

On Boundary Value Problems for Micro-hyperbolic Systems of Differential Equations

MOTOO UCHIDA

Osaka University, Graduate School of
Science, Department of Mathematics

内田素夫 (大阪大学大学院理学研究科)

In [KK], Kashiwara and Kawai formulate boundary value problems for elliptic systems of differential equations from a microlocal point of view, where they describe the obstruction of extension beyond the boundary in terms of a system of micro-differential equations induced on the boundary. In this short paper, we prove the same formula as established in [KK] for (semi-)micro-hyperbolic systems of differential equations. This enables us to understand boundary value problems for elliptic systems and for semi-hyperbolic systems in a unified manner.

The results proved in this paper¹ are more or less known to specialists, but are not found in the literature.

Notations. In this paper, we freely use the notations of [KS1] for sheaves and functors. For a complex manifold X , T^*X denotes the cotangent bundle of X . $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X , $\mathcal{D}_X$ the sheaf of rings of differential operators, and $\mathcal{E}_X$ the sheaf of rings of microdifferential operators. If M is a closed real submanifold of X , T_M^*X denotes the conormal bundle of M , $\pi_M : T_M^*X \rightarrow M$ the projection to the base space. We denote by H the Hamiltonian map $T^*T^*X \rightarrow TT^*X$. If M is a real submanifold of X , H induces an isomorphism $T^*T_M^*X \rightarrow T_{T_M^*X}^*T^*X$, which is also denoted simply by H .

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1. Main Theorems

Let M be a real analytic manifold of dimension $n \geq 1$, N a submanifold of M of codimension 1 defined by equation $f = 0$ for a real-valued analytic function f with $df|_N \neq 0$. Let Z_+ denote the closed subset $\{f \geq 0\}$ of M ; then Z_+ is a real analytic submanifold of M with boundary. We set $N^+ = \{k \cdot df(x) \mid x \in N, k > 0\}$; then $N^+ \subset T_N^*M$. Let X be a complex neighborhood of M , Y a closed complex submanifold of X of codimension 1 such that $M \cap Y = N$. Denote by φ the closed embedding $Y \hookrightarrow X$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -module. $\text{Ch}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$. We assume the following conditions :

(A.1) $\varphi : Y \rightarrow X$ is non characteristic for $\mathcal{M}$.

(A.2) At any point p of $(T_M^*X \cap T_N^*X \setminus N) \cap \text{Ch}(\mathcal{M})$,

$$(1.1) \quad -H(\pi^*df) \notin C_p(\text{Ch}(\mathcal{M}), Z_+ \times_M T_M^*X)/T_p T_M^*X,$$

where $\pi : T_M^*X \rightarrow M$ and $\pi^* : T_{\pi(p)}^*M \rightarrow T_p^*T_M^*X$.

In the right-hand side of (1.1), $C_p(\text{Ch}(\mathcal{M}), Z_+ \times_M T_M^*X)$ denotes the normal cone at p (cf. [KS1, Def.4.1.1]), which is a closed cone in $T_p T_M^*X$, and $C_p(\cdot, \cdot)/T_p T_M^*X$ the image of the normal cone in $(T_{T_M^*X}^*T_M^*X)_p$ for short.

Let $(T_N^*X)^+$ be an open subset of T_N^*X defined by $(T_N^*X)^+ = q^{-1}(N^+)$, with q being the canonical projection $T_N^*X \rightarrow T_N^*M$. Let ${}^t\varphi' : T^*X \times_X Y \rightarrow T^*Y$ the induced map of φ , $\rho : T_N^*X \rightarrow T_N^*Y$ the projection induced from ${}^t\varphi'$ on N .

Let $\widetilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$, with $\pi : T^*X \rightarrow X$. Denoting by $\varphi^*\widetilde{\mathcal{M}}$ the induced $\mathcal{E}_Y$ -module of $\widetilde{\mathcal{M}}$ on Y , we have :

Lemma 1.1. *If we assume (A.1) and (A.2), there exists a coherent $\mathcal{E}_Y$ -module $\mathcal{N}^+$ defined on $T_N^*Y \setminus N$ and an $\mathcal{E}_Y$ -homomorphism $\mathcal{N}^+ \rightarrow \varphi^*\widetilde{\mathcal{M}}$ such that*

$$(1.2) \quad \mathcal{N}_q^+ \cong \bigoplus_{p \in (T_N^*X)^+ \cap \text{Ch}(\mathcal{M}) \cap \rho^{-1}(q)} (\mathcal{E}_{Y \rightarrow X} \otimes_{\mathcal{E}_X} \widetilde{\mathcal{M}})_p$$

for any $q \in T_N^*Y \setminus N$.

Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on M , $\mathcal{C}_N$ the sheaf of microfunctions on N (cf. [SKK]). Let $\text{or}_{N|M}$ be the relative orientation sheaf of N in M as $\mathbb{C}$ -module.

Theorem 1.2. *Assume (A.1) and (A.2). There is an isomorphism*

$$(1.3) \quad \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\dot{\pi}_N^* \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N),$$

where $\dot{\pi}_N : T_N^*Y \setminus N \rightarrow N$.

Remark 1. Theorem 1.2 is first proved for elliptic $\mathcal{D}_X$ -modules by Kashiwara and Kawai [KK]. Note that (A.1) and (A.2) are automatically satisfied if $\mathcal{M}$ is elliptic. Let $(x_1, \dots, x_n)$ be a system of local coordinates of M , $Z_+ = \{x_1 \geq 0\}$. A classical example of non-elliptic differential operators which satisfy condition (A.2) is $D_1^2 - x_1^k A(x, D')$, with $k \in \mathbf{Z}$, $k \geq 2$, where $D_1 = \partial/\partial x_1$ and $A(x, D')$ is a differential operator of order 2 such that $[x_1, A] = 0$ and its principal symbol $\sigma(A)$ is negative valued on $T_M^*X \cap T_N^*X \setminus \rho^{-1}(0_N)$, 0_N being the zero section of T_N^*Y (i.e. $\sigma(A)(x, i\eta') < 0$ if $\eta' \neq 0$).

Remark 2. Condition (1.1) is an analogue of micro-hyperbolicity [KS2] and naturally appears in microlocal study of boundary value problems (cf. [S2, SZ]). It is well known that, if we assume

$$+H(\pi^*df) \notin C_p(\mathrm{Ch}(\mathcal{M}), Z_+ \times_M T_M^*X)/T_p T_M^*X$$

at $p \in T_M^*X \cap T_N^*X$, this entails propagation of regularity up to the boundary point p from the positive side of N (see [Kt2, S1, S2, SZ]).

Let $\mathcal{A}_M$ be the sheaf of real analytic functions on M . In place of (A.1) and (A.2), consider the following slightly stronger assumption. ((B.1) is the same as (A.1).)

(B.1) $\varphi : Y \rightarrow X$ is non characteristic for $\mathcal{M}$.

(B.2) φ is micro-hyperbolic for $\mathcal{M}$ at all $p \in T_M^*X \cap T_N^*X \setminus N$ [KS2, Def.2.1.2]:
For both $\pm$,

$$\pm H(\pi^*df) \notin C_p(\mathrm{Ch}(\mathcal{M}), T_M^*X)/T_p T_M^*X.$$

Theorem 1.3. *Assume (B.1) and (B.2). There is an isomorphism*

$$(1.4) \quad \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)|_N \otimes \mathrm{or}_{N|M}[1] \cong \mathrm{R}\dot{\pi}_N^* \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N)$$

as well as isomorphism (1.3), where $\mathcal{N}^+$ is the coherent $\mathcal{E}_Y$ -module on $T_N^*Y \setminus N$ given in Lemma 1.1.

2. Proof of Theorem 1.2 and 1.3

As in [KK], the proof of Theorem 1.2 is divided into two steps. In the first step, we relate the left-hand side of (1.3) to a differential complex with coefficients in $\mathcal{C}_{N|X}$ induced from $\mathcal{M}$. In the second step, proving Lemma 1.1, we complete the proof of Theorem 1.2.

Let us recall the notion of the $\mathcal{E}_X$ -module $\mathcal{C}_{Z_+|X}$ due to Kataoka [Kt1] and Schapira [S2]. Following [S2], let

$$\mathcal{C}_{Z_+|X} = \mu\mathrm{hom}(\mathbf{C}_{Z_+}, \mathcal{O}_X) \otimes \mathrm{or}_{M|X}[n].$$

Then all the cohomology groups $H^k(\mathcal{C}_{Z_+|X})$, $k \neq 0$, are zero and $H^0(\mathcal{C}_{Z_+|X})$ is an $\mathcal{E}_X$ -module. We identify $\mathcal{C}_{Z_+|X}$ with its zero-th cohomology $H^0(\mathcal{C}_{Z_+|X})$. For the $\mathcal{E}_X$ -module $\mathcal{C}_{N|X}$, refer to [KK], [KS2] and also [S1, S2]. (In this paper, we follow the definition of [KK, KS2] : $\mathcal{C}_{N|X} = H^n \mu_N(\mathcal{O}_X) \otimes \mathrm{or}_{N|X}$.) We prepare two lemmas.

Lemma 2.1.

- (1) $R\pi_* \mathcal{C}_{Z_+|X}|_M \cong R\Gamma_{Z_+} \mathcal{B}_M$.
- (2) $\mathrm{supp}(\mathcal{C}_{Z_+|X}) \cap T_N^* X \subset \overline{(T_N^* X)^+}$.
- (3) *There is an $\mathcal{E}_X$ -homomorphism $\mathcal{C}_{N|X} \otimes \mathrm{or}_{N|M} \rightarrow \mathcal{C}_{Z_+|X}$, and this is an isomorphism on $(T_N^* X)^+$.*

For the proof, see [Kt3, Sect.4] and [S2, S3].

Lemma 2.2. *If we assume (1.1) at a point p of $T_M^* X \cap T_N^* X$, we have*

$$R\mathrm{Hom}_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^* X} = 0$$

in a neighborhood of p .

Proof. (Cf. the proof of Corollary 3.3 of [SZ].) Let g be a real-valued smooth function defined on X such that $g|_M = f$. We set $h = g \circ \pi$, with $\pi : T^* X \rightarrow X$. From (1.1), we have

$$-H(dh) \notin C_p(\mathrm{Ch}(\mathcal{M}), Z_+ \times_M T_M^* X).$$

Hence we can find an open subset U of $T^* X$ so that $U \cap \mathrm{Ch}(\mathcal{M}) = \emptyset$,

$$-H(dh) \notin C_p(T^* X \setminus U, Z_+ \times_M T_M^* X),$$

and $-H(dh) \notin C_p(T^*X \setminus U, U)$. Let $T_{Z_+}^*X$ denote the micro-support $\text{SS}(\mathbf{C}_{Z_+})$ of the sheaf $\mathbf{C}_{Z_+}$ on X (cf. [KS1, Sect.5.1]). Since $T_{Z_+}^*X \subset Z_+ \times_M T_M^*X \cup U$ on a neighborhood of p , we have $-H(dh) \notin C_p(T^*X \setminus U, T_{Z_+}^*X)$. This yields

$$-H(dh) \notin C_p(\text{Ch}(\mathcal{M}), T_{Z_+}^*X).$$

Since

$$\text{SS}(\text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})) \subset C(\text{Ch}(\mathcal{M}), T_{Z_+}^*X),$$

it follows from the definition of micro-supports that

$$\text{R}\Gamma_{\{h \geq 0\}} \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} = 0$$

in a neighborhood of p . Since $\mathcal{C}_{Z_+|X}$ is supported on $T_{Z_+}^*X$ and $T_{Z_+}^*X \subset \{h \geq 0\}$, we have

$$\text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} \cong \text{R}\Gamma_{\{h \geq 0\}} \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_{\{h=0\}} \cong 0.$$

Q.E.D.

Since $\mathbf{C}_{Z_+}$ is cohomologically constructible, if we set

$$F = \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

it follows from [KS1, Prop.4.4.2] that

$$\begin{aligned} \text{R}\pi_* \text{R}\Gamma_{T_X^*X} \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N &\cong \text{R}\pi_* \text{R}\Gamma_{T_X^*X} \mu\text{hom}(\mathbf{C}_{Z_+}, F)|_N[n] \\ &\cong \text{RHom}_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_X) \otimes F|_N[n] \\ &\cong F \otimes \mathbf{C}_{Z_+ \setminus N}|_N \\ &\cong 0. \end{aligned}$$

Hence, from Lemma 2.1, we have

$$\begin{aligned} \text{R}\Gamma_{Z_+} \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N &\cong \text{R}\pi_* \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N \\ &\xrightarrow{\sim} \text{R}\pi_* \text{R}\Gamma_{T^*X \setminus X} \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{Z_+|X})|_N \\ &\cong \text{R}\pi'_* \left(\text{RHom}_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^*X \setminus N} \right), \end{aligned}$$

where $\pi' : T_N^*X \setminus N \rightarrow N$. It then follows from Lemma 2.1(2), (3) and 2.2 that

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^*X \setminus N} &\cong \mathrm{R}\Gamma_{(T_N^*X)^+}(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{Z_+|X})|_{T_N^*X \setminus N}) \\ &\cong \mathrm{R}\Gamma_{(T_N^*X)^+} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}) \otimes \mathrm{or}_{N|M}. \end{aligned}$$

Thus we have

$$\begin{aligned} (2.0) \quad \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M} \\ \cong \mathrm{R}\pi'_* \mathrm{R}\Gamma_{(T_N^*X)^+} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X}). \end{aligned}$$

Since $T_Y^*X \cap \mathrm{Supp}(\widetilde{\mathcal{M}}) \subset T_X^*X$, we have

the right-hand side of (2.0)

$$\begin{aligned} &\cong \mathrm{R}\dot{\pi}_{N*} \left[\mathrm{R}\rho_* \mathrm{R}\Gamma_{(T_N^*X)^+} \mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{T_N^*Y \setminus N} \right] \\ &= \mathrm{R}\dot{\pi}_{N*} \left[\mathrm{R}\rho_*^+ \left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+} \right) |_{T_N^*Y \setminus N} \right], \end{aligned}$$

where we denote by $\rho^+ : (T_N^*X)^+ \rightarrow T_N^*Y$ the restriction of ρ . Hence, in summary, we have²

$$\begin{aligned} (2.1) \quad \mathrm{R}\Gamma_{Z_+} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \mathrm{or}_{N|M} \\ \cong \mathrm{R}\dot{\pi}_{N*} \mathrm{R}\rho_*^+ \left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+} \right). \end{aligned}$$

In the rest of this section, we prove

$$(2.2) \quad \mathrm{R}\rho_*^+ \left(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+} \right) [1] \cong \mathrm{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N)$$

on $T_N^*Y \setminus N$. Combining (2.1) and (2.2), we get isomorphism (1.3).

We prepare two lemmas for the second part of the proof. Lemma 1.1 follows from the following Lemma 2.3 with $I = T_N^*Y \setminus N$.

² Takeuchi also proves (2.1) in the case where (B.1) and (B.2) are fulfilled; see K. Takeuchi : Edge of the wedge type theorems for hyperfunction solutions, preprint (Jan. 1996). If we assume (B.2), $M \hookrightarrow X$ is non characteristic for F on $N^+(\subset T^*M)$, and we immediately obtain (2.0) by applying Theorem 6.7.1 of [KS1] (see also Corollary 6.7.3).

Lemma 2.3. *Let I be a conic open subset of $T_N^*Y \setminus N$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$ -module on a conic neighborhood of $\rho^{-1}(I)$, with $\rho : T_N^*X \rightarrow T_N^*Y$. Assume the following :*

- (a.1) $\varphi : Y \rightarrow X$ is non characteristic for $\mathcal{M}$ on a neighborhood of I in the sense of [SKK, II, Def.3.5.4].
- (a.2) For a conic neighborhood U of $\rho^{-1}(I) \cap T_M^*X$,

$$U \cap (T_N^*X)^+ \cap \text{Supp}(\mathcal{M}) = \emptyset.$$

Then (1) ρ is finite on $\rho^{-1}(I) \cap (T_N^*X)^+ \cap \text{Supp}(\mathcal{M})$. (2) If we set

$$\mathcal{N}^+ = \rho_*((\mathcal{E}_{Y \rightarrow X} \otimes_{\mathcal{E}_X} \mathcal{M}) \otimes \mathbf{C}_{(T_N^*X)^+}),$$

$\mathcal{N}^+$ is a coherent $\mathcal{E}_Y|_I$ -module.

(We omit the proof. Cf. [SKK, II, Thm.3.5.3].)

Lemma 2.4. *Let $\mathcal{M}, \mathcal{N}^+$ be as in Lemma 2.3. Then there exists a commutative diagram on I*

$$\begin{array}{ccc} \text{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{E}_Y) & \xrightarrow{\sim} & \text{R}\rho_*^+(\text{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X \leftarrow Y})|_{(T_N^*X)^+})[1] \\ \uparrow & & \uparrow \\ \text{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^*\mathcal{M}, \mathcal{E}_Y) & \xrightarrow{\sim} & \text{R}\rho_*(\text{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X \leftarrow Y})|_{T_N^*X})[1] \end{array}$$

and every horizontal arrow is an isomorphism, where $\rho^+ = \rho|(T_N^*X)^+$.

Proof. This follows from the definition of $\mathcal{N}^+$ and [SKK, II, Thm.3.5.6].
Q.E.D.

Since $\mathcal{N}^+$ is coherent over $\mathcal{E}_Y|_{T_N^*Y}$ and ρ^+ is finite on $\text{Supp}(\widetilde{\mathcal{M}}) \cap (T_N^*X)^+$, by Lemma 2.4, we have

$$\begin{aligned} \text{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) &\cong \text{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{E}_Y) \otimes_{\mathcal{E}_Y}^L \mathcal{C}_N \\ &\cong \rho_*^+[\text{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X \leftarrow Y})|_{(T_N^*X)^+}] \otimes_{\mathcal{E}_Y}^L \mathcal{C}_N[1] \\ &\cong \rho_*^+[\text{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X \leftarrow Y})|_{(T_N^*X)^+}] \otimes_{\rho^{-1}\mathcal{E}_Y}^L \rho^{-1}\mathcal{C}_N[1]. \end{aligned}$$

Using the $\mathcal{E}_X$ -homomorphism $\mathcal{E}_{X \leftarrow Y} \otimes_{\rho^{-1}\mathcal{E}_Y} \rho^{-1}\mathcal{C}_N \rightarrow \mathcal{C}_{N|X}$ [KK, II], we have

$$\begin{aligned} &\text{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) \\ (2.3) \quad &\rightarrow \rho_*^+[\text{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{E}_{X \leftarrow Y} \otimes_{\rho^{-1}\mathcal{E}_Y}^L \rho^{-1}\mathcal{C}_N)|_{(T_N^*X)^+}][1] \\ &\rightarrow \rho_*^+[\text{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_{N|X})|_{(T_N^*X)^+}][1]. \end{aligned}$$

Let $q \in T_N^*Y \setminus N$. For $k \in \mathbf{Z}$, looking at the stalk on q , we have from (2.3)

$$\mathcal{E}xt_{\mathcal{E}_Y}^k(\mathcal{N}_q^+, \mathcal{C}_{Nq}) \rightarrow \bigoplus_{p \in (T_N^*X)^+ \cap \text{Supp}(\widetilde{\mathcal{M}}) \cap \rho^{-1}(q)} \mathcal{E}xt_{\mathcal{E}_X}^{k+1}(\widetilde{\mathcal{M}}_p, (\mathcal{C}_{N|X})_p).$$

It follows from the division theorem for the $\mathcal{E}_X$ -module $\mathcal{C}_{N|X}$ [KK, II, Prop.3; KS2, 6.3.1] and the definition of $\mathcal{N}^+$ that this is an isomorphism for any $k \in \mathbf{Z}$; therefore (2.3) is an isomorphism in $D^b(T_N^*Y \setminus N)$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. If φ is micro-hyperbolic for $\widetilde{\mathcal{M}}$ at $p \in T_M^*X \times_M N$, we have [KS2]

$$\text{R}\Gamma_{\pi_M^{-1}(Z_+)} \text{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, \mathcal{C}_M)_p = 0.$$

Since this holds at all $p \in (T_M^*X \setminus M) \times_M N$ by assumption (B.2), we have an isomorphism

$$\text{R}\Gamma_{Z_+} \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)|_N \xrightarrow{\sim} \text{R}\Gamma_{Z_+} \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N.$$

Combining this and (1.3), we get (1.4). Q.E.D.

3. Application

Let $M_+ = Z_+ \setminus N$. Isomorphism (1.3) gives a description of the structure of the sheaf $\mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \Gamma_{M_+} \mathcal{B}_M)|_N$ in terms of a system of micro-differential equations on the boundary.

Theorem 3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -module. Assume (A.1) and (A.2). Assume moreover $\mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \mathcal{A}_M) = 0$ for all $k > 0$. Then*

$$(3.1) \quad \mathcal{E}xt_{\mathcal{D}_X}^0(\mathcal{M}, \Gamma_{M_+} \mathcal{B}_M)|_N \cong \text{Ker}(\mathcal{H}om_{\mathcal{D}_Y}(\varphi^* \mathcal{M}, \mathcal{B}_N) \rightarrow \dot{\pi}_{N*} \mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N)),$$

where $\varphi^* \mathcal{M} = \mathcal{D}_{Y \rightarrow X} \otimes_{\varphi^{-1} \mathcal{D}_X} \varphi^{-1} \mathcal{M}$ and $\mathcal{N}^+$ is the coherent $\mathcal{E}_Y$ -module on $T_N^*Y \setminus N$ given in Lemma 1.1, and

$$(3.2) \quad \mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \Gamma_{M_+} \mathcal{B}_M)|_N \cong H^k \text{R}\dot{\pi}_{N*} \text{R}\mathcal{H}om_{\mathcal{E}_Y}(\varphi^* \widetilde{\mathcal{M}} / \mathcal{N}^+, \mathcal{C}_N)$$

for $k \neq 0$.

Proof. Let us first recall that, if $\varphi : Y \rightarrow X$ is non characteristic for a $\mathcal{D}_X$ -module $\mathcal{M}$, we have a canonical isomorphism

$$\text{R}\Gamma_N \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \text{or}_{N|M}[1] \cong \text{R}\mathcal{H}om_{\mathcal{D}_Y}(\varphi^* \mathcal{M}, \mathcal{B}_N)$$

[SKK, II, Cor.3.5.8]. By the proof of Theorem 1.2, the following diagram is commutative :

$$\begin{array}{ccc}
 R\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \text{or}_{N|M}[1] & \longrightarrow & R\dot{\pi}_{N*} R\mathcal{H}om_{\mathcal{E}_Y}(\varphi^* \widetilde{\mathcal{M}}, \mathcal{C}_N) \\
 \downarrow & & \downarrow \\
 R\Gamma_{Z_+} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \otimes \text{or}_{N|M}[1] & \xrightarrow[(1.3)]{\sim} & R\dot{\pi}_{N*} R\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N).
 \end{array}$$

Hence, from the Mayer-Vietoris cohomological sequence, we have a long exact sequence

$$\begin{aligned}
 \cdots \rightarrow \mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}, \Gamma_{M_+} \mathcal{B}_M)|_N \otimes \text{or}_{N|M} &\rightarrow \mathcal{E}xt_{\mathcal{D}_Y}^k(\varphi^* \mathcal{M}, \mathcal{B}_N) \\
 &\rightarrow H^k R\dot{\pi}_{N*} R\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N) \rightarrow \cdots,
 \end{aligned}$$

where the second arrow is factorized as follows :

$$\begin{aligned}
 \mathcal{E}xt_{\mathcal{D}_Y}^k(\varphi^* \mathcal{M}, \mathcal{B}_N) &\xrightarrow{\alpha} H^k R\dot{\pi}_{N*} R\mathcal{H}om_{\mathcal{E}_Y}(\varphi^* \widetilde{\mathcal{M}}, \mathcal{C}_N) \\
 &\xrightarrow{\beta} H^k R\dot{\pi}_{N*} R\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{N}^+, \mathcal{C}_N).
 \end{aligned}$$

Since $\mathcal{E}xt_{\mathcal{D}_Y}^k(\varphi^* \mathcal{M}, \mathcal{A}_N) = 0$ for $k > 0$ by assumption, α is surjective for all $k \in \mathbf{Z}$ and is an isomorphism for $k > 0$. On the other hand, since $\mathcal{N}^+$ is a direct summand of $\varphi^* \widetilde{\mathcal{M}}$ as an $\mathcal{E}_Y$ -module, β is surjective and

$$\text{Ker}(\beta) = H^k R\dot{\pi}_{N*} R\mathcal{H}om_{\mathcal{E}_Y}(\varphi^* \widetilde{\mathcal{M}}/\mathcal{N}^+, \mathcal{C}_N).$$

Hence, using an isomorphism $\text{or}_{N|M} \cong \mathbf{C}_N$ (see Remark 1 below), we obtain (3.1) and (3.2). Q.E.D.

Remark 1. The following diagram is commutative and every vertical arrow is an isomorphism :

$$\begin{array}{ccc}
 \mathbf{C}_{M_+} & \longrightarrow & \mathbf{C}_{Z_+} \\
 \downarrow & & \downarrow \\
 R\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_M) & \longrightarrow & R\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{M_+}, \mathbf{C}_M).
 \end{array}$$

Hence we have an isomorphism $\eta : \mathbf{C}_N \rightarrow \text{or}_{N|M}$ such that

$$\begin{array}{ccccc}
 \mathbf{C}_{Z_+} & \longrightarrow & \mathbf{C}_N & \longrightarrow & \mathbf{C}_{M_+}[1] \\
 \downarrow & & \downarrow \eta & & \downarrow \\
 R\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{M_+}, \mathbf{C}_M) & \longrightarrow & \text{or}_{N|M} & \longrightarrow & R\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{Z_+}, \mathbf{C}_M)[1]
 \end{array}$$

becomes commutative. (This corresponds to choosing a non-degenerate section df of T_N^*M as positive orientation.) Note that the following diagram is then commutative for $F \in \text{Ob}(\mathcal{D}^b(M))$:

$$\begin{array}{ccccc}
 R\Gamma_{M_+} F|_N & \xrightarrow{1 \otimes \eta} & R\Gamma_{M_+} F|_N \otimes \text{or}_{N|M} & \longrightarrow & \\
 \downarrow \cong & & & & \\
 R\mathcal{H}om_{\mathbf{C}}(\mathbf{C}_{M_+}, F)|_N & \longrightarrow & R\mathcal{H}om_{\mathbf{C}}(\text{or}_{N|M}[-1], F)|_N & \xrightarrow{\cong} & \\
 & & & \longrightarrow & R\Gamma_N F|_N[1] \otimes \text{or}_{N|M} \\
 & & & & \downarrow \cong \\
 & & & \xrightarrow{\cong} & R\Gamma_N F|_N[1] \otimes \text{or}_{N|M}^{\vee}
 \end{array}$$

with $\text{or}_{N|M}^{\vee} = \mathcal{H}om_{\mathbf{C}}(\text{or}_{N|M}, \mathbf{C}_N)$, which is canonically isomorphic to $\text{or}_{N|M}$. (The topological boundary value morphism for F is defined [S2, S3] as anti-clockwise composition of morphisms, from $R\Gamma_{M_+} F|_N$ to $R\Gamma_N F|_N[1] \otimes \text{or}_{N|M}^{\vee}$, in this diagram.)

Remark 2. For single differential equations, Oaku [O, Sect.3] extends (3.1) to the case where condition (A.2) is satisfied locally on T_N^*Y . If $\mathcal{N}^+ = 0$ in that case, this has been first treated by Kaneko [Kn].

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