

Remarks on High Linear Syzygy

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In this note we explain some properties that follow from a high linear syzygy. We consider the r -th, $(r-1)$ -st, and $(r-2)$ -nd linear syzygies over a polynomial ring in r variables. The most interesting, and the only nontrivial, case is the $(r-2)$ -nd linear syzygy which produces skew-symmetric matrices that are helpful in understanding certain geometric situations.

Let $S = K[x_1, \dots, x_r] = \bigoplus_{d \geq 0} S_d$ be a polynomial ring over a field K with the usual $\mathbb{N}$ -grading. Let $M = \bigoplus_{d \geq t} M_d$ be a finitely generated graded S -module. As usual, $M(n)$ denotes the same module M with its degrees shifted to the left by n units, i.e., $M(n)_d := M_{d+n}$. Let $F_\bullet$ denote the minimal graded free resolution of M over S , i.e.,

$$F_\bullet : 0 \rightarrow F_r \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_p \rightarrow \cdots \rightarrow F_0 \rightarrow 0,$$

where $F_p = \bigoplus_{q \in \mathbb{Z}} S(-q-p)^{b_{p,q}(M)}$.

The reason for the extra degree shift of $-p$ in the p -th free module F_p is because the entries of the maps in the minimal resolution are all of positive degrees. We say that M has a q -linear p -th syzygy if the graded betti number $b_{p,q}(M) \neq 0$. When $q = 0$ we drop 0- to call it a linear p -th syzygy. The most important result concerning the

linear syzygy is the vanishing theorem of Green ([G, Theorem 3.a.1]) which asserts that if M has a linear p -th syzygy, then $\dim M_0 \geq p$ under certain conditions, which are satisfied in geometric situations. Some progress in finding more precise algebraic conditions affecting the linear syzygies were made in [EK1] and [EK2], but much more remains a mystery.

Tor -modules of the graded modules are also graded and can be computed in the usual way using $M(n) \otimes_S N(q) \cong (M \otimes_S N)(n+q)$. Let K denote the graded S -module S/S_+ , where $S_+ := \bigoplus_{d>0} S_d$ is the unique homogeneous maximal ideal. We note that K is a graded module concentrated in degree 0. Using $F_\bullet \otimes_S K$, we compute

$$Tor_p^S(M, K) = \bigoplus_{q \in \mathbb{Z}} K(-q-p)^{b_{p,q}(M)},$$

which implies that

$$b_{p,q}(M) = \dim_K Tor_p^S(M, K)_{q+p}.$$

We may also compute $Tor_p^S(M, K)$ using the Koszul resolution $G_\bullet$ of K , where

$$G_\bullet : 0 \rightarrow S(-r) \rightarrow S(-r+1)^{\binom{r}{r-1}} \rightarrow \cdots \rightarrow S(-p)^{\binom{r}{p}} \rightarrow \cdots \rightarrow S \rightarrow 0.$$

Using $M \otimes_S G_\bullet$, we again compute

$$Tor_p^S(M, K) = \text{homology}(M(-p-1)^{\binom{r}{p+1}} \rightarrow M(-p)^{\binom{r}{p}} \rightarrow M(-p+1)^{\binom{r}{p-1}}),$$

and hence

$$(*) \quad Tor_p^S(M, K)_{q+p} = \text{homology}(M_{q-1}^{\binom{r}{p+1}} \rightarrow M_q^{\binom{r}{p}} \rightarrow M_{q+1}^{\binom{r}{p-1}}).$$

Since the differential maps in the Koszul complex is given by the natural maps between the wedge products, it is customary to write the right-hand-side of $(*)$ above as:

$$(**) \quad \text{homology} \left(\bigwedge^{p+1} S_1 \otimes_K M_{q-1} \xrightarrow{d_{p+1}} \bigwedge^p S_1 \otimes_K M_q \xrightarrow{d_p} \bigwedge^{p-1} S_1 \otimes_K M_{q+1} \right).$$

We remark that $\mathcal{K}_{p,q}(M)$ was the notation for $Tor_p^S(M, K)_{q+p}$ Green used in [G] in his systematic study of the relationship between the graded resolution and the geometry of projective algebraic variety.

Let $\{x_1, \dots, x_r\}$ be a basis of S_1 . To simplify notation we write $x_{i_1 \dots i_j}^*$ to denote the wedge product of $\{x_1, \dots, x_r\} - \{x_{i_1}, \dots, x_{i_j}\}$. We first consider some trivial cases.

p=r. Suppose that M has a q -linear r -th syzygy. Since $\bigwedge^{r+1} S_1 = 0$, this syzygy corresponds to a nonzero element $a \in M_q$ in the kernel of d_r . Since

$$d_r(x_1 \wedge \dots \wedge x_r \otimes a) = \sum_{1 \leq i \leq r} x_i^* \otimes (-1)^i x_i a,$$

$x_i a = 0$, for all $1 \leq i \leq r$. Hence a is a nonzero element of degree q that is killed by S_+ . The converse is equally trivial for us to state:

M has a q -linear r -th syzygy if and only if $(\text{Soc } M)_q \neq 0$.

p=r-1. Suppose now that M has a q -linear $(r-1)$ -st syzygy. By **(**)** above this syzygy is determined by an element in the kernel of d_{r-1} that is not in the image of d_r . Let a_i , $1 \leq i \leq r$, be elements of M_q such that $\sum_{1 \leq i \leq r} x_i^* \otimes a_i$ is in the kernel of d_{r-1} . Using

$$d_{r-1} \left(\sum_{1 \leq i \leq r} x_i^* \otimes a_i \right) = \sum_{1 \leq i < j \leq r} x_{ij}^* \otimes \pm(x_i a_j - x_j a_i),$$

We can easily check the validity of the following statement:

M has a q -linear $(r-1)$ -st syzygy if and only if there is a $2 \times r$ matrix

$$\begin{pmatrix} x_1 & \dots & x_r \\ a_1 & \dots & a_r \end{pmatrix}$$

such that

- i) $a_i \in M_q$, for all $1 \leq i \leq r$,
- ii) all of its 2×2 minors are 0, and
- iii) there is no element $a \in M_{q-1}$ such that $a_i = (-1)^i x_i a$ for all $1 \leq i \leq r$.

We now consider the main case.

p=r-2. Let M has a q -linear $(r-2)$ -nd syzygy. As before, we can find elements a_{ij} , $1 \leq i < j \leq r$, of M_q such that $\sum_{1 \leq i < j \leq r} x_{ij}^* \otimes a_{ij}$ is in the kernel of d_{r-2} . Since

$$d_{r-2} \left(\sum_{1 \leq i < j \leq r} x_{ij}^* \otimes a_{ij} \right) = \sum_{1 \leq i < j < k \leq r} x_{ijk}^* \otimes \pm(x_i a_{jk} - x_j a_{ik} + x_k a_{ij}),$$

$x_i a_{jk} - x_j a_{ik} + x_k a_{ij} = 0$, for all $1 \leq i < j < k \leq r$. Since these are nothing other than 4×4 pfaffians of Q below involving the first row and column, we have the following characterization:

M has a q -linear $(r-2)$ -nd syzygy if and only if there is a $(r+1) \times (r+1)$ skew symmetric matrix

$$Q = \begin{pmatrix} 0 & x_1 & \cdots & \cdots & x_r \\ -x_1 & 0 & \cdots & a_{ij} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & -a_{ji} & \cdots & \ddots & \vdots \\ -x_r & \vdots & \cdots & \cdots & 0 \end{pmatrix} \quad (1)$$

such that

- i) the first row spans S_1 ,
- ii) $a_{ij} \in M_q$ for $1 \leq i < j \leq r$,
- iii) each 4×4 pfaffian of Q involving the first row and column is zero, and
- iv) there are no elements $a_i \in M_{q-1}$ such that $a_{ij} = \pm(x_i a_j - x_j a_i)$ for all $i < j$.

We consider two geometric situations where all, not just the ones involving the first row and column, 4×4 pfaffians are zero. To consider general pfaffians the products of elements in M have to be defined. The first situation deals with the homogeneous coordinate ring of a set of points in, or more generally, a 0-dimensional subscheme of, $\mathbb{P}^{r-1}$, and the second deals with the canonical image of a nonsingular projective curve. We assume that the field K is algebraically closed in the rest of this note.

X is a set of points. Let X be a 0-dimensional subscheme of $\mathbb{P}^{r-1}$ in "general" position. Our discussion of this case is not rigorous because we use "general" to mean the argument below works. Let S be the homogeneous coordinate ring of $\mathbb{P}^{r-1}$, and I the saturated ideal defining X . Suppose that S/I has a 1-linear $(r-2)$ -nd syzygy. Then we may view Q in (1) above as a matrix of linear forms of S . The following trick expresses any 4×4 pfaffian of Q in terms of those involving the first row and column: for $1 \leq i < j < k < l \leq r$,

$$x_i(a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk})$$

$$\begin{aligned}
&= a_{ij}(x_l a_{ik} - x_k a_{il} + x_i a_{kl}) - a_{ik}(x_l a_{ij} - x_j a_{il} + x_i a_{jl}) \\
&+ a_{il}(x_k a_{ij} - x_j a_{ik} + x_i a_{jk}) \in I.
\end{aligned} \tag{2}$$

Since S/I is a 1-dimensional Cohen-Macaulay ring, we may assume that each x_i is a nonzero divisor on S/I , and hence the 4×4 pfaffian $a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}$ determined by $i < j < k < l$ is in I .

Since the vector space spanned by the entries of Q is of dimension r , the following result forces Q to have a generalized zero, i.e., one can produce a 0 off the diagonal after performing suitable (symmetric) row and column operations on Q .

Lemma ([KS, Lemma 1.5]). Let T be a $v \times v$ skew symmetric matrix of linear forms. If $\dim T < 2v - 3$, then T has a generalized zero.

We may, after a suitable row and column operations, put Q in the form

$$\left(\begin{array}{cc|c} 0 & 0 & A \\ 0 & 0 & \\ \hline & & \\ -A^t & & * \end{array} \right),$$

where A is a $2 \times (r-1)$ matrix of linear forms. We assume that the points in X are in "general" position so that if A is not 1-generic, i.e., one can produce a 0 after performing suitable row and column operations, then the whole column of A containing zero is zero. Since the 2×2 minors of A are 4×4 pfaffians of Q , this assumption is satisfied when I doesn't contain too many rank 2 quadrics, e.g., when X contains at least $2r-1$ reduced points in linearly general position because I can't contain a product of linear forms in this case. Under this assumption we may put Q in the form

$$\left(\begin{array}{c|c} 0 & A \\ \hline -A^t & * \end{array} \right),$$

where A is a $m \times n$ 1-generic matrix. Since $m+n = r+1$ and $\dim A = r$, a result of Eisenbud ([E, Theorem 5.1]) implies that 2×2 minors of A define a rational

normal curve. This argument provides a reason for one, more involved, direction of the following result of Green ([G, Theorem 3.c.6]).

Theorem (Strong Castelnuovo Lemma). Let X be a set of points in $\mathbb{P}^{r-1}$ in general position. Then X lies on a rational normal curve if and only if S/I has a 1-linear $(r-2)$ -nd syzygy.

We remark here that Yanagawa used the same result of Eisenbud in proving his Generalized Castelnuovo's Lemma ([Y, Theorem 2.1]).

X is a nonsingular projective curve. We sketch the argument given in [KS] to prove a result of Green and Lazarsfeld ([GL]) on normal generation of line bundles. Let X be a nonsingular projective curve in $\mathbb{P}^{r-1}$. Let $\mathcal{L}$ be a very ample line bundle on X . Write $r = h^0(\mathcal{L})$, the dimension of $H^0(X, \mathcal{L})$, and $S = \text{Sym } H^0(X, \mathcal{L})$, the symmetric algebra. For a line bundle $\mathcal{F}$ on X , let $M(\mathcal{F})$ denote the graded S -module $\bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}\mathcal{L}^n)$. There is a natural map $\varphi : S \rightarrow M(\mathcal{O})$ whose kernel is the ideal I of the image of the morphism f defined by $\mathcal{L}$. $\mathcal{L}$ is said to be normally generated if $f(X)$ is a normal subvariety of $\mathbb{P}^{r-1}$, or equivalently, the map φ is onto. In terms of the graded betti numbers, this condition is equivalent to $b_{0,q}(M(\mathcal{O})) = 0$, for all $q > 0$. (In fact, for all $q \geq 2$ because φ is onto in degree 1.) To obtain a $(r-2)$ -nd syzygy we apply the following result of Green.

Duality Theorem ([G] or [EKS]). Let ω denote the canonical bundle on X . For any line bundle $\mathcal{F}$ on X ,

$$b_{p,q}(M(\mathcal{F})) = b_{r-2-p, r-q}(M(\mathcal{F}^{-1}\omega)).$$

Suppose that $\mathcal{L}$ is not normally generated. Since $b_{0,q}(M(\mathcal{O})) \neq 0$ for some $q \geq 2$, $b_{r-2, r-q}(M(\omega)) \neq 0$ by the Duality Theorem. We now assume that $\text{Cliff}(X)$ will be defined below.)

$$\deg \mathcal{L} \geq 2g + 1 - \text{Cliff}(X). \quad (3)$$

This implies that $H^0(X, \mathcal{L}^n\omega) = 0$ for all $n \leq -2$ and $h^1(\mathcal{L}) := \dim H^1(X, \mathcal{L}) \leq 1$ (see [GL] or [KS]). Hence $b_{r-2, r-2}(M(\omega))$ is the only nonzero graded betti numbers

for $q \geq 2$, and $M(\omega)$ has a (0-)linear $(r-2)$ -nd syzygy. As in the previous case we get a skew symmetric matrix Q in (1), where a_{ij} are sections of the canonical bundle. Since X is irreducible, the similar argument as in (2) shows that all 4×4 pfaffians of Q are zero when viewed as elements either in $H^0(\mathcal{L}\omega)$ or $H^0(\omega^2)$. If $h^1(\mathcal{L}) = 0$, we take B to be the $r \times r$ skew symmetric submatrix of Q without the first row and the first column. If $h^1(\mathcal{L}) = 1$, we let $B = Q$. When $h^1(\mathcal{L}) = 1$, we may choose a nonzero section of $H^0(X, \mathcal{L}^{-1}\omega) \cong H^1(\mathcal{L})$ to define an injection $H^0(X, \mathcal{L}) \rightarrow H^0(X, \omega)$ so that each x_i can be viewed as a section of ω . Thus B is a $(h^0(\mathcal{L}) + h^1(\mathcal{L})) \times (h^0(\mathcal{L}) + h^1(\mathcal{L}))$ skew symmetric matrix with entries in $H^0(\omega)$ such that all of its 4×4 pfaffians are in the ideal of the canonical curve. Since $\dim B \leq g$, where g is the genus, the degree bound in (3) and the earlier lemma imply that B has a generalized zero. Since X is irreducible, the ideal of the canonical curve can't have a rank 2 quadric. Hence we may, after suitable row and column operations, transform B to

$$\left(\begin{array}{c|c} 0 & A \\ \hline - & - \\ -A^t & * \end{array} \right),$$

where A is 1-generic.

It is not hard to check that if A is of size $m \times n$, then $m + n = h^0(\mathcal{L}) + h^1(\mathcal{L})$ and $m, n \geq 2$. Let $\mathcal{F} := \text{Im}(A : \mathcal{O}^m \rightarrow \omega^n)$. Since all 2×2 minors vanish on the canonical image of X , $\mathcal{F}$ is a rank one subsheaf of ω^n , and hence a line bundle because X is nonsingular. Since the rows of a 1-generic matrix is linearly independent $h^0(\mathcal{F}) \geq m$. It can further be shown ([KS, Claim 2]) that $h^1(\mathcal{F}) \geq n$. We now recall the definition of the Clifford index of X :

$$\text{Cliff}(X) := \inf \{g + 1 - (h^0(\mathcal{G}) + h^1(\mathcal{G})) : \mathcal{G} \text{ is a line bundle with } h^0(\mathcal{G}), h^1(\mathcal{G}) \geq 2\}.$$

Our discussion on $\mathcal{F}$ above shows that

$$\text{Cliff}(X) \leq g + 1 - (h^0(\mathcal{F}) + h^1(\mathcal{F})) \leq g + 1 - (h^0(\mathcal{L}) + h^1(\mathcal{L})).$$

Applying Riemann-Roch Theorem, $h^0(\mathcal{L}) = \deg \mathcal{L} - g + 1 + h^1(\mathcal{L})$, we get

$$\deg \mathcal{L} \leq 2g - 2h^1(\mathcal{L}) - \text{Cliff}(X),$$

which contradicts the assumption on the degree of $\mathcal{L}$ in (3). We have thus proved the following result of Green and Lazarsfeld ([GL]):

Theorem. Let $\mathcal{L}$ be a very ample line bundle on a nonsingular projective curve X of genus g . If $\deg \mathcal{L} \geq 2g + 1 - 2h^1(\mathcal{L}) - \text{Cliff}(X)$, then $\mathcal{L}$ is normally generated.

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