j \in \mathbb{Z}_{\geq 0} \mid \tilde{\eta}_j^t + \tilde{u}_{j+1}^{t-1} \geq \tilde{L}\}. \end{aligned} \quad (2.12)$$

For a subset $I' = \{i'_0, i'_1, \dots, i'_m\} \subset J'(\boldsymbol{\eta}^t)$, let $l_{I'}(\tilde{\eta}_j^t)$ be the binary sequence of length $\tilde{L}$ as

$$l_{I'}(\tilde{\eta}_j^t) = \begin{cases} 0^{\tilde{L} - \tilde{\eta}_j^t} 1^{\tilde{\eta}_j^t}, & j \in I', \\ 1^{\tilde{\eta}_j^t} 0^{\tilde{L} - \tilde{\eta}_j^t}, & j \notin I', \end{cases} \quad (2.13)$$

and let $\mathbf{l}_{I'}(\boldsymbol{\eta}^t)$ be the concatenation of $l_{I'}(\tilde{\eta}_j^t)$ as

$$\mathbf{l}_{I'}(\boldsymbol{\eta}^t) = l_{I'}(\tilde{\eta}_0^t) \cdot l_{I'}(\tilde{\eta}_1^t) \cdot l_{I'}(\tilde{\eta}_2^t) \cdots. \quad (2.14)$$

Example 2.1.2. $L = 2$, $\eta^t = 0, 3, 0, -1, 3, 0, 2, -1, 0, 0, \dots$

$M = \max_{j \in \mathbb{Z}_{\geq 0}} \max(-\eta_j^t, \eta_j^t - L) = 1$, $\tilde{L} = 4$, and $\tilde{\eta}^t = 1, 4, 1, 0, 4, 1, 3, 0, 1, 1, \dots$. Using time evolution equations (2.2)–(2.4), we get

$$\begin{aligned} u^t &= 0, 0, 4, 2, -1, 3, 1, 3, -1, 0, \dots, \\ \eta^{t+1} &= 0, -1, 2, 2, -1, 2, 0, 3, -1, 0, \dots, \\ \tilde{u}^t &= 1, 1, 5, 3, 0, 4, 2, 4, 0, 1, \dots, \\ \tilde{\eta}^{t+1} &= 1, 0, 3, 3, 0, 3, 1, 4, 0, 1, \dots \end{aligned}$$

Then, $J(\eta^t) = \{j \in \mathbb{Z}_{\geq 0} \mid \tilde{\eta}_j^t + \tilde{u}_j^t \geq \tilde{L}\}$ becomes

$$J(\eta^t) = \{1, 2, 4, 5, 6, 7\}.$$

Choosing a subset of $J(\eta^t)$ as $I = \{2, 5\}$, we obtain

$$\begin{aligned} r_I(\eta^t) &= 000111111000000011111000011100000001 \dots, \\ T_1^1(r_I(\eta^t)) &= 100000000111111000000111100011110000 \dots, \\ l_I(T_L(\eta^t)) &= 100000000111111000000111100011110000 \dots. \end{aligned}$$

Theorem 2.1.3. For $\eta^t \in \mathcal{S}_{L,M}$ and $I = \{i_0, i_1, \dots, i_m\} \subset J(\eta^t)$,

$$T_1^M(r_I(\eta^t)) = l_I(T_L(\eta^t)). \quad (2.15)$$

Proof. From equation (2.4), we have $\eta_j^t + u_j^t = \eta_j^{t+1} + u_{j+1}^t$ and $J(\eta^t) = J'(T_L(\eta^t))$. Consider the carrier passing through $\tilde{L}$ boxes from the $j\tilde{L}$ th to the $((j+1)\tilde{L} - 1)$ th position in $r_I(\eta^t)$.

- Case 1: $j \notin I$ and $\tilde{\eta}_j^t + \tilde{u}_j^t < \tilde{L}$

The carrier drops off $\tilde{u}_j^t$ balls into the first $\tilde{u}_j^t$ empty boxes and picks up $\tilde{\eta}_j^t$ balls as shown in Fig 2.1. Then, the number of balls in the boxes in this interval becomes $\tilde{u}_j^t$, which is equal to $\tilde{\eta}_j^{t+1} = \min(\tilde{L} - \tilde{\eta}_j^t, \tilde{u}_j^t)$.

- Case 2: $j \notin I$ and $\tilde{\eta}_j^t + \tilde{u}_j^t \geq \tilde{L}$

The carrier drops off $\tilde{L} - \tilde{\eta}_j^t$ balls into $\tilde{L} - \tilde{\eta}_j^t$ empty boxes and picks up $\tilde{\eta}_j^t$ balls as shown in Fig 2.1. Then, the number of balls in the boxes in this interval becomes $\tilde{L} - \tilde{\eta}_j^t$, which is equal to $\tilde{\eta}_j^{t+1} = \min(\tilde{L} - \tilde{\eta}_j^t, \tilde{u}_j^t)$.

- Case 3: $j \in I$ (in this case $\tilde{\eta}_j^t + \tilde{u}_j^t \geq \tilde{L}$)

The carrier picks up $\tilde{\eta}_j^t$ balls. Then, the number of balls in the carrier is $\tilde{u}_j^t + \tilde{\eta}_j^t$. Since $\tilde{u}_j^t + \tilde{\eta}_j^t \geq \tilde{u}_j^t \geq \tilde{L} - \tilde{\eta}_j^t$, the carrier drops off $\tilde{L} - \tilde{\eta}_j^t$ balls as shown in Fig 2.1. The number of balls in the boxes in this interval becomes $\tilde{L} - \tilde{\eta}_j^t$, which is equal to $\tilde{\eta}_j^{t+1} = \min(\tilde{L} - \tilde{\eta}_j^t, \tilde{u}_j^t)$.

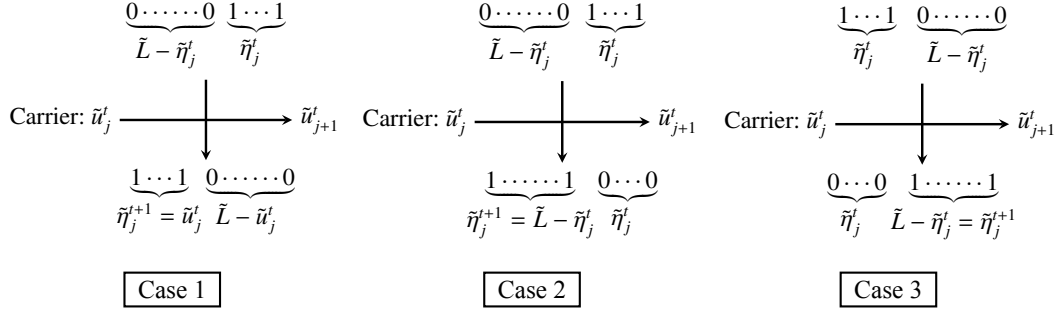


Fig 2.1: The carrier passes through $\tilde{L}$ boxes from the $j\tilde{L}$ th to the $((j+1)\tilde{L}-1)$ th position in $\mathbf{r}_I(\boldsymbol{\eta}^t)$.

□

2.2 Linearization

2.2.1 Decomposition of the box-ball system state to two sequences

Let us decompose the BBS(L) to the “trivial” dynamics, which simply moves at speed one, and the original Takahashi–Satsuma’s BBS.

We define the following sets of semi-infinite sequences:

$$\mathcal{S}_{L,M}^{(b)} = \left\{ \mathbf{c} = (c_j)_{j=0}^\infty \in \mathcal{S}_{L,M} \mid c_j + c_{j+1} \leq L \text{ for } j = 0, 1, 2, \dots \right\}, \quad (2.16)$$

$$\mathcal{S}_L^{(f)} = \left\{ \mathbf{d} = (d_j)_{j=0}^\infty \in \mathcal{S}_1 \mid \text{the length of every block of contiguous numbers that contains only 0s/1s is larger than } L \right\}. \quad (2.17)$$

First, we decompose $\boldsymbol{\eta} \in \mathcal{S}_{L,M}$ into two sequences $\mathbf{c} \in \mathcal{S}_{L,M}^{(b)}$ and $\mathbf{d} \in \mathcal{S}_L^{(f)}$ through the decomposition map $\beta_L : \mathcal{S}_{L,M} \rightarrow \mathcal{S}_{L,M}^{(b)} \oplus \mathcal{S}_L^{(f)}; \boldsymbol{\eta} \mapsto (\mathbf{c}, \mathbf{d})$ defined by the following procedure:

- (1) Let $\tilde{\boldsymbol{\eta}} = (\tilde{\eta}_0, \tilde{\eta}_1, \dots)$ be as defined above, $\tilde{\eta}_j = \eta_j + M$.
- (2) Define a *soliton flag sequence* $\mathbf{a} = (a_0, a_1, \dots) \in \mathcal{S}_1$ as

$$\begin{aligned} a_0 &= 0, \\ a_{j+1} &= \begin{cases} 0, & \tilde{\eta}_j + \tilde{\eta}_{j+1} < \tilde{L}, \\ a_j, & \tilde{\eta}_j + \tilde{\eta}_{j+1} = \tilde{L}, \\ 1, & \tilde{\eta}_j + \tilde{\eta}_{j+1} > \tilde{L}, \end{cases} \\ &= \begin{cases} 0, & \eta_j + \eta_{j+1} < L, \\ a_j, & \eta_j + \eta_{j+1} = L, \\ 1, & \eta_j + \eta_{j+1} > L. \end{cases} \quad (j = 0, 1, 2, \dots) \end{aligned} \quad (2.18)$$

We say j is in a *0-segment* if $a_j = 0$, and in a *1-segment* if $a_j = 1$.

- (3) Let $I = \{j \in \mathbb{Z}_{\geq 0} \mid a_j = 1, a_{j+1} = 0\}$, which is the set of the right ends of consecutive l -segments. We define

$$\widetilde{\text{bin}}(\boldsymbol{\eta}) = \mathbf{r}_I(\boldsymbol{\eta}), \quad (2.19)$$

where $\mathbf{r}_I$ is defined in equations (2.10) and (2.11). In Example 2.2.1 below, we underline each group of $\tilde{L}$ numbers to make it easy to see the boxes.

- (4) We write “ $\times$ ” on the subsequence corresponding to the j th and the $(j-1)$ th boxes where $j \in I$. This means that we mark the right end of every consecutive 1-segment as well as the position immediately to the left.
- (5) We iteratively draw 10-arc lines from 1s in the j th box without “ $\times$ ” and 0s in the $(j+1)$ th box. Then, the number of arc lines that connect the j th box and the $(j+1)$ th box is $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j+1})$.
- (6) Let $\tilde{\mathbf{b}} = (\tilde{b}_0, \tilde{b}_1, \dots)$, where $\tilde{b}_j$ is the number of 10-arc lines that connect 1s in the j th box and 0s in the $(j+1)$ th box. We will use this sequence in the proof of Theorem 2.2.9. If $a_{j+1} = 0$, $\tilde{\eta}_j + \tilde{\eta}_{j+1} \leq \tilde{L}$, then $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j+1}) = \tilde{\eta}_j$. If $a_{j+1} = 1$, $\tilde{\eta}_j + \tilde{\eta}_{j+1} \geq \tilde{L}$, then $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j+1}) = \tilde{L} - \tilde{\eta}_{j+1}$. Thus, we get

$$\tilde{b}_j = \begin{cases} \tilde{\eta}_j, & a_j = a_{j+1} = 0, \\ \tilde{L} - \tilde{\eta}_{j+1}, & a_{j+1} = a_{j+2} = 1, \\ 0, & (a_j, a_{j+1}) = (1, 0) \text{ or } (a_{j+1}, a_{j+2}) = (1, 0). \end{cases} \quad (2.20)$$

- (7) Define a raised background sequence $\tilde{\mathbf{c}} = (\tilde{c}_0, \tilde{c}_1, \dots)$ as a sequence obtained by skipping terms in the sequence $\tilde{\mathbf{b}}$ if “ $\times$ ” is written.
- (8) Define a raised soliton sequence $\tilde{\mathbf{d}} = (\tilde{d}_0, \tilde{d}_1, \dots)$ as a binary sequence that is obtained by eliminating 1s and 0s connected with the 10-arc lines from $\widetilde{\text{bin}}(\boldsymbol{\eta})$. Let $\tilde{s}_j$ be the number of consecutive 0s and $\tilde{t}_j$ be the number of consecutive 1s in $\tilde{\mathbf{d}}$ as $\tilde{\mathbf{d}} = 0^{\tilde{s}_1} 1^{\tilde{t}_1} 0^{\tilde{s}_2} 1^{\tilde{t}_2} \dots$.
- (9) Let $\mathbf{c} = (c_0, c_1, \dots) \in \mathcal{S}_{L,M}^{(b)}$, where $c_j = \tilde{c}_j - M$ ($j = 0, 1, \dots$). This sequence is called the *background sequence* [5].
- (10) Let $\mathbf{d} = (d_0, d_1, \dots) = 0^{s_1} 1^{t_1} 0^{s_2} 1^{t_2} \dots$, where $s_1 = \tilde{s}_1 - M$, $s_j = \tilde{s}_j - 2M$ ($j = 2, 3, \dots$), $t_j = \tilde{t}_j - 2M$ ($j = 1, 2, \dots$). We call this a *soliton sequence*.

Example 2.2.1. $L = 2$, $M = 1$ and $\tilde{L} = 4$.

$$\begin{aligned}
\eta &= 0 & 0 & 3 & 1 & -1 & 2 & 1 & 1 & 0 & 0 \dots \\
a &= 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \dots \\
\tilde{\eta} &= 1 & 1 & 4 & 2 & 0 & 3 & 2 & 2 & 1 & 1 \dots \\
\widetilde{\text{bin}}(\eta) &= \underbrace{00010001}_{\times} \underbrace{111111}_{\times} \underbrace{10000000}_{\times} \underbrace{11100111}_{\times} \underbrace{10000010001}_{\times} \dots \\
\tilde{b} &= 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 \dots \\
\tilde{c} &= 1 & 0 & & & 0 & 2 & & & 1 & 1 \dots \\
c &= 0 & -1 & & & -1 & 1 & & & 0 & 0 \dots \\
\tilde{d} &= 000 & 001 & 111111 & 1000000 & 01 & 11 & 1100 & 000 & 00 \dots \\
&= 0^5 1^7 0^7 1^5 0^\infty \\
d &= 0^4 1^5 0^5 1^3 0^\infty
\end{aligned}$$

Define the forward-shift operator Λ on $\mathcal{S}_{L,M}^{(b)}$ as

$$(\Lambda(c))_j = \begin{cases} 0, & j = 0, \\ (c)_{j-1}, & \text{otherwise.} \end{cases} \quad (2.21)$$

Theorem 2.2.2. For $\eta \in \mathcal{S}_{L,M}$, let $c(\eta)$ be the background sequence and $d(\eta)$ be the soliton sequence, as defined above.

$$\Lambda(c(\eta)) = c(T_L(\eta)) \quad (2.22)$$

$$T_1(d(\eta)) = d(T_L(\eta)) \quad (2.23)$$

The background sequence c is shifted to the right at speed one, and the soliton sequence d follows the time evolution of the BBS(1). Because the time evolution of the BBS(1) can be linearized [13], we can linearize the time evolution of the BBS(L). This claim will be proved from Theorems 2.2.7 and 2.2.9.

Example 2.2.3. $L = 2$, and $\eta = 0, 0, 3, 1, -1, 2, 1, 1, 0, 0, \dots \in \mathcal{S}_{2,1}$ (as in Example 2.2.1)

$$c(\eta) = 0, -1, -1, 1, 0, 0, \dots$$

$$d(\eta) = 000011111100000011100000 \dots$$

$$T_2(\eta) = 0, 0, -1, 1, 3, 0, 1, 1, 2, 0, \dots$$

$$c(T_2(\eta)) = 0, 0, -1, -1, 1, 0, \dots$$

$$d(T_2(\eta)) = 0000000001111100011100 \dots$$

$$\begin{array}{ccc}
\eta & \xrightarrow{T_2} & T_2(\eta) \\
\downarrow \beta_2 & & \downarrow \beta_2 \\
(c(\eta), d(\eta)) & \xrightarrow[\substack{c(T_2(\eta)) = \Lambda(c(\eta)) \\ d(T_2(\eta)) = T_1(d(\eta))}]{} & (c(T_2(\eta)), d(T_2(\eta)))
\end{array}$$

To prove Theorem 2.2.2, we define another procedure to obtain the background sequence and the soliton sequence.

- (1) Define a soliton flag sequence $\mathbf{a}' = (a'_{-1}, a'_0, \dots) \in \mathcal{S}_1$ as

$$\begin{aligned} a'_{-1} &= 0, \\ a'_{j+1} &= \begin{cases} 0, & \tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} < \tilde{L}, \\ a'_j, & \tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} = \tilde{L}, \\ 1, & \tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} > \tilde{L}, \end{cases} \\ &= \begin{cases} 0, & \eta_{j+1} + \eta_{j+2} < L, \\ a_j, & \eta_{j+1} + \eta_{j+2} = L, \\ 1, & \eta_{j+1} + \eta_{j+2} > L. \end{cases} \quad (j = 0, 1, 2, \dots) \end{aligned} \quad (2.24)$$

We say j is in a *right-0-segment* if $a'_j = 0$, and in a *right-1-segment* if $a'_j = 1$. By definition, it is clear that

$$a'_j = a_{j+1} \quad (j = 0, 1, \dots). \quad (2.25)$$

- (2) Let $I' = \{j \in \mathbb{Z}_{\geq 0} \mid a'_j = 1, a'_{j-1} = 0\}$, which is the set of the left ends of consecutive *right-1-segments*. We define

$$\widetilde{\text{bin}}'(\boldsymbol{\eta}) = \mathbf{l}_{I'}(\tilde{\boldsymbol{\eta}}), \quad (2.26)$$

where $\mathbf{l}_{I'}$ is defined in equations (2.13) and (2.14).

- (3) We write “ $\times$ ” on the subsequence corresponding to the j th and the $(j+1)$ th boxes where $j \in I'$. This means that we mark the left end of every consecutive *right-1-segment* as well as the position immediately to the right.
- (4) We iteratively draw 01-arc lines from 0s in the $(j-1)$ th box without “ $\times$ ” and 1s in the j th box repeatedly. Then, the number of arc lines that connect the $(j-1)$ th box and the j th box is $\min(\eta_j, L - \eta_{j-1})$.
- (5) Let $\tilde{\mathbf{b}}' = (\tilde{b}'_0, \tilde{b}'_1, \dots)$, where $\tilde{b}'_j$ is the number of 01-arc lines that connect 0s in the $(j-1)$ th box and 1s in the j th box. If $a'_{j-1} = 0, \tilde{\eta}_{j-1} + \tilde{\eta}_j \leq \tilde{L}$, then $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j-1}) = \tilde{\eta}_j$. If $a'_{j-1} = 1, \tilde{\eta}_{j-1} + \tilde{\eta}_j \geq \tilde{L}$, then $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j-1}) = \tilde{L} - \tilde{\eta}_{j-1}$. Thus, we get

$$\tilde{b}'_j = \begin{cases} \tilde{\eta}_j, & a'_{j-1} = a'_j = 0, \\ \tilde{L} - \tilde{\eta}_{j-1}, & a'_{j-2} = a'_{j-1} = 1, \\ 0, & (a'_{j-2}, a'_{j-1}) = (0, 1) \text{ or } (a'_{j-1}, a'_j) = (0, 1). \end{cases} \quad (2.27)$$

- (6) Let $\tilde{\mathbf{c}}' = (\tilde{c}'_0, \tilde{c}'_1, \dots)$ be a sequence obtained by skipping terms in the sequence $\tilde{\mathbf{b}}'$ if “ $\times$ ” is written.

- (7) Let $\tilde{\mathbf{d}}' = (\tilde{d}'_0, \tilde{d}'_1, \dots)$ be a binary sequence that is obtained by eliminating 1s and 0s connected with the 01-arc lines from $\widetilde{\text{bin}}'(\boldsymbol{\eta})$. Let $\tilde{s}'_j$ be the number of consecutive 0s and $\tilde{t}'_j$ be the number of consecutive 1s in $\tilde{\mathbf{d}}'$ as $\tilde{\mathbf{d}}' = 0^{\tilde{s}'_1} 1^{\tilde{t}'_1} 0^{\tilde{s}'_2} 1^{\tilde{t}'_2} \dots$.
- (8) Let $\mathbf{c}' = (c'_0, c'_1, \dots) \in \mathcal{S}_{L,M}^{(b)}$, where $c'_j = \tilde{c}'_j - M$ ($j = 0, 1, \dots$).
- (9) Let $\mathbf{d}' = (d'_0, d'_1, \dots) = 0^{s'_1} 1^{t'_1} 0^{s'_2} 1^{t'_2} \dots$, where $s'_j = \tilde{s}'_j - 2M, t'_j = \tilde{t}'_j - 2M$ ($j = 1, 2, \dots$).

Example 2.2.4. $L = 2$, and $\boldsymbol{\eta} = 0, 0, 3, 1, -1, 2, 1, 1, 0, 0, \dots \in \mathcal{S}_{2,1}$ (as in Examples 2.2.1 and 2.2.3)

$$\begin{array}{rcl}
T_2(\boldsymbol{\eta}) & = & 0 \quad 0 \quad -1 \quad 1 \quad 3 \quad 0 \quad 1 \quad 1 \quad 2 \quad 0 \dots \\
\mathbf{a}'(T_2(\boldsymbol{\eta})) & = & 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \dots \\
T_2(\tilde{\boldsymbol{\eta}}) & = & 1 \quad 1 \quad 0 \quad 2 \quad 4 \quad 1 \quad 2 \quad 2 \quad 3 \quad 1 \dots \\
\widetilde{\text{bin}}'(T_2(\boldsymbol{\eta})) & = & \underbrace{10001000000000}_{\times} \underbrace{111111}_{\times} \underbrace{1000110000}_{\times} \underbrace{1111101000}_{\times} \dots \\
\tilde{\mathbf{b}}'(T_2(\boldsymbol{\eta})) & = & 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 1 \dots \\
\tilde{\mathbf{c}}'(T_2(\boldsymbol{\eta})) & = & 1 \quad 1 \quad 0 \quad \quad \quad 0 \quad 2 \quad \quad \quad 1 \dots \\
\mathbf{c}'(T_2(\boldsymbol{\eta})) & = & 0 \quad 0 \quad -1 \quad \quad \quad -1 \quad 1 \quad \quad \quad 0 \dots \\
\tilde{\mathbf{d}}'(T_2(\boldsymbol{\eta})) & = & 00 \quad 000 \quad 000000 \quad 111111 \quad 10 \quad 00 \quad 0011 \quad 111 \quad 00 \dots \\
& & = 0^{11} 1^7 0^5 1^5 0^\infty \\
\mathbf{d}' & = & 0^9 1^5 0^3 1^3 0^\infty
\end{array}$$

Examples 2.2.1 and 2.2.4 imply the following Lemmas 2.2.5, 2.2.6, and Theorem 2.2.7.

Lemma 2.2.5. For $\boldsymbol{\eta} \in \mathcal{S}_{L,M}$, the following conditions are equivalent.

- (i) $(\mathbf{a}(\boldsymbol{\eta}))_j = 1$ and $(\mathbf{a}(\boldsymbol{\eta}))_{j+1} = 0$
- (ii) $(\mathbf{a}'(T_L(\boldsymbol{\eta})))_j = 1$ and $(\mathbf{a}'(T_L(\boldsymbol{\eta})))_{j-1} = 0$

Proof. Let $\mathbf{a}(\boldsymbol{\eta}) = (a_0, a_1, \dots)$, $\mathbf{a}'(T_L(\boldsymbol{\eta})) = (a'_0, a'_1, \dots)$. Here, we prove that (i) $\Rightarrow$ (ii) by using equations (2.3) and (2.4). The reverse (ii) $\Rightarrow$ (i) can be proved similarly.

- Case 1: $\eta'_{j-1} + \eta'_j > L, \eta'_j + \eta'_{j+1} < L$ (See Fig 2.2.)

$$\eta'_j + u'_j - L \geq \eta'_j + \eta'_{j-1} - L > 0, \text{ and}$$

$$\begin{aligned}
\eta_j^{t+1} &= \min(u'_j, L - \eta'_j) \\
&= L - \eta'_j.
\end{aligned}$$

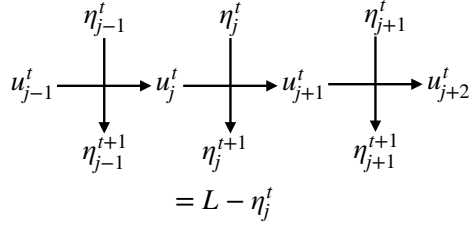


Fig 2.2: Case 1 in Lemma 2.2.5.

Therefore, we have

$$\begin{aligned}
\eta_{j-1}^{t+1} + \eta_j^{t+1} - L &= \min(u_{j-1}^t, L - \eta_{j-1}^t) + (L - \eta_j^t) - L \\
&= \min(u_{j-1}^t - \eta_j^t, L - \eta_{j-1}^t - \eta_j^t) \\
&< 0.
\end{aligned}$$

By using $u_{j+1}^t = \eta_j^t + \max(0, \eta_j^t + u_j^t - L) > \eta_j^t$,

$$\begin{aligned}
\eta_j^{t+1} + \eta_{j+1}^{t+1} - L &= (L - \eta_j^t) + \min(u_{j+1}^t, L - \eta_{j+1}^t) - L \\
&= \min(u_{j+1}^t - \eta_j^t, L - \eta_j^t - \eta_{j+1}^t) \\
&> 0.
\end{aligned}$$

- Case 2: The case $\eta_{k-1}^t + \eta_k^t > L$, $\eta_{i-1}^t + \eta_i^t = L$ ($i = k+1, \dots, j$), $\eta_j^t + \eta_{j+1}^t < L$ (See Fig 2.3.)

$\eta_i^t + u_i^t - L \geq \eta_i^t + \eta_{i-1}^t - L \geq 0$ for $i = k, k+1, \dots, j$ and

$$\begin{aligned}
\eta_i^{t+1} &= \min(u_i^t, L - \eta_j^t) \\
&= L - \eta_i^t \quad (i = k, k+1, \dots, j).
\end{aligned}$$

From $\eta_k^t + u_k^t - L \geq \eta_k^t + \eta_{k-1}^t - L > 0$, we have $u_{k+1}^t > \eta_k^t$. Then, we have $\eta_{k+1}^t + u_{k+1}^t - L > \eta_{k+1}^t + \eta_k^t - L = 0$, and $u_{k+2}^t > \eta_{k+1}^t$. Through an induction procedure, we also obtain $u_{j+1}^t > \eta_j^t$.

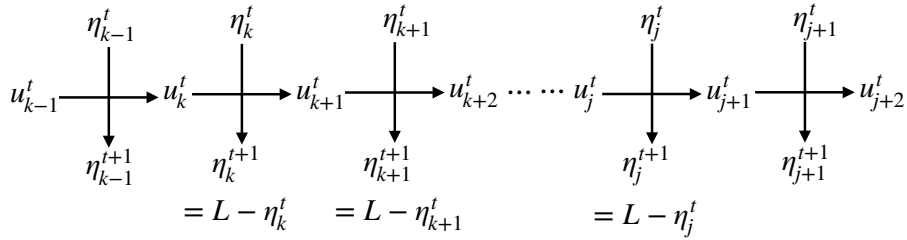


Fig 2.3: Case 2 in Lemma 2.2.5.

Therefore, we have

$$\begin{aligned}\eta_{k-1}^{t+1} + \eta_k^{t+1} - L &= \min(u_{k-1}^t, L - \eta_{k-1}^t) + (L - \eta_k^t) - L \\ &= \min(u_{k-1}^t - \eta_k^t, L - \eta_{k-1}^t - \eta_k^t) \\ &< 0,\end{aligned}$$

$$\begin{aligned}\eta_{i-1}^{t+1} + \eta_i^{t+1} - L &= (L - \eta_{i-1}^t) + (L - \eta_i^t) - L \\ &= L - \eta_{i-1}^t - \eta_i^t \\ &= 0 \quad (i = k + 1, \dots, j),\end{aligned}$$

$$\begin{aligned}\eta_j^{t+1} + \eta_{j+1}^{t+1} - L &= (L - \eta_j^t) + \min(u_{j+1}^t, L - \eta_{j+1}^t) - L \\ &= \min(u_{j+1}^t - \eta_j^t, L - \eta_{j+1}^t - \eta_j^t) \\ &> 0.\end{aligned}$$

□

Lemma 2.2.6. For $\eta \in \mathcal{S}_{L,M}$,

$$T_1^M(\widetilde{\text{bin}}(\eta)) = \widetilde{\text{bin}}'(T_L(\eta)). \quad (2.28)$$

Proof. Let I be the set of indices

$$\begin{aligned}I &= \{i \in \mathbb{Z}_{\geq 0} \mid (\mathbf{a}(\eta))_i = 1, (\mathbf{a}(\eta))_{i+1} = 0\} \\ &\subset \{i \in \mathbb{Z}_{\geq 0} \mid \eta_i + u_i \geq L\}.\end{aligned}$$

Then, from Lemma 2.2.5,

$$I = \{i \in \mathbb{Z}_{\geq 0} \mid (\mathbf{a}'(T_L(\eta)))_{i-1} = 0, (\mathbf{a}'(T_L(\eta)))_i = 1\}.$$

From Theorem 2.1.3, we have $T_1^M(\widetilde{\text{bin}}(\eta)) = \widetilde{\text{bin}}'(T_L(\eta))$. □

Theorem 2.2.7. For $\eta \in \mathcal{S}_{L,M}$,

$$\Lambda(\mathbf{c}(\eta)) = \mathbf{c}'(T_L(\eta)), \quad (2.29)$$

$$T_1(\mathbf{d}(\eta)) = \mathbf{d}'(T_L(\eta)). \quad (2.30)$$

To prove Theorem 2.2.7, we introduce 10-arc lines on the binary sequence $\eta \in \mathcal{S}_1$ [40] which can express the time evolution of the BBS(1) according to the following rules.

- i) For $\eta \in \mathcal{S}_1$, connect all 10 pairs with arc lines.
- ii) Neglecting the 1s and 0s which were connected already, connect all the remaining 10 pairs with arc lines.

iii) Repeat the above procedure until all the 1s are connected to 0s.

iv) $T_1(\eta)$ is the state obtained by exchanging the 1s and 0s in every connected 10 pair.

We can draw 01-arc lines in the same fashion, and the following lemma is obvious from the definition.

Lemma 2.2.8. (Lemma 1 in [13]) The 10-arc lines for $\eta \in \mathcal{S}_1$ coincide with the 01-arc lines for $T_1(\eta)$.

(Proof of Theorem 2.2.7.) From Lemmas 2.2.5 and 2.2.6, the 10-arc lines for $\widetilde{\text{bin}}(\eta)$ coincide with the 01-arc lines for $\widetilde{\text{bin}}'(T_L(\eta))$. Using the 01/10-arc lines, $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j+1})$ can be described as the number of 10-arc lines on $\widetilde{\text{bin}}(\eta)$ that connect the balls in the j th box and the vacancies in the $(j+1)$ th box. Similarly, $\min((\widetilde{T_L(\eta)})_j, \tilde{L} - (\widetilde{T_L(\eta)})_{j-1})$ can be explained as the number of 01-arc lines on $\widetilde{\text{bin}}'(T_L(\eta))$ that connect the balls in the j th box and the vacancies in the $(j-1)$ th box. Thus, it follows that $\Lambda(\tilde{\mathbf{b}}(\eta)) = \tilde{\mathbf{b}}'(T_L(\eta))$ and $\Lambda(\tilde{\mathbf{c}}(\eta)) = \tilde{\mathbf{c}}'(T_L(\eta))$.

When we obtain the raised soliton sequences $\tilde{\mathbf{d}}(\eta), \tilde{\mathbf{d}}'(T_L(\eta))$ deleting 1s, 0s and 10/01-arc lines that correspond to background materials, other 10-arc lines on $\widetilde{\text{bin}}(\eta)$ and 01-arc lines on $\widetilde{\text{bin}}'(T_L(\eta))$ do not change. Thus, we obtain $T_1^M(\tilde{\mathbf{d}}(\eta)) = \tilde{\mathbf{d}}'(T_L(\eta))$, and $T_1(\mathbf{d}(\eta)) = \mathbf{d}'(T_L(\eta))$.

Theorem 2.2.9.

$$\mathbf{c}(\eta) = \mathbf{c}'(\eta) \quad (2.31)$$

$$\mathbf{d}(\eta) = \mathbf{d}'(\eta) \quad (2.32)$$

Proof. For the soliton flag sequence $\mathbf{a}$, define indices $i_1, i_2, \dots, i_N, k_1, k_2, \dots, k_N \in \mathbb{Z}_{\geq 0}$ as

$$\{j \in \mathbb{Z}_{\geq 0} \mid a_j = 0, a_{j+1} = 1\} = \{i_1 < i_2 < \dots < i_N\} \quad (2.33)$$

$$\{j \in \mathbb{Z}_{\geq 0} \mid a_j = 1, a_{j+1} = 0\} = \{k_1 < k_2 < \dots < k_N\}. \quad (2.34)$$

These indices satisfy the interlacing condition $0 < i_1 < k_1 < i_2 < k_2 < \dots < i_N < k_N$. Similarly, for the soliton flag sequence $\mathbf{a}'$, define indices $i'_1, i'_2, \dots, i'_N, k'_1, k'_2, \dots, k'_N \in \mathbb{Z}_{\geq 0}$ as

$$\{j \in \mathbb{Z}_{\geq 0} \mid a'_{j-1} = 0, a'_j = 1\} = \{i'_1 < i'_2 < \dots < i'_N\} \quad (2.35)$$

$$\{j \in \mathbb{Z}_{\geq 0} \mid a'_{j-1} = 1, a'_j = 0\} = \{k'_1 < k'_2 < \dots < k'_N\}. \quad (2.36)$$

From equation (2.25), we have $i'_m = i_m, k'_m = k_m (m = 1, 2, \dots, N)$.

Letting $k_0 = k'_0 = -1$, we have

$$\tilde{b}_j = \begin{cases} \tilde{\eta}_j, & k_m + 1 \leq j \leq i_{m+1} - 1, \\ \tilde{L} - \tilde{\eta}_{j+1}, & i_m \leq j \leq k_m - 2, \\ 0, & j = k_m - 1, k_m. \end{cases} \quad (m = 0, 1, 2, \dots) \quad (2.37)$$

from equation (2.20), and

$$\tilde{b}'_j = \begin{cases} \tilde{\eta}_j, & k'_m + 1 \leq j \leq i'_{m+1} - 1, \\ \tilde{L} - \tilde{\eta}_{j-1}, & i'_m + 2 \leq j \leq k'_m, \\ 0, & j = i'_m, i'_m + 1. \end{cases} \quad (m = 0, 1, 2, \dots) \quad (2.38)$$

from (2.27). We obtain $\tilde{\mathbf{c}}$ from $\tilde{\mathbf{b}}$ by skipping the terms $j = k_m - 1, k_m$, and $\tilde{\mathbf{c}}'$ from $\tilde{\mathbf{b}}'$ by skipping the terms $j = i'_m, i'_m + 1$. Thus, we obtain $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}'$ and $\mathbf{c} = \mathbf{c}'$.

Letting $\tilde{\eta}_{-1} = \tilde{L}$, we can write the soliton sequence $\tilde{\mathbf{d}}$ as $\tilde{\mathbf{d}} = 0^{\tilde{s}_1} 1^{\tilde{t}_1} 0^{\tilde{s}_2} 1^{\tilde{t}_2} \dots 1^{\tilde{t}_N} 0^\infty$, where

$$\tilde{s}_m = \sum_{j=k_{m-1}}^{i_m} (\tilde{L} - \tilde{\eta}_j) - \sum_{j=k_{m-1}+1}^{i_m-1} \tilde{b}_j, \quad (2.39)$$

$$\tilde{t}_m = \sum_{j=i_m}^{k_m} \tilde{\eta}_j - \sum_{j=i_m}^{k_m-1} \tilde{b}_j, \quad (2.40)$$

and $\mathbf{d}'$ as $\mathbf{d}' = 0^{s'_1} 1^{t'_1} 0^{s'_2} 1^{t'_2} \dots 1^{t'_N} 0^\infty$, where

$$\tilde{s}'_1 = \sum_{j=0}^{i'_1} (\tilde{L} - \tilde{\eta}_j) - \sum_{j=0}^{i'_1-1} \tilde{b}'_j + M, \quad (2.41)$$

$$\tilde{s}'_m = \sum_{j=k'_{m-1}}^{i'_m} (\tilde{L} - \tilde{\eta}_j) - \sum_{j=k'_{m-1}+1}^{i'_m-1} \tilde{b}'_j \quad (m = 2, 3, \dots, N), \quad (2.42)$$

$$\tilde{t}'_m = \sum_{j=i'_m}^{k'_m} \tilde{\eta}_j - \sum_{j=i'_m+1}^{k'_m} \tilde{b}'_j \quad (m = 1, 2, \dots, N). \quad (2.43)$$

Using equations (2.37), (2.38) and $i'_m = i_m, k'_m = k_m$, we have

$$\tilde{s}_1 = \tilde{s}'_1 - M = \tilde{L} + \sum_{j=-1}^{i_1-1} (\tilde{L} - \tilde{\eta}_j - \tilde{\eta}_{j+1}), \quad (2.44)$$

$$\tilde{s}_m = \tilde{s}'_m = \tilde{L} + \sum_{j=k_{m-1}}^{i_m-1} (\tilde{L} - \tilde{\eta}_j - \tilde{\eta}_{j+1}) \quad (m = 2, 3, \dots, N), \quad (2.45)$$

$$\tilde{t}_m = \tilde{t}'_m = \tilde{L} + \sum_{j=i_m}^{k_m-1} (\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L}) \quad (m = 1, 2, \dots, N). \quad (2.46)$$

Therefore, we get $\tilde{\mathbf{d}}(\eta) = \tilde{\mathbf{d}}'(\eta)$. Since $\tilde{L} - \tilde{\eta}_j - \tilde{\eta}_{j+1} = L - \eta_j - \eta_{j+1}$, the soliton sequences $\mathbf{d} = 0^{s_1} 1^{t_1} 0^{s_2} 1^{t_2} \dots 1^{t_N} 0^\infty$ and $\mathbf{d}' = 0^{s'_1} 1^{t'_1} 0^{s'_2} 1^{t'_2} \dots 1^{t'_N} 0^\infty$ can be written as

$$s_m = s'_m = L + \sum_{j=k_{m-1}}^{i_m-1} (L - \eta_j - \eta_{j+1}), \quad (2.47)$$

$$t_m = t'_m = L + \sum_{j=i_m}^{k_m-1} (\eta_j + \eta_{j+1} - L) \quad (m = 1, 2, \dots, N). \quad (2.48)$$

□

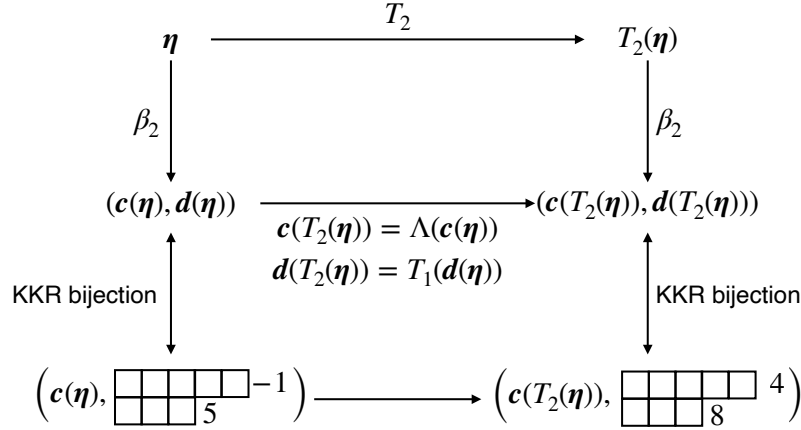
From Theorems 2.2.7 and 2.2.9, we obtain equations (2.22) and (2.23) in Theorem 2.2.3 as

$$\begin{aligned}\Lambda(c(\eta)) &= c'(T_L(\eta)) = c(T_L(\eta)), \\ T_1(d(\eta)) &= d'(T_L(\eta)) = d(T_L(\eta)).\end{aligned}$$

A binary sequence $\mathbf{d} \in \mathcal{S}_L^{(\text{f})}$ is associated with a rigged configuration by the KKR bijection, and the time evolution of the BBS(1) can be linearized (see Section 1.2.1).

Example 2.2.10. $L = 2$

$$\begin{aligned}\eta &= 0, 0, 3, 1, -1, 2, 1, 1, 0, 0, \dots \\ c(\eta) &= 0, -1, -1, 1, 0, 0, \dots \\ d(\eta) &= 0000111110000011100000 \dots \\ T_2(\eta) &= 0, 0, -1, 1, 3, 0, 1, 1, 2, 0, \dots \\ c(T_2(\eta)) &= 0, 0, -1, -1, 1, 0, \dots \\ d(T_2(\eta)) &= 00000000011111000111000000 \dots\end{aligned}$$



2.2.2 Reconstruction of box-ball system state from two sequences

In this section, we prove the bijectivity of the map β_L . First, we define the reconstruction map $\beta_L^{-1} : \mathcal{S}_{L,M}^{(\text{b})} \oplus \mathcal{S}_L^{(\text{f})} \rightarrow \mathcal{S}_{L,M}; (c, d) \mapsto \eta$ as follows.

Let $M = \max_{j \in \mathbb{Z}_{\geq 0}} \max(-c_j, c_j - L)$, $\tilde{L} = L + 2M$. Let $\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \dots)$ where $\tilde{c}_j = c_j + M$, and $\tilde{c}_{-1} = 0$. Let $\tilde{d} = 0^{s_1+M} 1^{t_1+2M} 0^{s_2+2M} 1^{t_2+2M} \dots 1^{t_N+2M} 0^\infty$, where $d = 0^{s_1} 1^{t_1} \dots 1^{t_N} 0^\infty$.

In the algorithm below, we will define variables $k^{(j)}$ and sequences $\tilde{c}^{(j)}$ for $j = 0, 1, 2, \dots$. The integer $k^{(j)}$ will denote the position in the subsequence of $\tilde{d}$ that corresponds to the left end of the j th box of η (or $\tilde{\eta}$), and let $k^{(0)} = 0$. The sequence $\tilde{c}^{(j)}$ will be obtained by inserting 0s into $\tilde{c}$ up to the j th box of η , with $\tilde{c}^{(0)} = \tilde{c}$. Repeat the following procedure for $j = 0, 1, 2, \dots$ in this order, and stop if $\tilde{c}_l^{(j)} = M$ for all $l > j$ and $\tilde{d}_m = 0$ for all $m > k^{(j)}$.

1). For the binary sequence $\tilde{\mathbf{d}}$, let $X_j = i - k^{(j)}$ where i is the minimal integer that satisfies $i \geq k^{(j)}$, $\tilde{d}_{i-1} = 1$, and $\tilde{d}_i = 0$. If there is no such integer i , let $X_j = +\infty$.

2). • Case 1: $X_j < 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$

Let $\tilde{\mathbf{c}}^{(j+1)}$ be the sequence obtained by inserting 0 between $\tilde{c}_{j-1}^{(j)}$ and $\tilde{c}_j^{(j)}$. The length of the subsequence of $\tilde{\mathbf{d}}$ that corresponds to the j th box of $\boldsymbol{\eta}$ is $\tilde{L} - \tilde{c}_{j-1}^{(j+1)} - \tilde{c}_j^{(j+1)} = \tilde{L} - \tilde{c}_{j-1}^{(j+1)}$ and let $k^{(j+1)}$ be

$$k^{(j+1)} = k^{(j)} + \left(\tilde{L} - \tilde{c}_{j-1}^{(j+1)} \right).$$

Underline the subsequence of $\tilde{\mathbf{d}}$, from $\tilde{d}_{k^{(j)}}$ to $\tilde{d}_{k^{(j+1)}-1}$.

• Case 2: $X_j \geq 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$

Let $\tilde{\mathbf{c}}^{(j+1)} = \tilde{\mathbf{c}}^{(j)}$. Then, the length of the subsequence of $\tilde{\mathbf{d}}$ that corresponds to the j th box of $\boldsymbol{\eta}$ is $\tilde{L} - \tilde{c}_{j-1}^{(j+1)} - \tilde{c}_j^{(j+1)}$, and let $k^{(j+1)}$ be

$$k^{(j+1)} = k^{(j)} + \left(\tilde{L} - \tilde{c}_{j-1}^{(j+1)} - \tilde{c}_j^{(j+1)} \right).$$

Underline the subsequence of $\tilde{\mathbf{d}}$, from $\tilde{d}_{k^{(j)}}$ to $\tilde{d}_{k^{(j+1)}-1}$.

After this procedure stops at step $j = J$, $\tilde{\eta}_i$ is obtained by summing $\tilde{c}_i^{(J)}$ with the number of 1s over the j th underline in $\tilde{\mathbf{d}}$ if $i = 0, 1, \dots, J-1$. For $i \geq J$, $\tilde{\eta}_i = \tilde{c}_i^{(J)}$.

Example 2.2.11. Consider $L = 2$, $\mathbf{c} = 0, -1, -1, 1, 0, \dots$, $\mathbf{d} = 0000111111000000111000\dots$. Then, $M = 1$, $\tilde{L} = 4$, $\tilde{\mathbf{c}} = 1, 0, 0, 2, 1, \dots$, $\tilde{\mathbf{d}} = 000001111111000000001111100000\dots$.

• ($j = 0$) $k^{(0)} = 0$, $i = 12$, and $X_0 = 12$. $2\tilde{L} - \tilde{c}_{-1}^{(0)} - \tilde{c}_0^{(0)} = 7 \leq 12$.

$$\tilde{\mathbf{c}}^{(1)} = 1, 0, 0, 2, 1, \dots, \tilde{L} - \tilde{c}_{-1}^{(1)} - \tilde{c}_0^{(1)} = 3.$$

$$\tilde{\mathbf{d}} = \underline{00000}1111111000000001111100000\dots$$

• ($j = 1$) $k^{(1)} = 3$, $i = 12$, and $X_1 = 9$. $2\tilde{L} - \tilde{c}_0^{(1)} - \tilde{c}_1^{(1)} = 7 \leq 9$.

$$\tilde{\mathbf{c}}^{(2)} = 1, 0, 0, 2, 1, \dots, \tilde{L} - \tilde{c}_0^{(2)} - \tilde{c}_1^{(2)} = 3.$$

$$\tilde{\mathbf{d}} = \underline{000}\underline{001}1111111000000001111100000\dots$$

• ($j = 2$) $k^{(2)} = 6$, $i = 12$, and $X_2 = 6$. $2\tilde{L} - \tilde{c}_1^{(2)} - \tilde{c}_2^{(2)} = 8 > 6$.

$$\tilde{\mathbf{c}}^{(3)} = 1, 0, 0, 0, 2, 1, \dots, \tilde{L} - \tilde{c}_1^{(3)} - \tilde{c}_2^{(3)} = 4.$$

$$\tilde{\mathbf{d}} = \underline{000}\underline{001}\underline{11111}11000000001111100000\dots$$

• ($j = 3$) $k^{(3)} = 10$, $i = 12$, and $X_3 = 2$. $2\tilde{L} - \tilde{c}_2^{(3)} - \tilde{c}_3^{(3)} = 8 > 2$.

$$\tilde{\mathbf{c}}^{(4)} = 1, 0, 0, 0, 0, 2, 1, \dots, \tilde{L} - \tilde{c}_2^{(4)} - \tilde{c}_3^{(4)} = 4.$$

$$\tilde{\mathbf{d}} = \underline{000}\underline{001}\underline{1111}\underline{1100000000}1111100000\dots$$

- $(j = 4) k^{(4)} = 14, i = 24$, and $X_4 = 10$. $2\tilde{L} - \tilde{c}_3^{(4)} - \tilde{c}_4^{(4)} = 8 \leq 10$.
 $\tilde{c}^{(5)} = 1, 0, 0, 0, 0, 2, 1, \dots, \tilde{L} - \tilde{c}_3^{(5)} - \tilde{c}_4^{(5)} = 4$.
 $\tilde{d} = \underline{000\ 001\ 1111\ 1100\ 00000}\underline{1111100000}\dots$.
- $(j = 5) k^{(5)} = 18, i = 24$, and $X_5 = 6$. $2\tilde{L} - \tilde{c}_4^{(5)} - \tilde{c}_5^{(5)} = 6 \leq 6$.
 $\tilde{c}^{(6)} = 1, 0, 0, 0, 0, 2, 1, \dots, \tilde{L} - \tilde{c}_4^{(6)} - \tilde{c}_5^{(6)} = 2$.
 $\tilde{d} = \underline{000\ 001\ 1111\ 1100\ 0000}\underline{01111100000}\dots$.
- $(j = 6) k^{(6)} = 20, i = 24$, and $X_6 = 4$. $2\tilde{L} - \tilde{c}_5^{(6)} - \tilde{c}_6^{(6)} = 5 > 4$.
 $\tilde{c}^{(7)} = 1, 0, 0, 0, 0, 2, 0, 1, \dots, \tilde{L} - \tilde{c}_5^{(7)} - \tilde{c}_6^{(7)} = 2$.
 $\tilde{d} = \underline{000\ 001\ 1111\ 1100\ 0000}\underline{01\ 11}\underline{1100000}\dots$.
- $(j = 7) k^{(7)} = 22, i = 24$, and $X_7 = 2$. $2\tilde{L} - \tilde{c}_6^{(7)} - \tilde{c}_7^{(7)} = 7 > 2$.
 $\tilde{c}^{(8)} = 1, 0, 0, 0, 0, 2, 0, 0, 1, \dots, \tilde{L} - \tilde{c}_6^{(8)} - \tilde{c}_7^{(8)} = 4$.
 $\tilde{d} = \underline{000\ 001\ 1111\ 1100\ 0000}\underline{01\ 11\ 1100000}\dots$.
- $(j = 8) k^{(8)} = 26, i = +\infty$, and $X_8 = +\infty$. $2\tilde{L} - \tilde{c}_7^{(8)} - \tilde{c}_8^{(8)} = 7 \leq +\infty$.
 $\tilde{c}^{(9)} = 1, 0, 0, 0, 0, 2, 0, 0, 1, \dots, \tilde{L} - \tilde{c}_7^{(9)} - \tilde{c}_8^{(9)} = 3$.
 $\tilde{d} = \underline{000\ 001\ 1111\ 1100\ 0000}\underline{01\ 11\ 1100}\underline{000}\dots$.

Then, we obtain $\tilde{\eta} = 1, 1, 4, 2, 0, 3, 2, 2, 1, \dots$ and $\eta = 0, 0, 3, 1, -1, 2, 1, 1, 0, \dots$

Theorem 2.2.12. (Injectivity of β_L)

For $\eta \in \mathcal{S}_{L,M}$, $(\beta_L^{-1} \circ \beta_L)(\eta) = \eta$.

Proof. If we have the background sequence $\tilde{\mathbf{b}} = (\tilde{b}_0, \tilde{b}_1, \dots) \in \mathcal{S}_{L,M}^{(b)}$ and a soliton sequence $\tilde{\mathbf{d}}$, we can get a binary subsequence that corresponds to the j th box of η by dividing $\tilde{\mathbf{d}}$ every $\tilde{L} - \tilde{b}_{j-1} - \tilde{b}_j$. Further, $\tilde{\eta}_j$ is the sum of $\tilde{b}_j$ and the number of 1s in the j th subsequence. Let $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$ respectively be the raised background sequence and soliton sequence constructed from η . Thus it is sufficient to show that $\tilde{\mathbf{b}}$ is uniquely determined from $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$. In the following, we will prove that $X_j < 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$ if and only if $(a_i, a_{i+1}) = (1, 0)$ or $(a_{i+1}, a_{i+2}) = (1, 0)$.

- Case 1: $a_{i+2k-1} = 1, a_{i+2k} = 0$ ($k = 1, 2, \dots, n$) and $(a_{i+2n+1}, a_{i+2n+2}) \neq (1, 0)$. (See Fig 2.4)

$$\begin{aligned}
 X_i &= (\tilde{L} - \tilde{\eta}_i - \tilde{c}_{i-1}) + \tilde{\eta}_i + \tilde{\eta}_{i+1} \\
 &= \tilde{L} - \tilde{c}_{i-1} + \tilde{\eta}_{i+1}, \\
 X_{i+1} &= \tilde{\eta}_{i+1}.
 \end{aligned}$$

From $a_{i+2k-1} = 1, a_{i+2k} = 0$ ($k = 1, 2, \dots, n$), we get $\tilde{\eta}_{i+2k-2} + \tilde{\eta}_{i+2k-1} > \tilde{L}, \tilde{\eta}_{i+2k-1} + \tilde{\eta}_{i+2k} < \tilde{L}$ and $\tilde{b}_{i+2n} = \tilde{c}_i \leq \tilde{\eta}_{i+2n}$. Therefore

$$\begin{aligned}
X_i &< (\tilde{L} - \tilde{c}_{i-1} + \tilde{\eta}_{i+1}) + \sum_{k=1}^n (\tilde{L} - \tilde{\eta}_{i+2k-1} - \tilde{\eta}_{i+2k}) \\
&\quad + \sum_{k=2}^n (\tilde{\eta}_{i+2k-2} + \tilde{\eta}_{i+2k-1} - \tilde{L}) \\
&= 2\tilde{L} - \tilde{c}_{i-1} - \tilde{\eta}_{i+2n} \\
&\leq 2\tilde{L} - \tilde{c}_{i-1} - \tilde{c}_i, \\
X_{i+1} &= X_i - (\tilde{L} - \tilde{c}_{i-1}) \\
&< \tilde{L} - \tilde{c}_i \\
&\leq 2\tilde{L} - \tilde{c}_i - \tilde{c}_{i+1}.
\end{aligned}$$

- Case 2: $\tilde{b}_i = \tilde{c}_i = \tilde{L} - \tilde{\eta}_{i+1}, a_{i+1} = \dots = a_k = 1, a_{k+1} = 0$. (We need not insert zeros. See Fig 2.5)

$$\begin{aligned}
X_i &= (\tilde{L} - \tilde{\eta}_i - \tilde{c}_{i-1}) + \sum_{j=i}^{k-2} (\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L}) + \tilde{\eta}_{k-1} + \tilde{\eta}_k \\
&= \tilde{L} - \tilde{c}_{i-1} + \tilde{\eta}_{i+1} + \sum_{j=i+1}^{k-1} (\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L}).
\end{aligned}$$

From $a_{j+1} = 1$, we get $\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L} \geq 0$ for $j = i+1, \dots, k-1$, and from $\tilde{b}_i = \tilde{c}_i = \tilde{L} - \tilde{\eta}_{i+1}$, we get $\tilde{\eta}_{i+1} = \tilde{L} - \tilde{c}_i$, and

$$\begin{aligned}
X_i &\geq \tilde{L} + (\tilde{L} - \tilde{c}_i) - \tilde{c}_{i-1} \\
&= 2\tilde{L} - \tilde{c}_{i-1} - \tilde{c}_i.
\end{aligned}$$

- Case 3: $\tilde{b}_i = \tilde{c}_i = \tilde{\eta}_i, \tilde{a}_i = \dots = \tilde{a}_l = 0, \tilde{a}_{l+1} = \dots = \tilde{a}_k = 1$. (We need not insert zeros. See Fig 2.6)

$$\begin{aligned}
X_i &= (\tilde{L} - \tilde{\eta}_i - \tilde{c}_{i-1}) + \sum_{j=i}^{l-1} (\tilde{L} - \tilde{\eta}_j - \tilde{\eta}_{j+1}) \\
&\quad + \sum_{j=l}^{k-2} (\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L}) + \eta_{k-1} + \eta_k \\
&= 2\tilde{L} - \tilde{\eta}_i - \tilde{c}_{i-1} + \sum_{j=i}^{l-1} (\tilde{L} - \tilde{\eta}_j - \tilde{\eta}_{j+1}) \\
&\quad + \sum_{j=l}^{k-1} (\tilde{\eta}_j + \tilde{\eta}_{j+1} - L).
\end{aligned}$$

For $j = i, \dots, l-1$, from $a_{j+1} = 0$, we get $\tilde{L} - \tilde{\eta}_j - \tilde{\eta}_{j+1} \geq 0$, and for $j = l, \dots, k-1$, from $a_{j+1} = 1$, we get $\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L} \geq 0$. Recalling $\tilde{\eta}_i = \tilde{c}_i$,

$$X_i \geq 2\tilde{L} - \tilde{c}_{i-1} - \tilde{c}_i.$$

$\tilde{\eta} :$	$\tilde{\eta}_i$	$\tilde{\eta}_{i+1}$	$\tilde{\eta}_{i+2}$	$\dots$	$\tilde{\eta}_{i+2n-1}$	$\tilde{\eta}_{i+2n}$	$\dots$
$\mathbf{a} :$	a_i	$a_{i+1} = 1$	$a_{i+2} = 0$	$\dots$	$a_{i+2n-1} = 1$	$a_{i+2n} = 0$	$\dots$
$\widetilde{\text{bin}}(\boldsymbol{\eta}) :$	$0^{\tilde{L}-\tilde{\eta}_i} 1^{\tilde{\eta}_i}$	$1^{\tilde{\eta}_{i+1}} 0^{\tilde{L}-\tilde{\eta}_{i+1}}$	$0^{\tilde{L}-\tilde{\eta}_{i+2}} 1^{\tilde{\eta}_{i+2}}$	$\dots$	$1^{\tilde{\eta}_{i+2n-1}} 0^{\tilde{L}-\tilde{\eta}_{i+2n-1}}$	$0^{\tilde{L}-\tilde{\eta}_{i+2n}} 1^{\tilde{\eta}_{i+2n}}$	$\dots$
$\tilde{\mathbf{b}} :$	0	0	0	$\dots$	0	$\tilde{b}_{i+2n} = \tilde{c}_i$	$\dots$
$\tilde{\mathbf{c}} :$	$\tilde{c}_i$	$\tilde{c}_{i+1}$	$\tilde{c}_{i+2}$	$\dots$	$\dots$	$\dots$	$\dots$
$\mathbf{d} :$	$0^{\tilde{L}-\tilde{\eta}_i-\tilde{c}_{i-1}} 1^{\tilde{\eta}_i}$	$1^{\tilde{\eta}_{i+1}} 0^{\tilde{L}-\tilde{\eta}_{i+1}}$	$0^{\tilde{L}-\tilde{\eta}_{i+2}} 1^{\tilde{\eta}_{i+2}}$	$\dots$	$1^{\tilde{\eta}_{i+2n-1}} 0^{\tilde{L}-\tilde{\eta}_{i+2n-1}}$	$0^{\tilde{L}-\tilde{\eta}_{i+2n}} 1^{\tilde{\eta}_{i+2n}-\tilde{c}_i}$	$\dots$

Fig 2.4: Case 1 in Theorem 2.2.12: $a_{i+2k-1} = 1, a_{i+2k} = 0$ ($k = 1, 2, \dots, n$) and $(a_{i+2n+1}, a_{i+2n+2}) \neq (1, 0)$

$\tilde{\eta} :$	$\tilde{\eta}_i$	$\tilde{\eta}_{i+1}$	$\dots$	$\tilde{\eta}_{k-2}$	$\tilde{\eta}_{k-1}$	$\tilde{\eta}_k$	$\tilde{\eta}_{k+1}$
$\mathbf{a} :$		1	$\dots$	1	1	1	0
$\widetilde{\text{bin}}(\boldsymbol{\eta}) :$	$0^{\tilde{L}-\tilde{\eta}_i} 1^{\tilde{\eta}_i}$	$0^{\tilde{L}-\tilde{\eta}_{i+1}} 1^{\tilde{\eta}_{i+1}}$	$\dots$	$0^{\tilde{L}-\tilde{\eta}_{k-2}} 1^{\tilde{\eta}_{k-2}}$	$0^{\tilde{L}-\tilde{\eta}_{k-1}} 1^{\tilde{\eta}_{k-1}}$	$1^{\tilde{\eta}_k} 0^{\tilde{L}-\tilde{\eta}_k}$	$0^{\tilde{L}-\tilde{\eta}_{k+1}} 1^{\tilde{\eta}_{k+1}}$
$\tilde{\mathbf{b}} :$	$\tilde{c}_i = \tilde{L} - \tilde{\eta}_{i+1}$	$\tilde{c}_{i+1} = \tilde{L} - \tilde{\eta}_{i+2}$	$\dots$	$\tilde{c}_{k-2} = \tilde{L} - \tilde{\eta}_{k-1}$	0	0	$\tilde{c}_{k-1}$
$\tilde{\mathbf{c}} :$	$\tilde{c}_i$	$\tilde{c}_{i+1}$	$\dots$	$\tilde{c}_{k-2}$	$\tilde{c}_{k-1}$	$\tilde{c}_k$	
$\mathbf{d} :$	$0^{\tilde{L}-\tilde{\eta}_i-\tilde{c}_{i-1}} 1^{\tilde{\eta}_i+\tilde{\eta}_{i+1}-\tilde{L}}$	$1^{\tilde{\eta}_{i+1}+\tilde{\eta}_{i+2}-\tilde{L}}$	$\dots$	$1^{\tilde{\eta}_{k-2}+\tilde{\eta}_{k-1}-\tilde{L}}$	$1^{\tilde{\eta}_{k-1}}$	$1^{\tilde{\eta}_k} 0^{\tilde{L}-\tilde{\eta}_k}$	$0^{\tilde{L}-\tilde{\eta}_{k+1}} 1^{\tilde{\eta}_{k+1}-\tilde{c}_{k-1}}$

Fig 2.5: Case 2 in Theorem 2.2.12: $\tilde{b}_i = \tilde{c}_i = \tilde{L} - \tilde{\eta}_{i+1}$, $a_{i+1} = \dots = a_k = 1, a_{k+1} = 0$

$$\begin{array}{cccccccccc}
\tilde{\eta} : & \tilde{\eta}_i & \tilde{\eta}_{i+1} & \cdots & \tilde{\eta}_l & \tilde{\eta}_{l+1} & \cdots & \tilde{\eta}_{k-1} & \tilde{\eta}_k & \tilde{\eta}_{k+1} \\
\mathbf{a} : & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & 0 \\
\widetilde{\text{bin}}(\boldsymbol{\eta}) : & 0^{\tilde{L}-\tilde{\eta}_i} 1^{\tilde{\eta}_i} & 0^{\tilde{L}-\tilde{\eta}_{i+1}} 1^{\tilde{\eta}_{i+1}} & \cdots & 0^{\tilde{L}-\tilde{\eta}_l} 1^{\tilde{\eta}_l} & 0^{\tilde{L}-\tilde{\eta}_{l+1}} 1^{\tilde{\eta}_{l+1}} & \cdots & 0^{\tilde{L}-\tilde{\eta}_{k-1}} 1^{\tilde{\eta}_{k-1}} & 1^{\tilde{\eta}_k} 0^{\tilde{L}-\tilde{\eta}_k} & 0^{\tilde{L}-\tilde{\eta}_{k+1}} 1^{\tilde{\eta}_{k+1}} \\
\tilde{\mathbf{b}} : & \tilde{c}_i = \tilde{\eta}_i & \tilde{c}_{i+1} = \tilde{\eta}_{i+1} & \cdots & \tilde{c}_l = \tilde{L} - \tilde{\eta}_{l+1} & \tilde{c}_{l+1} = \tilde{L} - \tilde{\eta}_{l+2} & \cdots & 0 & 0 & \tilde{c}_{k-1} \\
\tilde{\mathbf{c}} : & \tilde{c}_i & \tilde{c}_{i+1} & \cdots & \tilde{c}_l & \tilde{c}_{l+1} & \cdots & \tilde{c}_{k-1} & \tilde{c}_k & \tilde{c}_{k+1} \\
\tilde{\mathbf{d}} : & 0^{\tilde{L}-\tilde{\eta}_i-\tilde{c}_{i-1}} & 0^{\tilde{L}-\tilde{\eta}_i-\tilde{\eta}_{i+1}} & \cdots & 0^{\tilde{L}-\tilde{\eta}_{l-1}-\tilde{\eta}_l} 1^{\tilde{\eta}_l+\tilde{\eta}_{l+1}-\tilde{L}} & 1^{\tilde{\eta}_{l+1}+\tilde{\eta}_{l+2}-\tilde{L}} & \cdots & 1^{\tilde{\eta}_{k-1}} & 1^{\tilde{\eta}_k} 0^{\tilde{L}-\tilde{\eta}_k} & 0^{\tilde{L}-\tilde{\eta}_{k+1}} 1^{\tilde{\eta}_{k+1}-\tilde{c}_{k-1}}
\end{array}$$

Fig 2.6: Case 3 in Theorem 2.2.12: $\tilde{b}_i = \tilde{c}_i = \tilde{\eta}_i$, $\tilde{a}_i = \cdots = \tilde{a}_l = 0$, $\tilde{a}_{l+1} = \cdots = \tilde{a}_k = 1$

□

Theorem 2.2.13. (Surjectivity of β_L) For $\mathbf{c} \in \mathcal{S}_{L,M}^{(b)}$, $\mathbf{d} \in \mathcal{S}_L^{(f)}$, $(\beta_L \circ \beta_L^{-1})(\mathbf{c}, \mathbf{d}) = (\mathbf{c}, \mathbf{d})$.

Proof. For $\mathbf{c} \in \mathcal{S}_{L,M}^{(b)}$ and $\mathbf{d} \in \mathcal{S}_L^{(f)}$, let $\boldsymbol{\eta} = \beta_L^{-1}(\mathbf{c}, \mathbf{d})$, and let $\mathbf{a}$ be the soliton flag sequence calculated from $\boldsymbol{\eta}$. Let Y_j be the number of consecutive 0s in soliton sequence $\mathbf{d}$ from $d_{k(j)}$ (If $d_{k(j)} = 1$, let $Y_j = 0$). We will prove the following four statements:

- (1) If $\tilde{L} \leq X_j < 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$, then $a_{j+1} = 1$ and $a_{j+2} = 0$.
- (2) If $X_j < \tilde{L}$, then $a_j = 1$ and $a_{j+1} = 0$.
- (3) If $X_j \geq 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$ and $Y_j < \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$, then $a_{j+1} = a_{j+2} = 1$.
- (4) If $X_j \geq 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$ and $Y_j \geq \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$, then $a_j = a_{j+1} = 0$.

In this proof, let D_k be the subsequence of $\mathbf{d}$ corresponding to the k th box of $\boldsymbol{\eta}$.

- (1) ($\tilde{L} \leq X_j < 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$. Shown in Fig 2.7.)

From the algorithm of reconstruction, we have $\tilde{c}_j^{(j+1)} = 0$ and $\tilde{c}_{j-1}^{(j+1)} = \tilde{c}_{j-1}^{(j)}$. Since $X_j \geq \tilde{L} - \tilde{c}_{j-1}^{(j+1)}$, X_{j+1} becomes

$$\begin{aligned}
X_{j+1} &= X_j - (\tilde{L} - \tilde{c}_{j-1}^{(j+1)}) \\
&< (2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}) - (\tilde{L} - \tilde{c}_{j-1}^{(j)}) \\
&= \tilde{L} - \tilde{c}_j^{(j)} \\
&\leq \tilde{L} - \tilde{c}_j^{(j)} + \tilde{c}_j^{(j)} - \tilde{c}_j^{(j+1)} + (\tilde{L} - \tilde{c}_{j+1}^{(j+1)}) \\
&= 2\tilde{L} - \tilde{c}_j^{(j+1)} - \tilde{c}_{j+1}^{(j+1)},
\end{aligned}$$

and we get $\tilde{c}_{j+1}^{(j+2)} = 0$. First, we prove $a_{j+1} = 1$. Let A denote the number of 1s in D_j .

- (i) If $A < \tilde{L} - \tilde{c}_{j-1}^{(j)}$: Since the number of consecutive 1s in $\mathbf{d}$ is larger than $\tilde{L}$, we get $A + X_j - (\tilde{L} - \tilde{c}_{j-1}^{(j)}) > \tilde{L}$. Then,

$$\begin{aligned}\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L} &= A + X_j - \tilde{L} + \tilde{c}_{j-1}^{(j)} - \tilde{L} \\ &> 0,\end{aligned}$$

and we get $a_{j+1} = 1$.

- (ii) If $A = \tilde{L} - \tilde{c}_{j-1}^{(j)}$: Let i denote the maximum index less than j such that the left end of consecutive 1s in $\mathbf{d}$ is at D_i . From (i), $a_{i+1} = 1$. For $k = i + 2, \dots, j + 1$,

$$\begin{aligned}\tilde{\eta}_{k-1} + \tilde{\eta}_k - \tilde{L} &= (\tilde{L} - \tilde{c}_{k-2}^{(j)}) + (\tilde{L} - \tilde{c}_{k-1}^{(j)}) - \tilde{L} \\ &= \tilde{L} - \tilde{c}_{k-2}^{(j)} - \tilde{c}_{k-1}^{(j)} \\ &\geq 0,\end{aligned}$$

and we get $a_k = 1$.

Next, we prove $a_{j+2} = 0$. Let B denote the number of 1s in D_{j+2} .

- (i) If $B > 0$: Since the number of consecutive 0s in $\mathbf{d}$ is larger than $\tilde{L}$, we get

$$\begin{aligned}(\tilde{L} - (X_j - \tilde{L} + \tilde{c}_{j-1}^{(j)})) + (\tilde{L} - \tilde{c}_j^{(j)} - B) \\ = 2\tilde{L} - X_j - B - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)} \\ > 0.\end{aligned}$$

Here, using $\tilde{c}_{j+2}^{(j+3)} = \tilde{c}_j^{(j)}$ or 0, we get

$$\begin{aligned}\tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - \tilde{L} &= (X_j - \tilde{L} + \tilde{c}_{j-1}^{(j)}) + (B + \tilde{c}_{j+2}^{(j+3)}) - \tilde{L} \\ &= -2\tilde{L} + X_j + B + \tilde{c}_{j-1}^{(j)} + \tilde{c}_{j+2}^{(j+3)} \\ &< 0,\end{aligned}$$

and $a_{j+2} = 0$.

- (ii) If $B = 0$:

$$\begin{aligned}\tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - \tilde{L} &= (X_j - \tilde{L} + \tilde{c}_{j-1}^{(j)}) + \tilde{c}_{j+2}^{(j+3)} - \tilde{L} \\ &< (2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}) + \tilde{c}_{j-1}^{(j)} + \tilde{c}_{j+2}^{(j+3)} - 2\tilde{L} \\ &= \tilde{c}_{j+2}^{(j+3)} - \tilde{c}_j^{(j)} \\ &\leq 0\end{aligned}$$

and $a_{j+2} = 0$.

(2) ($X_j < \tilde{L}$. Shown in Fig 2.8.)

If we assume that $X_{j-1} \geq 2\tilde{L} - \tilde{c}_{j-2}^{(j-1)} - \tilde{c}_{j-1}^{(j-1)}$ and $\tilde{c}_{j-1}^{(j)} = \tilde{c}_{j-1}^{(j-1)}$, then

$$\begin{aligned} X_{j-1} &= (\tilde{L} - \tilde{c}_{j-2}^{(j)} - \tilde{c}_{j-1}^{(j)}) + X_j \\ &< \tilde{L} - \tilde{c}_{j-2}^{(j-1)} - \tilde{c}_{j-1}^{(j-1)} + \tilde{L} \\ &= 2\tilde{L} - \tilde{c}_{j-2}^{(j-1)} - \tilde{c}_{j-1}^{(j-1)}. \end{aligned}$$

Therefore, we get $X_{j-1} < 2\tilde{L} - \tilde{c}_{j-2}^{(j-1)} - \tilde{c}_{j-1}^{(j-1)}$ and from Case (1) in Theorem 2.2.13, $a_j = 1$ and $a_{j+1} = 0$.

(3) ($X_j \geq 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$ and $Y_j < \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$. Shown in Fig 2.9.)

First,

$$\begin{aligned} \tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L} &= (\tilde{L} - \tilde{c}_{j-1}^{(j)} - Y_j) + (\tilde{\eta}_{j+1} = \tilde{L} - \tilde{c}_j^{(j)}) - \tilde{L} \\ &= \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)} - Y_j \\ &> 0, \end{aligned}$$

and we get $a_{j+1} = 1$. Next, we prove $a_{j+2} = 1$. Let B be the number of 1s in D_{j+2} .

(i) If $B < \tilde{L} - \tilde{c}_{j+1}^{(j+2)} - \tilde{c}_{j+2}^{(j+3)}$: Since

$$\begin{aligned} B &= X_j - (\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}) - (\tilde{L} - \tilde{c}_j^{(j)} - \tilde{c}_{j+1}^{(j+2)}) \\ &= \{X_j - (2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)})\} + \tilde{c}_j^{(j)} + \tilde{c}_{j+1}^{(j+2)} \\ &\geq \tilde{c}_j^{(j)} + \tilde{c}_{j+1}^{(j+2)}, \end{aligned}$$

we obtain

$$\begin{aligned} \tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - \tilde{L} &= B - \tilde{c}_j^{(j)} - \tilde{c}_{j+2}^{(j+3)} \\ &\geq \tilde{c}_{j+1}^{(j+2)} + \tilde{c}_{j+2}^{(j+3)} \\ &\geq 0, \end{aligned}$$

and then, $a_{j+2} = 1$.

(ii) If $B = \tilde{L} - \tilde{c}_{j+1}^{(j+2)} - \tilde{c}_{j+2}^{(j+3)}$:

$$\begin{aligned} \tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - \tilde{L} &= (\tilde{L} - \tilde{c}_j^{(j)}) + (\tilde{L} - \tilde{c}_{j+1}^{(j+2)}) - \tilde{L} \\ &= \tilde{L} - \tilde{c}_j^{(j+2)} - \tilde{c}_{j+1}^{(j+2)} \\ &\geq 0 \end{aligned}$$

and $a_{j+2} = 1$.

(4) ($X_j \geq 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$ and $Y_j \geq \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}$. Shown in Fig 2.10.)

First, we prove $a_j = 0$.

- (i) If $X_{j-1} \leq \tilde{L} - \tilde{c}_{j-2}^{(j)} - \tilde{c}_{j-1}^{(j)}$: Since $X_{j-1} \leq \tilde{L}$ and Case (2) in Theorem 2.2.13, we get $a_{j-1} = 1, a_j = 0$.
- (ii) If $X_{j-1} > \tilde{L} - \tilde{c}_{j-2}^{(j)} - \tilde{c}_{j-1}^{(j)}$: Let i denote the maximum index less than j such that the left end of consecutive 0s in $\mathbf{d}$ is at D_i . From (i), $a_i = 1, a_{i+1} = 0$. For $k = i + 2, \dots, j$,

$$\begin{aligned}\tilde{\eta}_{k-1} + \tilde{\eta}_k - \tilde{L} &= \tilde{c}_{k-1}^{(j)} + \tilde{c}_k^{(j)} - \tilde{L} \\ &\leq 0,\end{aligned}$$

and we get $a_k = 0$.

Next, we prove $a_{j+1} = 0$. Let B the number of 1s in D_{j+1} .

- (i) If $B > 0$: Since Y_j is the number of consecutive 0s in soliton sequence $\mathbf{d}$ from $d_{k(j)}$, we have

$$Y_j + B = (\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)}) + (\tilde{L} - \tilde{c}_j^{(j)} - \tilde{c}_{j+1}^{(j+2)}).$$

Using this, we obtain

$$\begin{aligned}\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L} &= B + \tilde{c}_j^{(j)} + \tilde{c}_{j+1}^{(j+2)} - \tilde{L} \\ &= \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)} - Y_j \\ &\leq 0.\end{aligned}$$

- (ii) If $B = 0$:

$$\begin{aligned}\tilde{\eta}_j + \tilde{\eta}_{j+1} - \tilde{L} &= \tilde{c}_j^{(j)} + \tilde{c}_{j+1}^{(j+2)} - \tilde{L} \\ &\leq 0,\end{aligned}$$

and we obtain $a_j = a_{j-1} = 0$.

□

$\tilde{\mathbf{c}}^{(j)}:$	$\tilde{c}_{j-1}^{(j)}$	$\tilde{c}_j^{(j)}$		
$\tilde{\mathbf{c}}^{(j+1)}:$	$\tilde{c}_{j-1}^{(j+1)} = \tilde{c}_{j-1}^{(j)}$	$\tilde{c}_j^{(j+1)} = 0$	$\tilde{c}_{j+1}^{(j+1)} = \tilde{c}_j^{(j)}$	
$\tilde{\mathbf{c}}^{(j+2)}:$	$\tilde{c}_{j-1}^{(j+2)} = \tilde{c}_{j-1}^{(j)}$	$\tilde{c}_j^{(j+2)} = 0$	$\tilde{c}_{j+1}^{(j+2)} = 0$	$\tilde{c}_{j+2}^{(j+2)} = \tilde{c}_j^{(j)}$
		$\underbrace{\tilde{L} - \tilde{c}_{j-1}^{(j)}}_{A}$		$\underbrace{\tilde{L} - \tilde{c}_j^{(j)}}_{B}$
$\tilde{\mathbf{d}}:$		$\underbrace{0 \dots 0 1 \dots 1}_A$	$\underbrace{1 \dots 1}_{X_j - (\tilde{L} - \tilde{c}_{j-1}^{(j)})} \quad 0 \dots 0$	$\underbrace{0 \dots 0 1 \dots 1}_B$
$\tilde{\boldsymbol{\eta}}:$		$\tilde{\eta}_j = A + \tilde{c}_j^{(j+1)}$	$\tilde{\eta}_{j+1} = X_j - \tilde{L} + \tilde{c}_{j-1}^{(j)} + \tilde{c}_{j+1}^{(j+2)}$	$\tilde{\eta}_{j+2} = B + \tilde{c}_{j+2}^{(j+3)}$

Fig 2.7: Case (1) in Theorem 2.2.13

$$\begin{array}{lcl}
\tilde{\mathbf{c}}^{(j)}: & \tilde{c}_{j-1}^{(j)} & \tilde{c}_j^{(j)} \\
\tilde{\mathbf{c}}^{(j+1)}: & \tilde{c}_{j-1}^{(j+1)} = \tilde{c}_{j-1}^{(j)} & \tilde{c}_j^{(j+1)} = 0 \\
& \tilde{L} - \tilde{c}_{j-2}^{(j)} - \tilde{c}_{j-1}^{(j)} & \\
\tilde{\mathbf{d}}: & \underbrace{0 \cdots 0 1 \cdots 1}_A & \underbrace{1 \cdots 1 0 \cdots 0}_{\substack{X_j \quad \tilde{L} - X_j}} \\
\tilde{\eta}: & \tilde{\eta}_{j-1} = A + \tilde{c}_{j-1}^{(j)} & \tilde{\eta}_j = X_j
\end{array}$$

Fig 2.8: Case (2) in Theorem 2.2.13

$$\begin{array}{lclcl}
\tilde{\mathbf{c}}^{(j)}: & \tilde{c}_j^{(j)} & & & \\
\tilde{\mathbf{c}}^{(j+1)}: & \tilde{c}_j^{(j+1)} = \tilde{c}_j^{(j)} & \tilde{c}_{j+1}^{(j+1)} & & \\
\tilde{\mathbf{c}}^{(j+2)}: & \tilde{c}_j^{(j+2)} = \tilde{c}_j^{(j)} & \tilde{c}_{j+1}^{(j+2)} & \tilde{c}_{j+2}^{(j+2)} & \\
\tilde{\mathbf{c}}^{(j+3)}: & \tilde{c}_j^{(j+3)} = \tilde{c}_j^{(j)} & \tilde{c}_{j+1}^{(j+3)} = \tilde{c}_{j+1}^{(j+2)} & \tilde{c}_{j+2}^{(j+3)} & \\
& \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)} & \tilde{L} - \tilde{c}_j^{(j)} - \tilde{c}_{j+1}^{(j+2)} & \tilde{L} - \tilde{c}_{j+1}^{(j+2)} - \tilde{c}_{j+2}^{(j+3)} & \\
\tilde{\mathbf{d}}: & \underbrace{0 \cdots 0 1 \cdots 1}_{Y_j} & \underbrace{1 \cdots 1} & \underbrace{1 \cdots 1 0 \cdots 0}_B & \\
\tilde{\eta}: & \tilde{\eta}_j = \tilde{L} - \tilde{c}_{j-1}^{(j)} - Y_j & \tilde{\eta}_{j+1} = \tilde{L} - \tilde{c}_j^{(j)} & \tilde{\eta}_{j+2} = B + \tilde{c}_{j+2}^{(j+3)} &
\end{array}$$

Fig 2.9: Case (3) in Theorem 2.2.13

$$\begin{array}{lclcl}
\tilde{\mathbf{c}}^{(j)}: & \tilde{c}_{j-1}^{(j)} & \tilde{c}_j^{(j)} & & \\
\tilde{\mathbf{c}}^{(j+1)}: & \tilde{c}_{j-1}^{(j+1)} = \tilde{c}_{j-1}^{(j)} & \tilde{c}_j^{(j+1)} = \tilde{c}_j^{(j)} & \tilde{c}_{j+1}^{(j+1)} & \\
\tilde{\mathbf{c}}^{(j+2)}: & \tilde{c}_{j-1}^{(j+2)} = \tilde{c}_{j-1}^{(j)} & \tilde{c}_j^{(j+2)} = \tilde{c}_j^{(j)} & \tilde{c}_{j+1}^{(j+2)} & \\
& \tilde{L} - \tilde{c}_{j-2}^{(j)} - \tilde{c}_{j-1}^{(j)} & \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_j^{(j)} & \tilde{L} - \tilde{c}_j^{(j)} - \tilde{c}_{j+1}^{(j+2)} & \\
\tilde{\mathbf{d}}: & \underbrace{1 \cdots 1 0 \cdots 0}_A & \underbrace{0 \cdots 0} & \underbrace{0 \cdots 0 1 \cdots 1}_B & \\
\tilde{\eta}: & \tilde{\eta}_{j-1} = A + \tilde{c}_{j-1}^{(j)} & \tilde{\eta}_j = \tilde{c}_j^{(j)} & \tilde{\eta}_{j+1} = B + \tilde{c}_{j+1}^{(j+2)} &
\end{array}$$

Fig 2.10: Case (4) in Theorem 2.2.13

From Theorems 2.2.12 and 2.2.13, the bijectivity of the map β_L is proved, and we have the following theorem.

Theorem 2.2.14. A BBS state with a box capacity of L can be decomposed into a soliton sequence and a background sequence.

$$\mathcal{S}_{L,M} = \mathcal{S}_{L,M}^{(b)} \otimes \mathcal{S}_L^{(f)}. \quad (2.49)$$

Chapter 3

Arc lines for the box–ball system related to the ultradiscrete relativistic Toda equation

In 1990, Ruijsenaars proposed the relativistic Toda equation, introducing a parameter of the speed of light, as a Poincaré invariant generalization of Toda equation [24]. In 1997, Kharchev et al. proved that the generalization of the relativistic Toda equation is derived from a special reduction of two-dimensional Toda lattice hierarchy [12]. They discussed the integrability of the relativistic Toda equation from the perspectives of the Lax pair, orthogonal polynomials, and the tau function. The discrete version of the relativistic Toda equation has also been studied using the Lax matrix or Hirota’s bilinear form [21, 25]. The ultradiscretization of the discrete relativistic Toda equation is interpreted as the time evolution of a soliton cellular automaton [10]. Note that the parameter of the speed of light is eliminated in the variable transformation used to discretize the equation, so the discrete/ultradiscrete relativistic Toda equation does not contain any relativistic factors. In this chapter, we introduce 10-arc lines for this automaton and obtain the conserved quantities corresponding to soliton sizes.

3.1 Time evolution

The *ultradiscrete relativistic Toda (urToda) equation* is given by

$$F_j^t = \min(Q_j^t, E_j^t), \quad (3.1)$$

$$Q_j^{t+1} = Q_{j-1}^t + F_j^t - F_{j-1}^t, \quad (3.2)$$

$$E_j^{t+1} = E_j^t + F_{j+1}^t - F_j^t. \quad (3.3)$$

As in Section 1.1, the urToda equation with $E_0^N = +\infty$ is regarded as a cellular automaton by interpreting Q_j as the length of a j th block of contiguous balls and E_j as the number of contiguous empty boxes between j th and $(j + 1)$ th blocks. We call it *relativistic box–ball system (rBBS)*.

The time evolution is described by the following rules:

- (i) Move $F_j^t = \min(Q_j^t, E_j^t)$ balls in the j th block to the nearest empty boxes on the right for $j = 1, 2, \dots, N$.
- (ii) Move the boxes that have the remaining $Q_j^t - F_j^t$ balls in the j th block to just before the $(j + 1)$ th block for $j = 1, 2, \dots, N$.

Example 3.1.1. $\eta^t = 111110000111111001111011000\dots$

- (i) $\rightarrow 00001\underline{1111}001\underline{1111}\underline{10111}\underline{00110}\dots$

The underlined ‘1’s represent the balls moved in step (i). Here, $F_1^t = 4, F_2^t = 2, F_3^t = 1, F_4^t = 2$.

- (ii) $\rightarrow \eta^{t+1} = T(\eta^t) = 0000\underline{1111}001\underline{110}\overline{1111}\underline{100}\overline{111}\underline{110}\dots$

The overlined ‘1’s represent the balls moved in step (ii).

The urToda equation (3.1)–(3.3) coincides with the uToda equation (1.7)–(1.9) by taking oblique coordinates as

$$\hat{Q}_j^t = F_j^{t+j-1}, \quad \hat{E}_j^t = E_j^{t+j}, \quad \hat{D}_j^t = Q_j^{t+j-1},$$

and rewriting independent variable $(t - j)$ as t . The initial value problem of the urToda equation can be solved by transforming it into the uToda equation. However, this transformation requires computing the time evolution as many times as the number of solitons. In this chapter, we aim to analyze the rBBS directly without transforming it into the uToda equation, with a view toward applications including probabilistic analysis when the number of balls is infinite.

3.2 10-arc lines for the box–ball system state and soliton configuration

3.2.1 10-arc lines

For the original BBS, the 10-arc line that connects each ball to an empty box was introduced by Yoshihara et al. [40] and used to linearize its time evolution [13] (see Chapter 1). Because the arc lines corresponding to balls that move together with boxes always intersect with other arc lines, it is impossible to distinguish solitons using arc lines drawn according to the same rules as the original BBS. Therefore, we introduce new rules for drawing arc lines. In the case of the rBBS, we define the 10-arc lines for a binary sequence η that describe the time evolution by the following rules:

- Connect the k th ‘1’ in Q_j to:

- the k th ‘0’ in E_j if $1 \leq k \leq F_j$ and
- the space between Q_{j+1} and E_{j+1} if $k > F_j$.
- The order of drawing arcs is as follows:
 - (a) (i) Draw an arc from the leftmost ‘1’ that has not been connected by an arc.
 - (ii) Draw an arc from the closest ‘1’ on the right of the ‘0’ or the space connected by an arc just before that has not been connected by an arc.
 - (iii) Repeat (ii) until no ‘1’ is on the right.
 - (b) Repeat (a) until all ‘1’s are connected by arcs.

An arc has *depth* i if it is drawn during the i th repetition of (a). The rule of constructing 10-arc lines above is similar to the “seat configuration” method for the original BBS proposed by Mucciconi et al. [22], which is applied to analysis from the perspective of probability theory [26].

The leftmost ‘1’ in each Q_j is connected by an arc of depth 1. We say the leftmost ‘1’ in Q_j and the arc connecting it belongs to the j th *soliton*. Consider the interval from the ‘1’ to the ‘0’ or the space connected by the arc of depth i belonging to the j th soliton. If a ‘1’ connected to the arc of depth $(i + 1)$ is positioned between the ‘1’ and ‘0’ connected to the arc of depth i belonging to the j th soliton, then we say that the arc of depth $(i + 1)$ belongs to the j th soliton.

Example 3.2.1. For the binary sequence in Fig 3.1, there are four solitons: one of size 6 (shown by solid lines), one of size 3 (long dashed lines), one of size 1 (short dashed lines), and one of size 2 (dotted lines). The vertical lines represent the positions where the boxes with balls are inserted.

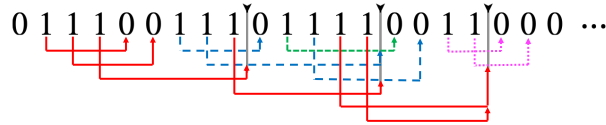


Fig 3.1: 10-arc lines for rBBS.

3.2.2 Soliton configuration

Consider a pair $(M = (M_1, M_2, \dots, M_N), p = (p_1, p_2, \dots, p_N))$ where $M_i = \{m_{i,1} \leq m_{i,2} \leq \dots \leq m_{i,l_i}\} \in \mathbb{Z}_{\geq 0}^{l_i}$, $p_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, \dots, N$, and $p_1 < p_2 < \dots < p_N$. We depict a pair (M, p) as a collection of boxes arranged in left-justified rows and integers written in the boxes and on the left of the rows: the length of the i th row is $l_i = \#M_i$, the number $m_{i,j}$ is written in the box of the i th row and j th column, and p_i is written on the left of the i th row as in Fig 3.2. For convenience in the discussions in Section 3.3, the ‘ $(i - 1)$ ’s up to $(p_{i+1} - p_i)$ th counting from the left are underlined. A pair (M, p) is called a *soliton configuration* if it satisfies the following three conditions:

- (A) $m_{i,1} = i - 1$ for $i = 1, 2, \dots, N$;
- (B) The ' i 's are placed from left to right, under the ' $(i - 1)$'s up to $(p_{i+1} - p_i)$ th counting from left to right or ' i' 's for $i' \leq i - 2$, where $p_{N+1} = \infty$; and
- (C) $m_{i,j} > m_{i',j+1}$ for $i > i'$.

Let n_i be the total number of ' i 's in the sets $M_1, M_2, \dots, M_N$, that is, $n_i = \#\{(k, \ell) \mid m_{k,\ell} = i\}$. From the conditions above, we easily find that the soliton configuration is uniquely determined by a pair of sequences $(\{n_i\}_{i=0}^{N-1}, \{p_i\}_{i=1}^N)$.

We give an example of constructing the soliton configuration in Fig 3.2 from the corresponding sequences ($n_0 = 3, n_1 = 3, n_2 = 4, n_3 = 2$) and ($p_1 = 1, p_2 = 3, p_3 = 4, p_4 = 6$). Three '0's must be placed in the first row as $m_{1,1} = m_{1,2} = m_{1,3} = 0$ and two '0's are underlined because $p_2 - p_1 = 2$. Next, we place three '1's from left to right. Because '0' in the third column is not underlined, '1' cannot be placed in the third column. Four '2's are placed in the same way. Two '3's are placed in the first and second columns. Because condition (C) does not allow $m_{4,1} = m_{3,2} = 3$, the '3' in the second column is placed in the fourth row.

1	<u>0</u>	<u>0</u>	0	1	2	2
3	<u>1</u>	<u>1</u>	<u>2</u>			
4	<u>2</u>					
6	<u>3</u>	<u>3</u>				

Fig 3.2: A soliton configuration, $(M = (\{0, 0, 0, 1, 2, 2\}, \{1, 1, 2\}, \{2\}, \{3, 3\}), p = (1, 3, 4, 6))$.

Proposition 3.2.2. Consider 10-arc lines for given $(E_0, Q_1, E_1, Q_2, \dots, Q_N, E_N = \infty)$. For every '1' connected by the arc of depth i belonging to the j th soliton, let $m_{j,i} = k - 1$ if the '1' belongs to Q_k . Let $p_j = \sum_{k=0}^{j-1} E_k$ for $j = 1, 2, \dots, N$. Then, (M, p) is a soliton configuration.

Noting $E_j = p_{j+1} - p_j$, we can easily check that the above proposition holds. The j th row corresponds to the j th soliton, while the i th column corresponds to the '1's connected by arcs of depth i . The underlined number $\underline{m}_{i,j}$ corresponds to the ball that moves to the adjacent block of vacant boxes, and the not-underlined number $m_{i,j}$ corresponds to the ball that moves with the boxes. The soliton configuration in Fig 3.2 is based on the arcs in Fig 3.1.

3.3 Conserved quantities of soliton sizes

We now prove that the size l_i of each soliton is conserved over the time evolution. First, we define the time evolution T_{sc} on a soliton configuration

$$(M^{t+1}, p^{t+1}) = T_{sc}(M^t, p^t)$$

as follows:

- (i) (*Dropping rule*) If there exist integers i, k, j such that $k > i$, $m_{k,j-1} = m_{i,j} + 1$, $m_{k,j-1}$ is underlined, and $m_{i,j}$ is not underlined, then drop the boxes to the right of the $(j-1)$ th column in the i th row to the k th row.
- (ii) Update p_j by $p_j^{t+1} = p_j^t + \min(n_{j-1}^t, p_{j+1}^t - p_j^t)$, where n_i is the total number of ' i 's that appear in the sets $M_1, M_2, \dots, M_N$.
- (iii) Replace every number m that is not underlined with $m + 1$. (Underlined numbers are not adjusted in this step.)
- (iv) Redraw the underlines to $m_{i,j}^{t+1}$ according to the new soliton configuration (M^{t+1}, p^{t+1}) .

Note that the value of $\min(n_{j-1}^t, p_{j+1}^t - p_j^t)$ in step (ii) means the number of underlined $(j-1)$ s in the soliton configuration. The time evolution for the example in Fig 3.2 is shown in Fig 3.3.

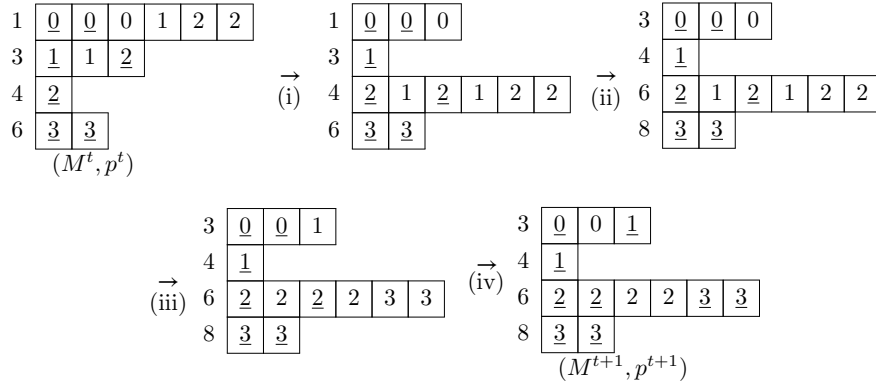


Fig 3.3: Time evolution of the soliton configuration.

The following proposition shows that the dropping rule can properly apply to any soliton configuration.

Proposition 3.3.1. (Validity of the dropping rule.) Let (M, p) be a soliton configuration. If there exist integers i, k, j such that $k > i$, $m_{k,j-1} = m_{i,j} + 1$, $m_{k,j-1}$ is underlined, and $m_{i,j}$ is not underlined, then the lengths l_i of the rows satisfy

$$l_{i'} < l_k = j - 1 \quad (i < i' < k).$$

Proof. First, because (M, p) satisfies condition (C) and the numbers in each column are different and increasing, we find $l_{i'} < j - 1$ for $i < i' < k$.

Next, we prove $l_k = j - 1$ by contradiction. Suppose $l_k \geq j$. Then, condition (B) implies $m_{k,j} = m_{k,j-1} + 1$. Because $m_{k,j-1}$ is underlined, condition (B) shows that the same value as $m_{k,j}$ is placed under $m_{k,j-1}$. This contradicts condition (C). $\square$

Theorem 3.3.2. The time evolution of the rBBS is equivalent to T_{sc} on the soliton configuration.

Proof. Using the time evolution equations (3.1) and (3.3), we obtain

$$\begin{aligned}
p_j^{t+1} &= p_j^t + \min(n_{j-1}^t, p_{j+1}^t - p_j^t) \\
&= p_j^t + \min(Q_j^t, E_j^t) \\
&= p_j^t + F_j^t \\
&= p_j^t + \sum_{k=0}^{j-1} (F_{k+1}^t - F_k^t) \\
&= p_j^t + \sum_{k=0}^{j-1} (E_k^{t+1} - E_k^t) \\
&= \sum_{k=0}^{j-1} E_k^{t+1}.
\end{aligned}$$

Because the number of underlined ‘ $(j-1)$ ’s at t is F_j^t and the number of ‘ $(j-2)$ ’s not underlined at t is $Q_{j-1}^t - F_{j-1}^t$, we can use equation (3.2) to show that the number of ‘ $(j-1)$ ’s at $(t+1)$ becomes

$$\begin{aligned}
n_{j-1}^{t+1} &= F_j^t + (Q_{j-1}^t - F_{j-1}^t) \\
&= Q_j^{t+1}.
\end{aligned}$$

Because the soliton configuration is uniquely determined by $(\{n_j\}_{j=0}^{N-1}, \{p_j\}_{j=1}^N)$, the theorem holds. $\square$

Theorem 3.3.2 and the dropping rule in the time evolution on a soliton configuration lead to the following corollary.

Corollary 3.3.3. The lengths l_i of the solitons are conserved through the time evolution of the rBBS.

Considering the state after a sufficient amount of time has passed and the interactions between solitons have completely ended, we see that each l_i corresponds to the length of each soliton.

We have also developed a method to determine the rigging of each soliton from the soliton configuration, and we can prove that the rigging increases linearly under the time evolution of the rBBS. In the soliton configuration, the data is arranged in the order in which solitons appear in the state of the rBBS. While we have confirmed that the original order of solitons can be reconstructed from the unordered set of pairs of soliton length and rigging, a formal proof has not yet been established.

Chapter 4

Solitons in 3-state Mealy automata

In 2014, Tsujimoto et al. found that the original BBS can be represented as a Mealy automaton with infinitely many states [30]. The BBS with a finite carrier capacity and the BBS with a box capacity larger than one can also be described as Mealy automata by either limiting the number of states or increasing the size of the alphabet. It is worth exploring the interrelation between the BBS and Mealy automata. We consider Mealy automata from the perspective of the BBS. We search for the Mealy automata interpreted as a soliton cellular automata based on certain *key properties* of the BBS automaton. In this chapter, we first search for the soliton Mealy automata where the alphabet size is two and the state set size is one, two, and three. Following this, we discuss the linearizability of these automata.

4.1 Mealy Automata

The definition of a Mealy automaton is given in Chapter 1.

In this chapter, the automaton (Q, S, φ, ψ) is called the automaton of the *type* $(|Q|, |S|)$. For a given pair of (Q, S) , φ and ψ are uniquely determined by the pair of $|Q| \times |S|$ matrices whose entries are in Q and S , respectively. Thus the total number of automata of the type $(|Q|, |S|)$ is given by $(|Q| \times |S|)^{|Q| \times |S|}$.

Assume $Q = \{q_0, q_1, \dots, q_{m-1}\}$ and $S = \{0, 1, \dots, n-1\}$. The automaton $\mathcal{A} = (Q, S, \varphi, \psi)$ is determined by a quadruple of integers $[m, n, k, l]$ as follows:

- Let $\varphi(q_i, j) := q_{f(i, j)}$ and $\psi(q_i, j) := g(i, j)$ where $f(i, j)$ and $g(i, j)$ are the elements of two $m \times n$ matrices, which are given by the m -ary expansion of k and the n -ary expansion of l , as

$$(f(i, j))_{0 \leq i < m, 0 \leq j < n}, (g(i, j))_{0 \leq i < m, 0 \leq j < n} \in \text{Mat}(m, n), \quad (4.1)$$

where

$$k = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(i, j) m^{mn-in-j-1}, \quad l = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} g(i, j) n^{mn-in-j-1}. \quad (4.2)$$

We call this automaton MA-[m, n, k, l].

Example 4.1.1. $m = 3, n = 2, 104 = 1 \cdot 3^4 + 2 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0, 11 = 1 \cdot 2^3 + 1 \cdot 2^1 + 1 \cdot 2^0$.

$$\text{MA-[3, 2, 104, 11]} \mapsto \left[Q = \{q_0, q_1, q_2\}, S = \{0, 1\}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \right] \quad (4.3)$$

This automaton is the BBS with a carrier capacity of two, described later.

Next, we introduce the isomorphism and minimality of Mealy automata. If there exists a pair of permutations $(\sigma, \rho) \in \mathfrak{S}_{|Q|} \times \mathfrak{S}_{|S|}$ such that

$$\begin{cases} \tilde{\varphi}(q, s) = \sigma(\varphi(\sigma^{-1}(q), \rho^{-1}(s))), \\ \tilde{\psi}(q, s) = \rho(\psi(\sigma^{-1}(q), \rho^{-1}(s))), \end{cases} \quad \text{for all } q \in Q, s \in S, \quad (4.4)$$

then we say that the automaton $(Q, S, \tilde{\varphi}, \tilde{\psi})$ is *isomorphic* to the automaton (Q, S, φ, ψ) . If for any distinct states $q \in Q$ and $r \in Q$, there exist $s \in S^*$ such that $\psi(q, s) \neq \psi(r, s)$, then $\mathcal{A}$ is called *reduced* or *minimal*. Here, S^* is the Kleene closure of S , which is an infinite set containing the empty string ε and all possible concatenations of letters in S . If $\mathcal{A}$ is not minimal, the behavior of $\mathcal{A}$ can be described by an automaton that has fewer states than $\mathcal{A}$.

4.2 The box–ball system with a finite carrier capacity

The BBS with a carrier capacity of k , derived by Takahashi and Matsukidaira from the modified KdV equation in [31], can be described as a Mealy automaton. This extension adds a new rule to the time evolution of BBS($L = 1$) explained in Section 1.1.1:

- If the carrier is already holding k balls when it passes in front of a box containing a ball, then it passes by the box without picking the ball up.

We call this automaton the $BBS-C(k)$. Kuniba et al. proved that the time evolution of the $BBS-C(k)$ is linearized by the KKR bijection [14].

Proposition 4.2.1. (Proposition 2.6. in [14]) Consider the $BBS-C(k)$. Let $s \in \{0, 1\}^{\mathbb{Z}_{\geq 0}}$ be an input sequence and $\tilde{s} \in \{0, 1\}^{\mathbb{Z}_{\geq 0}}$ be an output sequence. Let $(\mu^{(1)}, J^{(1)})$ and $(\tilde{\mu}^{(1)}, \tilde{J}^{(1)})$ be the rigged configurations corresponding to s and $\tilde{s}$ by the KKR bijection, respectively. Then, the following relations hold.

$$\tilde{\mu}^{(1)} = \mu^{(1)}, \quad (4.5)$$

$$\tilde{J}_i^{(1)} = J_i^{(1)} + \min(\mu_i^{(1)}, k). \quad (4.6)$$

Additionally, Kakei et al. provided an alternative linearization method based on the 01-arc lines [13]. In this section, we define a soliton automaton after pointing out some *nice* properties of the $BBS-C(2)$ automaton.

The BBS-C(2) is described by MA-[3,2,104,11]. The transition function φ and the exit function ψ are as seen in Example 4.1.1 and Fig 4.1. The state q_k of this Mealy automaton corresponds to the state in the BBS-C(k) where the carrier is holding k balls.

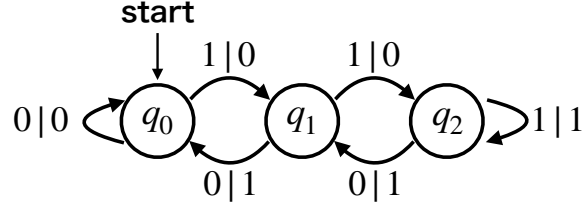


Fig 4.1: The MA-[3,2,104,11] of the BBS with a carrier capacity of $k = 2$. ‘start’ means that q_0 is the initial state.

The time evolution of the BBS with a box capacity of L and a carrier capacity of k is described by the equations as

$$u'_0 = 0, \quad (4.7)$$

$$\eta_j^{t+1} = \eta_j^t - \min(k - u_j^t, \eta_j^t) + \min(L - \eta_j^t, u_j^t), \quad (4.8)$$

$$u_{j+1}^t = u_j^t + \min(k - u_j^t, \eta_j^t) - \min(L - \eta_j^t, u_j^t). \quad (4.9)$$

If we take $k = \infty$, equations (4.7)–(4.9) coincide with equations (2.2)–(2.4). Furthermore, setting $L = 1$ results in the time evolution of the BBS-C(k).

4.2.1 Properties of the BBS-C(k)

We introduce several important properties of Mealy automata: transitive, time-reversible, particle-preserving, and locally interacting.

If for every pair of two states q and q' there exists a finite sequence $(s_1, s_2, \dots, s_n) \in S^n$ such that

$$\varphi_{s_n} \circ \varphi_{s_{n-1}} \circ \dots \circ \varphi_{s_1}(q) = q',$$

the transition function φ is called *transitive*. In other words, for (Q, S, φ, ψ) to be transitive means that it is strongly connected as a directed graph. In this chapter, we say the automaton (Q, S, φ, ψ) is transitive if φ is transitive. The following lemma and proposition are derived from the graph’s strong connectivity. We use them to check whether the automaton is transitive or not.

Lemma 4.2.2. Let φ be a transition function. If φ is transitive, there doesn’t exist $q' \in Q$ such that

$$q' \notin \{\varphi(q, s) \mid q \in Q \setminus \{q'\}, s \in S\}. \quad (4.10)$$

Proof. If q' is not in $\{\varphi(q, s) \mid q \in Q \setminus \{q'\}, s \in S\}$, it is not possible to reach q' from any other state q , and φ is not transitive. $\square$

Proposition 4.2.3. The transition function φ is transitive if and only if for all non-empty proper subset $R \subsetneq Q$, $\{\varphi(q, s) \mid q \in R, s \in S\}$ is not a subset of R .

Proof. First, we assume φ is transitive, and let R be a non-empty proper subset of Q . If $\{\varphi(q, s) \mid q \in R, s \in S\}$ is a subset of R , then it is not possible to reach $q' \in Q \setminus R$ from $q \in R$.

Inversely, we assume that for all $R \subsetneq Q$, $\{\varphi(q, s) \mid q \in R, s \in S\}$ is not a subset of R . For each $q \in Q$, let R_q be the set of all reachable states from q as $R_q = \{\varphi_{s_n} \circ \varphi_{s_{n-1}} \circ \cdots \circ \varphi_{s_1}(q) \mid n \in \mathbb{Z}_{\geq 0}, s_i \in S \ (i = 1, 2, \dots, n)\}$. Because of $\{\varphi(q', s) \mid q' \in R_q, s \in S\} = R_q$, R_q is equal to Q . Therefore φ is transitive. $\square$

Next, we discuss the time-reversibility of Mealy automata. There are two types of time-reversibility: left-invertible automaton and right-invertible automaton. If the mapping $\psi_q : S \rightarrow S; s \mapsto \psi(q, s)$ is a bijection on S for all $q \in Q$, the automaton (Q, S, φ, ψ) is said to be *left-invertible*. Note that left-invertible automata lead to the automata group [6]. In the BBS-C(k), the exit function at state q_0 outputs 0 regardless of the input, indicating it is not left-invertible. We introduce another time-reversibility. If the mapping $(q, s) \mapsto (\varphi(q, s), \psi(q, s))$ is a bijection on $Q \times S$, the automaton (Q, S, φ, ψ) is said to be *right-invertible* or *bijective*. The total number of bijective automata of type (m, n) is $(m \times n)!$.

The automaton $\mathcal{A}$ is said to be *particle-preserving* if there exists a weight function $w : Q \cup S \rightarrow \mathbb{R}$ such that the function w is not identically constant for $s \in S$ and

$$w(q) + w(s) = w(\tilde{q}) + w(\tilde{s}) \quad \text{for all } (q, s) \in Q \times S, \quad (4.11)$$

where $\tilde{q} = \varphi(q, s)$ and $\tilde{s} = \psi(q, s)$.

Because we can consider whether an automaton is particle-preserving, even if it is not minimal, the condition that the weight function w is identically constant for $q \in Q$ is unnecessary.

To define locally interacting, we define the vacuum alphabet and the final state set. Let F and V be a non-empty subset of Q and S , respectively, such that for all $q, q' \in F, q'' \in Q \setminus F$ and $s \in V^*$,

$$\varphi(q, s) \neq q'', \quad \varphi(q, s') = q' \quad (4.12)$$

where s' is some element of V^* . We call V the *vacuum alphabet* and F the *final state set* with respect to V . In the case that the final state set with respect to V is unique, then we call F the *vacuum state set* and the pair (F, V) the *vacuum pair*.

Finally, we define the locally interacting automata. Let V be a subset of S . We call $\mathcal{A} = (Q, S, \varphi, \psi)$ the *locally interacting automaton* with respect to V if for all $s_1, s_2 \in S^*$, there exist $s, s' \in V^*$ and $q \in Q$ such that

- (a). The length of $s_1 s$ equals to the length of $s_2 s'$, and

(b). For all $s_3 \in S^*$,

$$\psi(\varphi(q, s_1 s), s_3) = \psi(\varphi(q, s_2 s'), s_3). \quad (4.13)$$

In a locally interacting automaton with initial state q , if any strings $s_1, s_3 \in S^*$ are separated by a sufficiently long string $s \in V^*$, then s_1 and s_3 do not influence each other under the action of $\mathcal{A}_q$. If $\mathcal{A}$ is not the locally interacting automaton, then we call it the non-locally interacting automaton with respect to V .

Proposition 4.2.4. Let $\mathcal{A} = (Q, S, \varphi, \psi)$ be a minimal, transitive, and locally interacting automaton. Then, $\mathcal{A}$ has a unique final state set with respect to V .

Proof. Suppose that $\mathcal{A}$ is a minimal, transitive, and locally interacting automaton. Let $q_1 = \varphi(q, s_1)$, and $q_2 = \varphi(q, s_2)$. Since $\mathcal{A}$ is transitive, q_1 and q_2 take arbitrary states in Q for s_1 and s_2 . Therefore, for all $q_1, q_2 \in Q$ and $s_3 \in S^*$ there exist $s, s' \in V^*$ such that

$$\psi(\varphi(q_1, s), s_3) = \psi(\varphi(q_2, s'), s_3). \quad (4.14)$$

The minimal automaton does not allow two different states $\varphi(q_1, s)$ and $\varphi(q_2, s')$ satisfying the relation above, thus for all q_1, q_2 there exist $s, s' \in V^*$ such that $\varphi(q_1, s) = \varphi(q_2, s')$. By introducing $q_0 = \varphi(q_2, s')$, it is shown that for all q_1 there exist $s \in V^*$ such that $\varphi(q_1, s) = q_0$. Hence the final state set is unique. $\square$

4.2.2 Soliton automata

Suppose the vacuum alphabet is $V = \{0\}$ here and hereafter. Since we consider the particle-preserving automata, we take the initial state $q = q_0$ and suppose q_0 is in the final state set F with respect to V . It is also possible to consider cases where the boundary conditions increase the number of balls in the system, in which the initial state can be set to a different state from q_0 . We define the solitary wave and the soliton automata.

A string v of a finite length is called the *solitary wave*, if there exists a pair of $(m, l) \in \mathbb{Z}_{>0}^2$ such that $(\psi_{q_0})^m(v0^{\mathbb{Z}_{>0}}) = 0^l v 0^{\mathbb{Z}_{>0}}$. This pair of nonnegative integers (m, l) determines the (average) speed $\text{sp}(v) = l/m$ and the fundamental period of v , denoted by $\text{period}(v)$, is given by the smallest value of m .

A bijective automaton $\mathcal{A}$ is called the *soliton automaton* if for all $s \in S^*$, there is some $t \in \mathbb{Z}_{>0}$ such that for some $n \in \mathbb{Z}_{>0}$ and solitary waves $v_1, v_2, \dots, v_n$,

$$\mathcal{A}_{q_0}^t(s0^{\mathbb{Z}_{>0}}) = 0^{k_1} v_1 0^{k_2} v_2 0^{k_3} \dots 0^{k_n} v_n 0^{\mathbb{Z}_{>0}}, \quad (4.15)$$

$$\text{sp}(v_1) < \text{sp}(v_2) < \dots < \text{sp}(v_n), \quad (4.16)$$

where $k_2, \dots, k_n \geq |Q| - 1$. The condition $k_2, \dots, k_n \geq |Q| - 1$ means the interaction between the solitary waves has already finished.

4.3 Classification of the 3-state Mealy automata over a 2-letter alphabet

In this section, we will classify the class of 3-state Mealy automata over a 2-letter alphabet, that is the class of automata of type $(Q = \{q_0, q_1, q_2\}, S = \{0, 1\}, *, *)$. Since the number of automata of type $(|Q|, |S|)$ is only $(|Q| \times |S|)^{|Q| \times |S|}$, we can use a computer to check whether they satisfy the properties of being bijective, transitive, particle-preserving, and locally interacting. Before going to this class, it is better to state on the simpler classes (1) $Q = \{q_0\}, S = \{0, 1\}$ and (2) $Q = \{q_0, q_1\}, S = \{0, 1\}$:

4.3.1 Case $Q = \{q_0\}, S = \{0, 1\}$

The total number of automata in this class is just $(1 \times 2)^{1 \times 2} = 4$: $[1, 2, 0, 0]$, $[1, 2, 0, 1]$, $[1, 2, 0, 2]$, and $[1, 2, 0, 3]$. Here the output binary sequences of MA- $[1, 2, 0, 0]$ (MA- $[1, 2, 0, 3]$) are all zeros (all ones) for any binary input sequence. The MA- $[1, 2, 0, 2]$ is the permutation of 0s and 1s, thus the time-evolution behavior is blinking manner $0 \rightarrow 1 \rightarrow 0$ or $1 \rightarrow 0 \rightarrow 1$. The MA- $[1, 2, 0, 1]$ identically acts on the binary sequence. Hence the MA- $[1, 2, 0, 1]$ is the only Mealy automaton of type (1, 2) of bijective, transitive, and particle-preserving.

4.3.2 Case $Q = \{q_0, q_1\}, S = \{0, 1\}$

The total number of automata in this class is $(2 \times 2)^{2 \times 2} = 256$: $[2, 2, 0, 0]$, $[2, 2, 0, 1]$, ..., $[2, 2, 15, 15]$. In this class, the number of bijective ones is given by $4! = 24$. Among these,

- the number of transitive ones is 20,
- the number of particle-preserving ones is 6,
- and the number of transitive and particle-preserving ones is 5: MA- $[2, 2, 5, 3]$, $[2, 2, 6, 5]$, $[2, 2, 9, 5]$, $[2, 2, 10, 12]$, and $[2, 2, 12, 5]$.

The automata $[2, 2, 6, 5]$, $[2, 2, 9, 5]$, and $[2, 2, 12, 5]$ identically act on binary sequences, so these are not minimal. The automata $[2, 2, 5, 3]$ and $[2, 2, 10, 12]$ are isomorphic by exchanging two states q_0 and q_1 . Therefore, $[2, 2, 5, 3]$ ($[2, 2, 10, 12]$) is the only automaton that is bijective, transitive, particle-preserving, and minimal. This automaton is known as the BBS with a carrier capacity of one (BBS-C(1)). Fig 4.2 shows the state translation diagram of MA- $[2, 2, 5, 3]$.

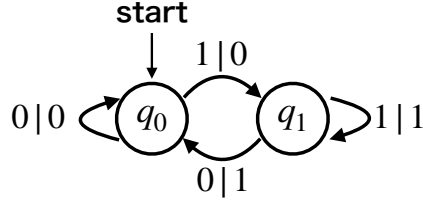


Fig 4.2: The bijective, transitive, particle-preserving, and minimal Mealy automaton of type $(2, 2)$: MA-[2,2,5,3] (BBS-C(1)).

4.3.3 Case $Q = \{q_0, q_1, q_2\}, S = \{0, 1\}$

The total number of automata in this class is $(3 \times 2)^{3 \times 2} = 46656$: $[3,2,0,0], [3,2,0,1], \dots, [3,2,728,63]$. In this class, the number of bijective ones is given by $6! = 720$. Among these,

- the number of transitive ones is 592,
- the number of particle-preserving ones is 84,
- and the number of transitive and particle-preserving ones is 68.

42 out of these 68 automata are minimal, and they can be divided into seven types up to the permutation of three states: $[3,2,104,11], [3,2,146,7], [3,2,128,19], [3,2,154,7], [3,2,318,52], [3,2,106,13], [3,2,266,19]$. Assuming the vacuum alphabet is $V = \{0\}$, we found only three types of locally interacting automata:

- (1) $[3,2,104,11]$ (BBS-C(2), carrier capacity of two),
- (2) $[3,2,146,7]$ (BBS-V(2), jump on to the second nearest vacant box),
- (3) $[3,2,154,7]$ (BBS-S(2), skip to the box two spaces ahead).

These three automata are candidates for soliton automata. Fig. 4.3 shows the state translation diagrams of these automata. BBS-C(2) is known to be a soliton automaton as a box-ball system with a carrier capacity of two (see Section 4.2). We will confirm that these candidates are soliton automata, and discuss their integrability.

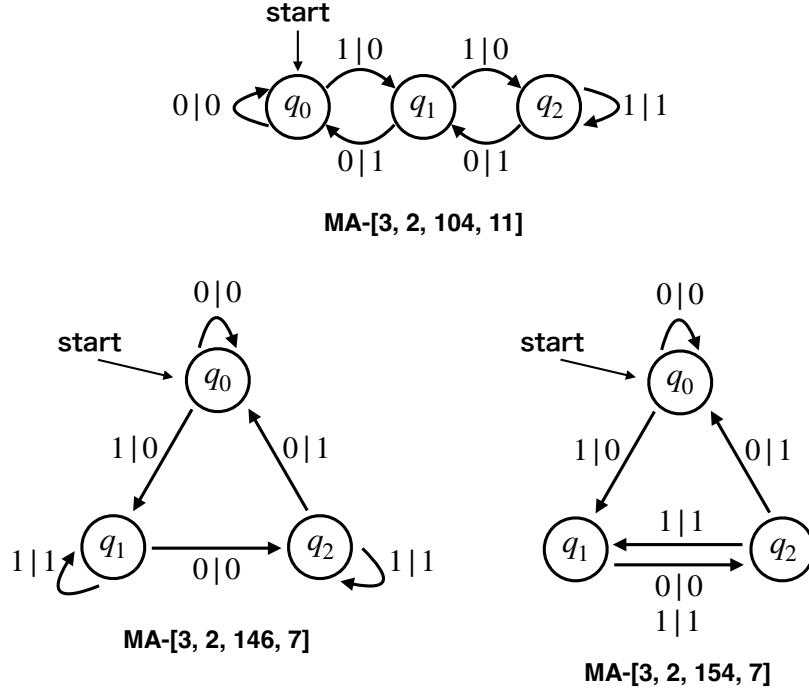


Fig 4.3: The bijective, transitive, particle-preserving, minimal, and locally interacting Mealy automaton of type (3, 2).

4.4 Linearization of MA-[3,2,154,7] (BBS-S(2))

In this section, we interpret MA-[3,2,154,7] as the box–ball system and prove the integrability by introducing the linearization method.

4.4.1 BBS-S(2): skip to the box two spaces ahead rule

Let us introduce the time evolution of the BBS-S(2): the BBS of skipping to the box two spaces ahead rule. Consider the boxes aligned in a semi-infinite row and finitely many balls in some boxes. The time evolution of the BBS-S(2) is described using a carrier as follows:

- (i) The carrier of capacity one moves from the left end to the right.
- (ii) The empty carrier moves right by one box until meeting the box occupied by a ball. If the carrier is empty and there are no balls in the boxes right to the carrier, then the carrier stops.
- (iii) When the empty carrier meets the box occupied by a ball, take the ball from the box to the carrier and move to the right by skipping every one box until meeting the vacant box. Meeting the vacant box, the carrier unloads the ball to the vacant box and follows Step (ii).

The MA-[3,2,154,7] with the initial state $q = q_0$ corresponds to the BBS-S(2). The automaton is given by

$$[Q, S, \phi, \psi] = \left[\{q_0, q_1, q_2\}, \{0, 1\}, \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \right]. \quad (4.17)$$

4.4.2 Linearization

We construct a bijection between the BBS-S(2) state sequence and two sequences of nonnegative integers.

Proposition 4.4.1. Let $S = \{0, 1\}$.

$$S^* \cong \{(b_1, b_2, \dots, b_n) \mid n \in \mathbb{Z}_{\geq 0}, b_j \in \mathbb{Z}_{\geq 0}, b_1 \leq b_2 \leq \dots \leq b_n\} \\ \times \{(c_1, c_2, \dots, c_n) \mid n \in \mathbb{Z}_{\geq 0}, c_j \in \mathbb{Z}_{\geq 0}, c_1 < c_2 < \dots < c_n\}. \quad (4.18)$$

The procedure for constructing two sequences from $s \in S^*$ is as follows:

- (1) For $s \in S^*$, replace contiguous '1's with 'b' or 'c' as follows:

$$\begin{cases} 1^{2k} \mapsto b^k & (\text{the length of '1's is even}), \\ 1^{2k+1} \mapsto b^k c & (\text{the length of '1's is odd}). \end{cases} \quad (4.19)$$

Here, when the length of '1's is odd, the position of 'c' is not necessarily at the end of 'b's, that is, 1^{2k+1} can be replaced with $b^j c b^{k-j}$ instead of $b^k c$. For example,

$$\begin{aligned} 1000100100011110011011011110000 \dots &\mapsto c000c00c000bb00b0b0bb0000 \dots \\ 000000101111001110110111100000 \dots &\mapsto 000000c0bb00cb0b0cbb00000 \dots \end{aligned}$$

- (2) We define two sequences $\bar{b} = (b_1, b_2, \dots)$ and $\bar{c} = (c_1, c_2, \dots)$ from the sequence of $\{0, b, c\}$ obtained above. Let b_j denote the number of 0s to the left of the j th b from the left. Similarly, let c_j denote the number of 0s to the left of the j th c from the left.

$$\begin{aligned} 1000100100011110011011011110000 \dots &\mapsto c000c00c000bb00b0b0bb0000 \dots \\ &\mapsto \bar{b} = (8, 8, 10, 11, 12, 12), \bar{c} = (0, 3, 5), \\ 000000101111001110110111100000 \dots &\mapsto 000000c0bb00cb0b0bbc00000 \dots \\ &\mapsto \bar{b} = (7, 7, 9, 10, 11, 11), \bar{c} = (6, 9, 11). \end{aligned}$$

By definition, $\bar{b}$ is an increasing integer sequence, and $\bar{c}$ is a strictly increasing integer sequence.

There are two types of solitary waves as

- The solitary wave of speed one: $0b^k0 \mapsto 00b^k$ $(0(11)^k0 \mapsto 00(11)^k)$,
- The solitary wave of speed two: $0(c0)^k0 \mapsto 00(0c)^k$ $(0100 \mapsto 0001)$.

We slightly modify the rule for the time evolution of BBS-S(2). Here, when considering the $\{b, c\}$ sequence corresponding to 1^{2k+1} , ‘c’ is assumed to be the rightmost. After the carrier loads a ball from the box, it unloads the ball into the second box. If the second box contains another ball, it is exchanged for the loaded ball. After that, the balls associated with each ‘c’ are exchanged with the balls associated with ‘b’ to the right of ‘c’ until the number of letters to the right of ‘c’ becomes zero. This change in procedure does not affect the final $\{0, 1\}$ sequence, as it only replaces ‘1’ with another ‘1’. Under this time evolution rule, each ‘1’ either moves to the second box or is overtaken by other balls. Therefore, the following holds.

- Every ‘1’ in ‘c’ moves to the second box. Because the letter to the right of ‘c’ is ‘0’, it does not overtake other ‘1’s. Since the number of ‘1’s on the left side of it does not change, the number of ‘0’s on the left side increases by two.
- The first ‘1’ in each ‘b’ overtakes the second ‘1’ and moves to the second box. Since the number of ‘1’s on the left side of it increases by one, the number of ‘0’s on the left side increases by one.
- The second ‘1’ in each ‘b’ is overtaken by the first ‘1’ and does not move. The number of ‘1’s on the left side of it decreases by one.

The first ‘1’s in ‘b’ becomes the second ‘1’s in ‘b’ the next time, and the second ‘1’s becomes the first ‘1’s. Thus, for every ‘b’, the number of balls on the left side increases by one. Let $\vec{b}^t = (b_1^t, b_2^t, \dots)$ and $\vec{c}^t = (c_1^t, c_2^t, \dots)$ be the corresponding integer sequences at time t . Then, we find

$$b_j^t = b_j^0 + t, \quad (4.20)$$

$$c_j^t = c_j^0 + 2t. \quad (4.21)$$

Example 4.4.2. In the following series of the time evolution of the BBS-S(2), a solitary wave with speed one is overtaking a solitary wave with speed two. We can observe that the behavior of the ‘10’ (‘11’) sequence becomes a simple parallel translation from left to right after the interaction.

Example 4.5.1. In the following time evolution series of the BBS-V(2), there are three solitons consisting of one, two, and five balls, respectively. A soliton of n balls moves by $n + 1$ boxes in n time steps. Therefore, the speed of a soliton consisting of n balls is $(n + 1)/n$.

```

t = 0 : 1001100000111110000000000000000000
t = 1 : 0010101000011110100000000000000000
t = 2 : 0000110010001110110000000000000000
t = 3 : 0000010100100110111000000000000000
t = 4 : 0000000110001010111100000000000000
t = 5 : 0000000010100011011101000000000000
t = 6 : 0000000000110001011110010000000000
t = 7 : 0000000000010100011111000100000000
t = 8 : 0000000000000110001111010001000000
t = 9 : 00000000000000010100111011000010000
t = 10 : 00000000000000000110011011100000100
t = 11 : 000000000000000000010101011110000001

```

4.5.2 Relationships between the BBS-V(2) and the original BBS

First, we recall the relationships between the original BBS and the ultradiscrete Lotka–Volterra equation [29]. The time evolution of the original BBS introduced in 1990 [32] is described by the piecewise linear equation called the uKdV equation (1.4) with $L = 1$:

$$\eta_j^{t+1} = \min \left\{ 1 - \eta_j^t, \sum_{i=0}^{j-1} (\eta_i^t - \eta_i^{t+1}) \right\}. \quad (4.23)$$

Through the variable transformation

$$\gamma_j^t = \sum_{i=0}^{j+1} \eta_i^t - \sum_{i=0}^j \eta_i^{t+1}, \quad (4.24)$$

we have

$$\gamma_{j+1}^{t+1} - \gamma_j^t = \max(0, \gamma_j^{t+1} - 1) - \max(0, \gamma_{j+1}^t - 1). \quad (4.25)$$

which is known as the ultradiscrete Lotka–Volterra (uLV) equation.

Next, we will explain the correspondence between the BBS-V(2) and the uLV equation [38]. There are two types of input/output sequences with state transitions from q_0 to q_0 :

1. input “0”, output “0”,
2. input “ $1^{n+1}01^m0$ ”, output “ 01^n01^{m+1} ”,

where $n, m \geq 0$, and 1^n denotes the sequence $\underbrace{11 \cdots 1}_n$.

Proposition 4.5.2. Let $X = \{0\} \cup \{1^{n+1}01^m0\}_{n,m \geq 0}$ be the set of sequence. Then, any sequence $s00$ ($s \in S^*$), which has suffix 00 , corresponds one-to-one with an element of X^* .

Because we assume that the input sequence has finitely many 1s and there are sufficiently many 0s at the right end of the sequence, the sequence can be divided uniquely as $x_1x_2 \cdots x_n$ ($x_i \in X$). Let $Y = \{0\} \cup \{(n+1, m)\}_{n,m \geq 0} \subset \mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{\geq 0}^2$, and define a map $\kappa : X \rightarrow Y$ as $0 \mapsto 0 (\in \mathbb{Z}_{\geq 0})$ and $1^{n+1}01^m0 \mapsto (n+1, m) (\in \mathbb{Z}_{\geq 0}^2)$. Introduce the extension of κ on X^* by

$$\kappa(x_1x_2 \cdots x_n) = \kappa(x_1)\kappa(x_2) \cdots \kappa(x_n). \quad (4.26)$$

The product of sequences of integers in the right-hand side of the equation (4.26) denotes the concatenation of sequences.

Example 4.5.3. The sequence

$$0011101010001101100 \cdots \in S^*$$

is divided as

$$\underline{0011101010001101100} \cdots \in X^*.$$

Applying κ for every underlined subsequence gives us

$$\underline{0031100220} \cdots \in Y^*,$$

and we finally get the sequence

$$0031100220 \cdots \in \mathbb{Z}_{\geq 0}^*.$$

Introduce the operator T on lattice (in Fig 4.4) as

$$\begin{aligned} T : \{0, 1\} \times \mathbb{Z}_{\geq 0} &\rightarrow \mathbb{Z}_{\geq 0} \times \{0, 1\}, \\ (i, j) &\mapsto (j', i') = (\max(2i + j - 1, 0), \min(1 - i, j)). \end{aligned} \quad (4.27)$$

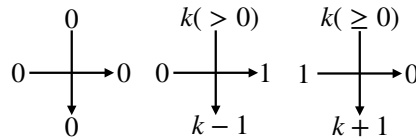


Fig 4.4: The action of T on $\{0, 1\} \times \mathbb{Z}_{\geq 0}$.

The operator T satisfies

$$\begin{aligned} T : (0, 0) &\mapsto (0, 0), \\ (T \otimes \mathbf{1})(\mathbf{1} \otimes T) : (0, n+1, m) &\mapsto (n, m+1, 0), \end{aligned}$$

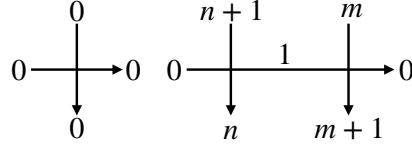


Fig 4.5: The action of T and $(T \otimes \mathbf{1})(\mathbf{1} \otimes T)$ that correspond to MA-[3,2,146,7].

and these correspond to the input/output of the MA-[3,2,146,7] by shifting one cell to the left, where $\mathbf{1} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ is an identity operator. Both two operators T and $(T \otimes \mathbf{1})(\mathbf{1} \otimes T)$ can be regarded as the map $\{0, 1\} \times Y \rightarrow Y \times \{0, 1\}$ shown in Fig 4.5.

Define the operator $\mathcal{T}$ as

$$\mathcal{T} = \mathcal{T}^{(n-1)} \dots \mathcal{T}^{(2)} \mathcal{T}^{(1)} \mathcal{T}^{(0)}, \quad (4.28)$$

$$\mathcal{T}^{(i)} = \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_i \otimes T \otimes \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_{n-i-1}. \quad (4.29)$$

Proposition 4.5.4. Let Δ be the map $S^* \rightarrow S^*$ of the time evolution of the MA-[3,2,146,7]. Then, $\kappa \circ \Delta = \Lambda \circ \mathcal{T} \circ \kappa$, where Λ is the forward-shift operator on $\mathbb{Z}_{\geq 0}^*$ as

$$(\Lambda(\eta))_j = \begin{cases} 0, & j = 0, \\ (\eta)_{j-1}, & \text{otherwise.} \end{cases}$$

Next, introduce an operator U (in Fig 4.6) like T as

$$\begin{aligned} U : \{0, 1\} \times \mathbb{Z}_{\geq 0} &\rightarrow \mathbb{Z}_{\geq 0} \times \{0, 1\}, \\ (i, j) &\mapsto (j', i') = (i + j, \min(1 - i, j)). \end{aligned} \quad (4.30)$$

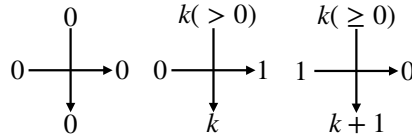


Fig 4.6: The action of U on $\{0, 1\} \times \mathbb{Z}_{\geq 0}$.

The operator U satisfies

$$\begin{aligned} U : (0, 0) &\mapsto (0, 0), \\ (U \otimes \mathbf{1})(\mathbf{1} \otimes U) : (0, n + 1, m) &\mapsto (n + 1, m + 1, 0), \end{aligned}$$

and both U and $(U \otimes \mathbf{1})(\mathbf{1} \otimes U)$ can be regarded as the map $\{0, 1\} \times Y \rightarrow Y \times \{0, 1\}$ shown in Fig 4.7.

Define the operator $\mathcal{U}$ like $\mathcal{T}$ as

$$\mathcal{U} = \mathcal{U}^{(n-1)} \dots \mathcal{U}^{(2)} \mathcal{U}^{(1)} \mathcal{U}^{(0)}, \quad (4.31)$$

$$\mathcal{U}^{(i)} = \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_i \otimes U \otimes \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_{n-i-1}. \quad (4.32)$$

Then, the following theorem holds.

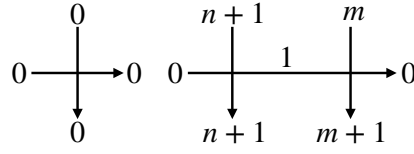


Fig 4.7: The action of U and $(U \otimes \mathbf{1})(\mathbf{1} \otimes U)$.

Theorem 4.5.5. Let uLV be the map $Y^* \rightarrow Y^*$ of the time evolution of the uLV equation given by the equation (4.25). Then, $\text{uLV} \circ \mathcal{U} \simeq \mathcal{U} \circ \mathcal{T}$.

Proof. For $x \in \mathbb{Z}_{\geq 0}^*$, let $y = \mathcal{U}x$, $x' = \mathcal{T}x$, and $y' = \mathcal{U}x' \in \mathbb{Z}_{\geq 0}^*$. We will prove that the map $y \rightarrow y'$ satisfies the uLV equation (4.25).

The terms of x are denoted by $x = (x_1, x_2, x_3, \dots)$, $x_i \in \mathbb{Z}_{\geq 0}^*$, and the same applied to y , x' , and y' . In addition, define the auxiliary variables for $y = \mathcal{U}x$, $x' = \mathcal{T}x$ and $y' = \mathcal{U}x'$ as in Fig 4.8. Note that the auxiliary variables for $\mathcal{U}$ are the same as those of $\mathcal{T}$ (see Fig 4.4 and Fig 4.6).

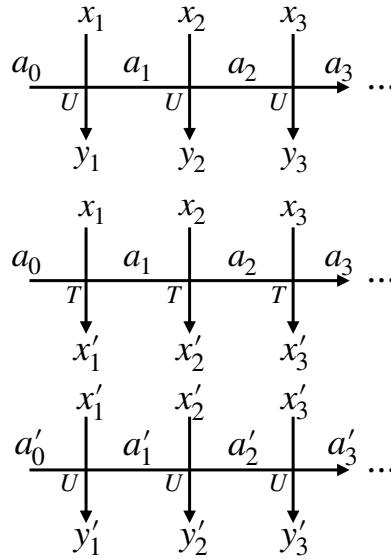


Fig 4.8: The auxiliary variables a_i, a'_i .

We can set $a_0 = a'_0 = 0$ for the boundary condition. Now, we have

$$y_{i+1} = x_{i+1} + a_i, \quad (4.33)$$

$$y'_{i+1} = x'_{i+1} + a'_i \quad (4.34)$$

from equation (4.30) (the definition of U), and

$$x'_{i+1} = x_{i+1} + a_i - a_{i+1} \quad (4.35)$$

from equation (4.27) (the definition of T) and $a_i, x_i \in \mathbb{Z}_{\geq 0}$. Here, by using $a_j \in \{0, 1\}$, we obtain

$$\begin{aligned}\min(y_i, 1) &= \min(x_i + a_{i-1}, 1) \\ &= a_{i-1} + \min(x_i, 1 - a_{i-1}) \\ &= a_{i-1} + a_i,\end{aligned}\tag{4.36}$$

and

$$\min(y'_i, 1) = a'_{i-1} + a'_i\tag{4.37}$$

in the same way. The equations (4.33)–(4.37) lead us to

$$\begin{aligned}y'_{i+1} + \max(y'_i - 1, 0) &= y'_{i+1} + y'_i - \min(1, y'_i) \\ &= y'_{i+1} + y'_i - a'_{i-1} - a'_i \\ &= x'_{i+1} + x'_i \\ &= (x_{i+1} + a_i - a_{i+1}) + (x_i + a_{i-1} - a_i) \\ &= y_{i+1} + y_i - a_{i+1} - a_i \\ &= y_{i+1} + y_i - \min(y_{i+1}, 1) \\ &= y_i + \max(y_{i+1} - 1, 0).\end{aligned}\tag{4.38}$$

Setting $y'_i = y_i, y'^{t+1}_i = y'_i, \gamma'_t = y'_i$, the variables γ'_i satisfies the uLV equation (4.25). $\square$

Corollary 4.5.6. $(\mathcal{U} \circ \kappa) \circ \Delta \simeq \Lambda \circ \text{uLV} \circ (\mathcal{U} \circ \kappa)$.

Chapter 5

Conclusions

In Chapter 2, we examined the BBS with a box capacity of L (BBS(L)) (described by the uKdV equation), including cases where the dependent variables take arbitrary integer values. Negative values cannot be interpreted as balls, so conventional methods such as KKR bijection and arc rules cannot be used directly. We constructed a bijection that associates the state of the BBS(L) with a pair of sequences: the background sequence and the soliton sequence. In correspondence with the time evolution of the BBS(L), the background sequence translates at speed one, while the soliton sequence evolves according to the time evolution of the BBS(1). The time evolution of the BBS(1) can be linearized by mapping the states to rigged configurations via the KKR bijection (that can also be constructed using an elementary method with the arc lines). Consequently, we can construct a bijection between the states of BBS(L) and the pair of the background sequence and rigged configuration, which linearizes the time evolution.

In Chapter 3, we introduced a new arc line rule for the state of the rBBS, whose time evolution is described by the ultradiscretization of the discrete relativistic Toda equation. By rewriting the state of the rBBS into a soliton configuration according to the new arc rule, the time evolution can be described using the ‘dropping rule’. This proves that conserved quantities corresponding to the sizes of the solitons can be obtained. The riggings (the positions of solitons) can be calculated from the soliton configuration. The rigging increases linearly under the time evolution of the rBBS. We have also confirmed that the rigged configuration, consisting of the lengths and riggings of solitons, corresponds one-to-one with the soliton configuration. However, formal theoretical proof of this correspondence has not yet been established. By applying the rule of arc lines introduced here, we have confirmed that the sizes of solitons and the riggings can also be computed for the original BBS and the BBS with the second vacant box rule introduced in Chapter 4.

In Chapter 4, we examined soliton cellular automata that are described by Mealy automata. Focusing on the BBS with finite carrier capacity, we introduced several key properties of Mealy automata, such as bijective, transitive (strongly connected), particle preserving, and locally interacting. Through computational and theoretical analysis based on these properties, we identified three classes of 3-state soliton Mealy automata over a 2-letter alphabet: MA-[3, 2, 104, 11], MA-[3, 2, 146, 7], and MA-[3, 2, 154, 7]. MA-[3, 2, 104, 11] is the BBS

with a carrier capacity of two (BBS-C(2)). The BBS-C(2) has been extensively studied and its time evolution is linearized through the KKR bijection. MA-[3, 2, 146, 7] is the jumping on to the second nearest vacant box rule (BBS-V(2)). We proved that the time evolution of the BBS-V(2) is equivalent to the uLV equation, which is closely related to the original BBS. MA-[3, 2, 154, 7] is the skipping to the box two spaces ahead rule (BBS-S(2)). We provided a simple method for linearizing the BBS-S(2). For Mealy automata with a state set size of $|Q| \geq 4$, candidates for soliton cellular automata that satisfy the above key properties can be enumerated using a computer. We checked whether they are soliton automata by computing the time evolution starting from several initial binary sequences. It was confirmed that some candidates do not exhibit solitonic behavior. A theoretical analysis of these candidates also remains for future work.

In this thesis, we conducted combinatorial analysis on three types of extensions of the box-ball systems, obtained from different perspectives. We hope these results will lead to a comprehensive understanding of various soliton automata.

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List of author's papers and works related to the thesis

Original Papers

- [1] A. Maeno and S. Tsujimoto, *Linearization of the box-ball system with box capacity L* , J. Phys. Soc. Jpn., **92** (2023), 024003.
- [2] A. Maeno and S. Tsujimoto, *Arc lines for the box-ball system related to the ultradiscrete relativistic Toda equation*, JSIAM Letters, **16** (2024), 93–96.
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Conference Presentation

- [5] A. Maeno and S. Tsujimoto, *Linearization of the box-ball system with box capacity L* , 2020 JSIAM Annual Meeting.
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