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ABSTRACT

We construct point vortex equilibria with strengths quantized by multiples of 2π in a fixed background vorticity field on the surface of a curved torus. The background vorticity consists of two terms: first, a term exponentially related to the stream function and a second term arising from the curvature of the torus, which leads to a Liouville-type equation for the stream function. By using a stereographic projection of the torus onto an annulus in a complex plane, the Liouville-type equation admits a class of exact solutions, given in terms of a loxodromic function on the annulus. We show that appropriate choices of the loxodromic function in the solution lead to stationary vortex patterns with $4\hat{n}$ point vortices of identical strengths, $\hat{n} \in \mathbb{N}$. The quantized point vortices are stationary in the sense that they are equilibria of a “one-way interaction” model where the evolution of point vortices is subject to the continuous background vorticity, while the background vorticity distribution is not affected by the velocity field induced by the point vortices. By choosing loxodromic functions continuously dependent on a parameter and taking appropriate limits with respect to this parameter, we show that there are solutions with inhomogeneous point vortex strengths, in which the exponential part of the background vorticity disappears. The point vortices are always located at the innermost and outermost rings of the torus owing to the curvature effects. The topological features of the streamlines are found to change as the modulus of the torus changes.

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I. INTRODUCTION

The formation of a vortex-like tube having its circulation quantized by the Planck constant $\hbar$ in superfluid helium was theoretically predicted by Feynman (1955). This prediction was confirmed later in a laboratory experiment of a rotating superfluid (two-dimensional) thin-film by Yarmchuk *et al.* (1979), who found identical vortex-like cores organized in polygonal patterns. More recently, many lattice patterns of quantized vortices have been observed in Bose–Einstein condensates (BECs) (Abo-Shaeer *et al.*, 2001; Engels *et al.*, 2002; and 2003). Abrisokov, in his Nobel lecture (Abrikosov, 2004), suggested *vortex lattice theory* as a theoretical model for investigating such quantized vortex patterns. Suppose that the velocity field in the azimuthal direction induced by a localized vortex core depends only on the distance s from the core and its magnitude decays as $1/s$ for $s \gg 1$. Vortex lattice patterns are, then, characterized by equilibrium states of vortex cores under their mutual interaction through the induced velocity field. Although it is a simple phenomenological model, the theoretical treatment of this model is much easier than that of physical models based on partial differential equations, such as the Gross–Pitaevskii equation and the Ginzburg–Landau equation.

If we assume that the vorticity in the vortex core is concentrated at a discrete point, i.e., Dirac’s δ -measure, then the vortex lattice is equivalent to a system of n point vortices with the circulation around a core corresponding to the strength of the point vortex there. The n -vortex problem has been utilized as a theoretical model to describe the equilibria and dynamics of vortices in the literature of

hydrodynamics. See the book by [Newton \(2001\)](#) and the review article by [Newton and Chamoun \(2009\)](#) for references. Stationary lattice patterns formed by point vortices are referred to as *vortex crystals* ([Aref et al., 2003](#)), whose investigation originated in the theory of vortex atoms due to Thomson in the 19th century. When we use vortex crystals as a model for vortex lattice patterns in superfluids and BECs, the strengths of the point vortices must be quantized. This means that the ratios of the strengths of all point vortex pairs must be integers. This is an important condition since it is in contrast to the vortex motion of hydrodynamics in which the strengths of point vortices can be chosen to be arbitrary real numbers. It is known that vortex crystals with the same strength in the plane tend to acquire a certain symmetry, but, in general, vortex crystals in an asymmetric configuration do not usually have quantized strengths. In addition, most of the vortex crystals obtained so far are *relative equilibria* that translate or rotate at a constant speed without changing their relative configuration. To study vortex lattice patterns, we need to obtain fixed (or stationary) equilibria where all of the point vortices remain completely stationary. However, a lesser number of fixed equilibria are known than relative equilibria.

Vortex crystals can also be considered on two-dimensional Riemannian manifolds. Vortex dynamics on a sphere ([Newton, 2001](#)) is well-studied due to its geophysical relevance. [Kimura \(1999\)](#) considered vortex dynamics on surfaces of constant curvature, including the sphere S^2 and a hyperbolic sphere H^2 . The review by [Turner et al. \(2010\)](#) discusses the general significance of vortices on surfaces with a non-constant Gaussian curvature and the effects of the non-constant curvature on vortex dynamics. We are interested in how the non-constant curvature and the handle structure of a torus affect vortex crystal patterns on the torus. On a compact surface such as a torus, just like on a sphere, the condition that the total vorticity needs to vanish has to be satisfied. [Sakajo \(2019b\)](#) formulated a numerical method to find vortex crystals on a torus and found that the sum of the vortex circulations does not vanish, in general, for such crystals. This means that constant background vorticity needs to be added in order for the total vorticity to vanish. [Sakajo \(2019b\)](#) further found that, in general, the point vortex strengths are not quantized when the background vorticity is constant. In order to overcome these difficulties and obtain vortex crystals with quantized strengths, in this paper, we introduce a non-constant background vorticity distribution, given by a modified Liouville equation on the torus.

Consider that the stream function ψ and the vorticity Ω associated with an incompressible velocity field in $\mathbb{R}^2$ satisfy $\Omega = -ce^{d\psi}$ for $c, d \in \mathbb{R}$ with $cd < 0$. Without loss of generality, we take $c = 1$ and $d = -2$ so that $cd = -2$. Since $\Omega = -\nabla^2\psi$, where ∇^2 is the planar Laplacian operator, this results in the planar Liouville equation

$$\nabla^2\psi = e^{-2\psi} \quad (1)$$

for the stream function ψ . In the vorticity-stream function formulation of ideal fluid flow in $\mathbb{R}^2$, solutions to (1) also provide steady solutions of the Euler equation ([Majda and Bertozzi, 2001](#)). This is because the vorticity equation in this case reads $\Omega_t + \mathcal{J}(\Omega, \psi) = 0$, where $\mathcal{J}$ is the Jacobian, and any vorticity functional of the form $\Omega = V(\psi)$, where V is a differentiable function, solves this vorticity equation. Furthermore, the Liouville equation is a fundamental nonlinear elliptic partial differential equation that appears in field theory and plasma physics. Mathematically, an important class of exact solutions to the planar Liouville equation are known ([Bhutani et al., 1994](#); [Calogero and Degasperis, 1982](#)), and [Crowdy \(1997\)](#) has obtained the most general form of these solutions.

Vortex solutions of the Liouville equation have been studied in a variety of contexts. For example, see the work of [Horváthy and Yéra \(1998\)](#) and [Akerblom et al. \(2011\)](#) for a discussion in the context of field theory. In the context of hydrodynamics, [Stuart \(1967\)](#) constructed singly-periodic, everywhere smooth solutions to (1) that he used to model planar shear flows. [Crowdy \(2003\)](#) constructed solutions with a single point vortex surrounded by polygonal patterns of smooth vortices. Such solutions have been described as forming a “rational necklace” due to the rational functions appearing in the theory ([Tur and Yanovsky, 2004](#); [Tur et al., 2011](#)). [Krishnamurthy et al. \(2019\)](#) constructed exact solutions with two embedded point vortices in stationary equilibrium with the background vorticity term $-e^{-2\psi}$. Indefinitely iterated solutions called “Liouville chains,” containing increasing numbers of embedded point vortices in each iteration, have been constructed recently by [Krishnamurthy et al. \(2021\)](#); see the work of [Krishnamurthy et al. \(2020\)](#) for the closely-related “pure” point vortex equilibria, which are limiting solutions in which the background vorticity vanishes.

The class of exact solutions to (1) can be written as

$$\psi(\zeta, \bar{\zeta}) = -\log \left[\frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \right]. \quad (2)$$

This formula is given in terms of an arbitrary complex-valued analytic function $f(\zeta)$ and its derivative $f'(\zeta)$, with various choices of this function allowing us to construct many exact solutions in the complex ζ -plane in a systematic manner. If the analytic function $f(\zeta)$ is chosen so that $f'(\zeta)$ has zeros at points $\zeta = \nu_k \in \mathbb{C}$, $k = 1, \dots, n$, then the stream function (2) has logarithmic singularities at each of the points ν_k . These logarithmic singularities represent embedded point vortices in the flow. For n point vortices with strengths $\Gamma_k \in \mathbb{R}$, $k = 1, \dots, n$, embedded in the background vorticity given by $-e^{-2\psi}$, we are, in fact, solving the Liouville-type equation

$$\nabla^2\psi = e^{-2\psi} - \sum_{k=1}^n \Gamma_k \delta_{\nu_k}. \quad (3)$$

To show that a solution ψ of (3) is a stationary solution, it is necessary and sufficient to confirm that the point vortices in (3) remain stationary under the interaction with the other point vortices and the Liouville-type background vorticity field.

When we construct solutions to the Liouville-type equation on a torus based on the analytic formula (2), Eq. (1) needs to be modified by adding a term corresponding to the curvature on the right-hand side (Sakajo, 2019a),

$$\nabla_{\mathbb{T}_\alpha}^2 \psi = e^{-2\psi} - \kappa(\mathbb{T}_\alpha). \quad (4)$$

Here, $\mathbb{T}_\alpha$ is the torus with aspect ratio α , $\nabla_{\mathbb{T}_\alpha}^2$ is the Laplace–Beltrami operator on $\mathbb{T}_\alpha$, and $\kappa(\mathbb{T}_\alpha)$ is the non-constant Gaussian curvature of $\mathbb{T}_\alpha$. The vorticity and stream function on $\mathbb{T}_\alpha$ are related by $\Omega = -\nabla_{\mathbb{T}_\alpha}^2 \psi$. This class of exact solutions is found by the same method as in the work of Crowdy (2004) who constructed solutions to a modified Liouville equation on the unit sphere. The class of solutions on the torus described by Sakajo (2019a) are smooth solutions except for two point vortices of equal strengths embedded at the two antipodal points.

As discussed in the planar case, when there are n point vortices on the torus, we have the following modified Liouville equation for the stream function:

$$\nabla_{\mathbb{T}_\alpha}^2 \psi = e^{-2\psi} - \kappa(\mathbb{T}_\alpha) - \sum_{j=1}^n \Gamma_j \delta_{v_j}. \quad (5)$$

By recasting the Euler equation on a torus as a vorticity equation in the vorticity-stream function formulation, it can be shown that this vorticity equation has the important Jacobian term $\mathcal{J}(\Omega, \psi)$. The vorticity $\Omega = -e^{-2\psi} + \kappa(\mathbb{T}_\alpha) + \sum_{j=1}^n \Gamma_j \delta_{v_j}$ is not a functional of ψ because of the non-constant Gaussian curvature $\kappa(\mathbb{T}_\alpha)$. This implies that solutions of (5) are not, in general, solutions of the Euler equation on $\mathbb{T}_\alpha$. However, the solutions of the Liouville equation (5) can still be regarded as stationary configurations of quantized point vortices in the following sense. We adopt (5) as a kinematic “one-way interaction” model of localized vortex structures in superfluids and BECs. This means that the evolution of point vortices is subject to the continuous background vorticity, while the background vorticity distribution is not affected by the velocity field induced by the point vortices. This is the same situation as in the study of point vortex dynamics on a rotating sphere as considered by Newton and Shokraneh (2005), who introduced a one-way model in which point vortices are advected by the fixed continuous vorticity field corresponding to the solid body rotation of the sphere. In contrast to such a one-way interaction model, we can consider a “two-way interaction” model where the background also evolves under the velocity field induced by the point vortices. The relation between the solutions of the Liouville equation and those of the Euler equation is further discussed in Sec. VI.

The purpose of the present paper is to create a catalog of stationary lattice patterns of quantized point vortices embedded in the background vorticity of Liouville-type on the surface of a curved torus. Since the toroidal surface is the simplest model of porous media having one handle structure, it is of theoretical significance to consider the formation of vortex lattice patterns in the superfluid thin film, as discussed by Corrada-Emmanuel (1994) and Machta and Guyer (1988). The exact solution gives rise to various quantized point vortices by specifying loxodromic functions that are doubly periodic in an annulus in the complex domain. The locations of quantized point vortices correspond to zeros and poles of the loxodromic functions. To show that these quantized point vortices form a fixed equilibrium, we need to confirm two conditions. First is the Gauss condition, which requires that the total vorticity vanishes on the toroidal surface. The second is a stationary condition, where the velocity field induced by point vortices and the Liouville-type background vorticity vanishes at the location of each quantized point vortex.

This paper is organized as follows. In Sec. II, we review the exact solution of the modified Liouville equation on the surface of a curved torus given by Sakajo (2019a). We, then, provide the two conditions to be confirmed. In Sec. III, we show the existence of stationary lattice patterns consisting of quantized point vortices with the choice of the Weierstrass $\wp$ -function. In Sec. IV, we obtain a continuous parametric family of stationary quantized point vortices when a loxodromic function is considered. After deriving a new solution formula of the modified Liouville equation, we construct another series of lattice patterns of quantized vortices in Sec. V, called “limit solutions.” Section VI shows a summary and discussion for future studies.

II. LIOUVILLE-TYPE EQUATION ON THE TOROIDAL SURFACE AND POINT VORTEX EQUILIBRIA

The geometry of a toroidal surface is generally characterized by the major radius R and the minor radius r with $R > r$. However, by its scale invariance, it is sufficient to deal with a canonical one, say $\mathbb{T}_\alpha$, with modulus $\alpha = R/r > 1$ in this paper. With the toroidal coordinates $(\theta, \phi) \in (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$, it is embedded in the three-dimensional Euclidean space $\mathbb{E}^3$, which is represented by

$$(\theta, \phi) \mapsto ((\alpha - \cos \theta) \cos \phi, (\alpha - \cos \theta) \sin \phi, \sin \theta) \in \mathbb{E}^3. \quad (6)$$

Equation (4) represents the Liouville-type quasi-linear elliptic equation with respect to a scalar function $\psi(\theta, \phi)$ on the torus. The Laplace–Beltrami operator $\nabla_{\mathbb{T}_\alpha}^2$ and the Gauss curvature of the toroidal surface $\kappa(\mathbb{T}_\alpha)$ are, respectively, represented by

$$\begin{aligned} \nabla_{\mathbb{T}_\alpha}^2 &\equiv \frac{1}{(\alpha - \cos \theta)} \frac{\partial}{\partial \theta} \left((\alpha - \cos \theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{(\alpha - \cos \theta)^2} \frac{\partial^2}{\partial \phi^2}, \\ \kappa(\mathbb{T}_\alpha) &\equiv -\frac{\cos \theta}{\alpha - \cos \theta}. \end{aligned}$$

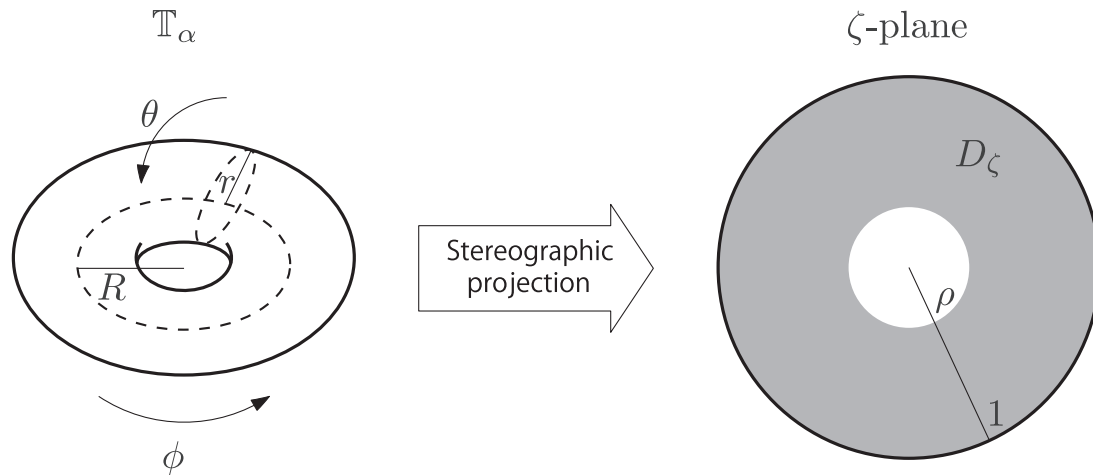


FIG. 1. Stereographic projection of the torus $\mathbb{T}_\alpha$, $\alpha = R/r > 1$, on the annulus $\{D_\zeta \mid \rho < |\zeta| \leq 1\}$. The toroidal angle $\phi \in [0, 2\pi)$ and the poloidal angle $\theta \in [0, 2\pi)$ are related to ζ through (7). Due to the symmetry, the lower half of the torus (viz. $(\theta, \phi) = [0, \pi) \times [0, 2\pi)$) is mapped to $\sqrt{\rho} < |\zeta| \leq 1$, whereas the upper half of the torus (viz. $(\theta, \phi) = [\pi, 2\pi) \times [0, 2\pi)$) is mapped to $\rho < |\zeta| \leq \sqrt{\rho}$. The identity of $\theta = 0$ and $\theta = 2\pi$ corresponds to the identity of $|\zeta| = 1$ and $|\zeta| = \rho$.

We also introduce a complex structure on the surface through a stereographic projection from the torus $\mathbb{T}_\alpha$ to the annulus $\{D_\zeta : \rho < |\zeta| \leq 1\}$ in a complex ζ -plane, as shown in Fig. 1. This stereographic projection is defined by

$$\zeta(\theta, \phi) = e^{i\phi} \exp(r_c(\theta)) \in \mathbb{C}, \quad r_c(\theta) = -\int_0^\theta \frac{d\hat{\theta}}{\alpha - \cos \hat{\theta}}. \quad (7)$$

The function $r_c(\theta)$ is monotonically decreasing owing to $r'_c(\theta) < 0$ for $1 < \alpha$ and satisfies the following quasiperiodicity (Sakajo and Shimizu, 2016):

$$r_c(\theta \pm 2\pi) = \mp 2\pi\mathcal{A} + r_c(\theta), \quad \mathcal{A} = (\alpha^2 - 1)^{-\frac{1}{2}}.$$

For later use, we evaluate $r_c(\theta)$ over the interval $[0, 2\pi)$,

$$r_c(0) = 0, \quad r_c(\theta) = 2\mathcal{A} \tan^{-1} \left[\mathcal{B} \cot \left(\frac{\theta}{2} \right) \right] - \pi\mathcal{A} \quad \text{for } \theta \neq 0, \quad (8)$$

where $\mathcal{B} = \sqrt{\frac{\alpha-1}{\alpha+1}}$. The parameter ρ associated with the conformal structure is related to the modulus α via the equation $\rho = \exp(-2\pi\mathcal{A}) = \exp(-2\pi/\sqrt{\alpha^2 - 1})$. Under the stereographic projection (7), $\theta = 0$ is mapped to the circle $|\zeta| = 1$, $\theta = 2\pi$ is mapped to the circle $|\zeta| = \rho$, and for $-\log \rho < r_c(\theta) \leq 0$, the torus is mapped to the annulus $\rho < |\zeta| \leq 1$. In particular, $\theta = \pi$ is mapped to the circle $|\zeta| = \sqrt{\rho}$. The periodicity of the torus in θ is, thus, transferred to the periodicity in ρ on the ζ -plane.

When the solution of (4) is regarded as the stream function, it gives rise to the incompressible velocity field on the toroidal surface: Suppose that a passive particle at $[\theta(t), \phi(t)]$ on the toroidal surface at time t is advected by the velocity field induced by the stream function ψ . Then, its governing equation is described by

$$\frac{d\bar{\zeta}}{dt} = -2i\lambda^{-2}(\zeta, \bar{\zeta}) \frac{\partial \psi}{\partial \zeta} \quad (9)$$

in the complex form Sakajo and Shimizu (2016). Here, $\bar{\zeta}$ is the conjugate of ζ and $\lambda(\zeta, \bar{\zeta})$ is the conformal factor associated with the metric g on the toroidal surface (Green and Marshall, 2013), which is given by

$$\lambda(\zeta, \bar{\zeta}) = \frac{\alpha - \cos \theta}{|\zeta|}. \quad (10)$$

Based on (9), we rewrite the angular velocity fields on the toroidal surface in the θ -direction and the ϕ -direction explicitly. Substituting (7) and (10) into (9), we have

$$\frac{d}{dt} \left[e^{-i\phi(t)} \exp(r_c(\theta(t))) \right] = -i\bar{\zeta} \frac{d\phi}{dt} - \frac{\bar{\zeta}}{\alpha - \cos \theta} \frac{d\theta}{dt} = -2i \frac{|\zeta|^2}{(\alpha - \cos \theta)^2} \frac{\partial \psi}{\partial \zeta},$$

which yields

$$\frac{i}{\alpha - \cos \theta} \frac{d\theta}{dt} - \frac{d\phi}{dt} = -\frac{2}{(\alpha - \cos \theta)^2} \zeta \frac{\partial \psi}{\partial \zeta}. \quad (11)$$

This indicates that the passive particle is at rest if $\zeta \frac{\partial \psi}{\partial \zeta} = 0$. On the other hand, it follows from

$$\left. \frac{\partial}{\partial \theta} \right|_{\phi} = -\frac{\zeta}{\alpha - \cos \theta} \frac{\partial}{\partial \zeta} - \frac{\bar{\zeta}}{\alpha - \cos \theta} \frac{\partial}{\partial \bar{\zeta}}, \quad \left. \frac{\partial}{\partial \phi} \right|_{\theta} = i\zeta \frac{\partial}{\partial \zeta} - i\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}$$

that

$$\zeta \frac{\partial}{\partial \zeta} = -\frac{\alpha - \cos \theta}{2} \frac{\partial}{\partial \theta} - \frac{i}{2} \frac{\partial}{\partial \phi}, \quad (12)$$

and thus, we have

$$-\frac{i}{\alpha - \cos \theta} \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{2}{(\alpha - \cos \theta)^2} \left(-\frac{\alpha - \cos \theta}{2} \frac{\partial \psi}{\partial \theta} - \frac{i}{2} \frac{\partial \psi}{\partial \phi} \right).$$

Accordingly, the angular velocity u_{θ} in the θ -direction and u_{ϕ} in the ϕ -direction of the toroidal surface $\mathbb{T}_{\alpha}$ are given by

$$u_{\theta} \equiv \frac{d\theta}{dt} = \frac{1}{\alpha - \cos \theta} \frac{\partial \psi}{\partial \phi}, \quad u_{\phi} \equiv (\alpha - \cos \theta) \frac{d\phi}{dt} = -\frac{\partial \psi}{\partial \theta}, \quad (13)$$

which is a divergence-free vector field on $\mathbb{T}_{\alpha}$. Introducing the vorticity Ω as the curl of the velocity field (13), we obtain the elliptic equation $\nabla_{\mathbb{T}_{\alpha}}^2 \psi = -\Omega$, in which the vorticity distribution is given by

$$-\Omega = e^{-2\psi} - \kappa(\mathbb{T}_{\alpha}). \quad (14)$$

The analytic solution of Eq. (4) was provided explicitly by Sakajo (2019a),

$$\psi(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4|f'(\zeta)|^2}{(1 + |f(\zeta)|^2)^2} \right] - \frac{1}{2} \log \left[\frac{|\zeta|^2}{(\alpha - \cos \theta)^2} \right], \quad (15)$$

in which $f(\zeta)$ is a given analytic function on a domain D_{ζ} . The second term originates from the conformal factor, representing the curvature effect of the toroidal surface. It is regular since we consider the function $f(\zeta)$ and the domain $D_{\zeta} = \{\zeta \in \mathbb{C} \mid \rho < |\zeta| \leq 1\}$ in the present paper. The first term is coming from the general solution to the Liouville equation in the plane (Liouville, 1853; Stuart, 1967). Crowdy (2004) made two remarks on the nature of the analytic function $f(\zeta)$ so that the first term becomes non-singular. First, $f(\zeta)$ is not necessarily analytic everywhere in D_{ζ} and it can admit the finite number of simple poles in D_{ζ} . Second, $f'(\zeta) \neq 0$ everywhere in D_{ζ} . Additionally, in order to define the solution on the toroidal surface, the function is required to be doubly periodic in the domain D_{ζ} .

As a matter of fact, as discussed in the work of Krishnamurthy et al. (2019; 2021), even if we allow for the existence of simple zeros in $f'(\zeta)$ and double poles in $f(\zeta)$, one can still define the stream function (15) as a physical solution. Suppose that $f'(\zeta)$ has a simple zero or $f(\zeta)$ has a double pole at $\zeta = v_0$. Then, the solution has the local behavior,

$$\psi(\zeta, \bar{\zeta}) \sim -\log |\zeta - v_0| + \text{regular}, \quad \zeta \rightarrow v_0.$$

On the other hand, the stream function of a point vortex located at $\zeta = v_0$ with the strength (circulation) Γ on the toroidal surface has the following asymptotic expansion (Sakajo and Shimizu, 2016):

$$\psi(\zeta, \bar{\zeta}) \sim -\frac{\Gamma}{2\pi} \log \widehat{d}(\zeta, v_0) = -\frac{\Gamma}{2\pi} \log |\zeta - v_0| + \text{regular}, \quad \zeta \rightarrow v_0, \quad (16)$$

in which the geodesic distance $\widehat{d}(\zeta, v_0)$ between ζ and v_0 is asymptotically represented by

$$\widehat{d}(\zeta, v_0) = \lambda(v_0, \bar{v}_0) |\zeta - v_0| + o(|\zeta - v_0|), \quad \zeta \rightarrow v_0.$$

This indicates that the stream function (15) contains a point vortex at $\zeta = v_0$ with the strength $\Gamma = 2\pi$. Hence, when there exist n simple zeros of $f'(\zeta)$ and double poles of $f(\zeta)$, say $v_j, j = 1, 2, \dots, n$, the solution contains n point vortices at v_j with the same strength $\Gamma_j = 2\pi$ in a field of Liouville-type vorticity distribution (14) on $\mathbb{T}_{\alpha}$. The vorticity distribution is, thus, given by

$$-\Omega = e^{-2\psi} - \kappa(\mathbb{T}_{\alpha}) - \sum_{j=1}^n \Gamma_j \delta_{v_j}, \quad (17)$$

where δ_{v_j} denotes the δ -measure at v_j . In this paper, we consider exact solutions to the modified Liouville equation (5), which corresponds to the vorticity distribution given by (17).

It is important to note that the strengths of the point vortices are identical to 2π . That is to say, the solution is regarded as the configuration of *quantized point vortices* on the surface of a torus in the background Liouville-type vorticity distribution. Such a stream function with point vortex singularities on the toroidal surface has already been obtained by Sakajo (2019a), in which $f(\zeta) = \text{sn}(\log \zeta)/\text{cn}(\log \zeta)$ with the Jacobi's elliptic functions $\text{sn}(z)$ and $\text{cn}(z)$ is chosen. The stream function yields a stationary configuration of identical point vortices located at two antipodals of the toroidal surface since $f'(\zeta)$ has two simple zeros in the fundamental domain.

Since $\mathbb{T}_\alpha$ is a compact surface without boundary, Gauss's divergence theorem gives rise to a global constraint on the vorticity distribution

$$\iint_{\mathbb{T}_\alpha} \Omega \, d\sigma = 0, \quad (18)$$

in which $d\sigma$ denotes the area element of the toroidal surface. When there are M point vortices located at v_k with the strength Γ_k , $k = 1, \dots, n$ on the surface of the torus, the Gauss constraint (18) for the vorticity distribution (17) is equivalent to

$$\iint_{\mathbb{T}_\alpha} e^{-2\psi} \, d\sigma - \sum_{j=1}^n \Gamma_j = 0 \quad (19)$$

since the Gauss–Bonnet theorem and the definition of the δ -measure assure that

$$\iint_{\mathbb{T}_\alpha} \kappa(\mathbb{T}_\alpha) \, d\sigma = 2\pi\chi(\mathbb{T}_\alpha) = 0, \quad \iint_{\mathbb{T}_\alpha} \delta_{v_j} \, d\sigma = 1 \quad \text{for } j = 1, \dots, n,$$

where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold $\mathcal{M}$. The first term in (19) is calculated as follows:

$$\begin{aligned} \iint_{\mathbb{T}_\alpha} e^{-2\psi} \, d\sigma &= 4 \iint_{\mathbb{T}_\alpha} \frac{|\zeta|^2 |f'(\zeta)|^2}{(1 + |f(\zeta)|^2)^2 (\alpha - \cos \theta)^2} \, d\sigma \\ &= -2i \iint_{D_\zeta} \frac{|f'(\zeta)|^2}{(1 + |f(\zeta)|^2)^2} \, d\zeta \wedge d\bar{\zeta} \\ &= -2i \iint_{D_\zeta} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{f'(\zeta) \bar{f}(\bar{\zeta})}{1 + |f(\zeta)|^2} \right) \, d\zeta \wedge d\bar{\zeta} \\ &= 2i \oint_{\partial D_\zeta} \frac{f'(\zeta) \bar{f}(\bar{\zeta})}{1 + |f(\zeta)|^2} \, d\zeta. \end{aligned} \quad (20)$$

The Gauss constraint (19) now takes the form

$$2i \oint_{\partial D_\zeta} \frac{f'(\zeta) \bar{f}(\bar{\zeta})}{1 + |f(\zeta)|^2} \, d\zeta - \sum_{j=1}^n \Gamma_j = 0 \quad (21)$$

and needs to be independently verified for every choice of the function $f(\zeta)$.

Finally, in order that the solution becomes a vortex crystal, it is sufficient and necessary that the configuration of point vortices is in an equilibrium state, which yields an additional constraint. Suppose that $v_0 \in D_\zeta$ is a zero of $f'(\zeta)$ with degree m . Then, there is a point vortex at $\zeta = v_0$ with the strength $2\pi m$. The velocity field at this point is, then, calculated by subtracting the self-induced singular component (16) from the stream function (Newton, 2001; Saffman, 1992; and Sakajo and Shimizu, 2016). That is to say, the evolution equation of the point vortex is given by

$$\frac{d\bar{v}_0}{dt} = 2i\lambda^{-2}(v_0, \bar{v}_0) \frac{\partial}{\partial v_0} \tilde{\psi}(v_0, \bar{v}_0), \quad (22)$$

in which the modified stream function $\tilde{\psi}(v_0, \bar{v}_0)$ is defined by

$$\begin{aligned} \tilde{\psi}(v_0, \bar{v}_0) &= \lim_{\zeta \rightarrow v_0} \left(\psi(\zeta, \bar{\zeta}) + m \log \widehat{d}(\zeta, v_0) \right) \\ &= \lim_{\zeta \rightarrow v_0} \left(\psi(\zeta, \bar{\zeta}) + m \log \left[\frac{(\alpha - \cos \theta_0)}{|v_0|} |\zeta - v_0| \right] \right). \end{aligned} \quad (23)$$

With the same calculation as (11), the point vortex at $\zeta = v_0$ stays at the same location when we have

$$v_0 \frac{\partial}{\partial v_0} \tilde{\psi}(v_0, \bar{v}_0) = 0. \quad (24)$$

For a given doubly periodic function $f(\zeta)$ on D_ζ , substituting it into formula (15), we identify the locations of point vortices from the solution and confirm whether the Gauss condition (19) and the stationary condition (24).

III. STEADY SOLUTIONS IN TERMS OF WEIERSTRASS $\wp$ -FUNCTION

A. A solution with four antipodal quantized point vortices

In this section, we describe solutions to the modified Liouville equation (5), given in terms of the doubly-periodic Weierstrass $\wp$ -function. A brief discussion of the $\wp$ -function is provided in Appendix A. We choose the analytic function $f(\zeta)$ in the stream function (15) as $f(\zeta) = \wp(-i \log \zeta)$, which is defined on the annular domain $D_\zeta = \{\zeta \in \mathbb{C} \mid \rho < |\zeta| \leq 1\}$ shown in Fig. 1. The domain D_ζ is related to the fundamental domain D_z given by (A1) via the simple mapping $z = -i \log \zeta$. The half-periods of $\wp(z)$ are $\omega_1 = \pi$ and $\omega_3 = -\frac{i}{2} \log \rho$. The function $f(\zeta)$ is doubly periodic in D_ζ since

$$\begin{aligned} f(\zeta(\theta + 2\pi, \phi)) &= f(\rho\zeta) = \wp(-i \log(\rho\zeta)) = \wp(-i \log \zeta - i \log \rho) = \wp(-i \log \zeta) = f(\zeta), \\ f(\zeta(\theta, \phi + 2\pi)) &= f(\zeta) \end{aligned}$$

by the definition of the stereographic projection (7). Substituting for $f(\zeta)$ and using $f'(\zeta) = -\frac{i}{\zeta} \wp'(-i \log \zeta)$ in (15), we obtain the stream function

$$\psi(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4|\wp'(-i \log \zeta)|^2}{(1 + |\wp(-i \log \zeta)|^2)^2 (\alpha - \cos \theta)^2} \right]. \quad (25)$$

Recall that $\wp(z)$ has a double pole at the origin $z = 0 \equiv \omega_0$ and $\wp'(z)$ has three simple zeros at the half-periods ω_1, ω_3 and at $\omega_2 = \omega_1 + \omega_3$. Hence, the stream function (25) has four logarithmic singularities, located at the points $v_k = \exp(i\omega_k) \in D_\zeta$, $k = 0, 1, 2, 3$, i.e., $v_0 = 1$, $v_1 = -1$, $v_2 = -\sqrt{\rho}$, and $v_3 = \sqrt{\rho}$. Each of these logarithmic singularities corresponds to a point vortex, and it can be verified using (A3) and (A4) that they all have the same strength $+2\pi$. There are, thus, four identical point vortices located at the antipodal positions of the toroidal surface $\mathbb{T}_\alpha$, namely, $(\theta_0, \phi_0) = (0, 0)$, $(\theta_1, \phi_1) = (0, \pi)$, $(\theta_2, \phi_2) = (\pi, \pi)$, and $(\theta_3, \phi_3) = (\pi, 0)$.

The Gauss condition (21) is calculated as in the work of Sakajo (2019a). The second term gives $\sum \Gamma_k = +8\pi$, and after substituting for $f(\zeta)$, the first term is given by

$$2i \oint_{\partial D_\zeta} \frac{f'(\zeta) \overline{f(\zeta)}}{1 + |f(\zeta)|^2} d\zeta = 2i \oint_{\partial D_\zeta} \frac{\wp'(-i \log \zeta) \overline{\wp(-i \log \zeta)}}{1 + |\wp(-i \log \zeta)|^2} \left(-\frac{i}{\zeta} \right) d\zeta = 2i \oint_{\partial D_z} \frac{\wp'(z) \overline{\wp(z)}}{1 + |\wp(z)|^2} dz. \quad (26)$$

Using (A3) and (A4), we find that

$$\wp'(z) \overline{\wp(z)} = -\frac{2}{z|z|^4} (1 + o(z^3))(1 + o(\bar{z}^3)), \quad 1 + |\wp(z)|^2 = \frac{1}{|z|^4} (1 + o(|z|^3))$$

so that we have

$$\frac{\wp'(z) \overline{\wp(z)}}{1 + |\wp(z)|^2} = -\frac{2}{z} (1 + o(|z|)), \quad |z| \rightarrow 0,$$

which means that the integrand of (26) has a simple pole at the origin with residue -2 . Accordingly, owing to the doubly periodicity of the integrand, integral (26) becomes

$$2i \oint_{\partial D_z} \frac{\wp'(z) \overline{\wp(z)}}{1 + |\wp(z)|^2} dz = 2i \cdot \lim_{\varepsilon \rightarrow 0} \oint_{|z|=\varepsilon} \frac{\wp'(z) \overline{\wp(z)}}{1 + |\wp(z)|^2} dz = 2i \cdot 2\pi i \cdot (-2) = 8\pi.$$

This shows that the Gauss condition (21) is satisfied.

We turn now to the stationary condition (24) for the point vortices at $\zeta = v_k$, $k = 0, 1, 2, 3$. The modified stream function (23) is given by (using $m = 1$)

$$\begin{aligned} \tilde{\psi}(v_k, \bar{v}_k) &= \lim_{\zeta \rightarrow v_k} \left(-\frac{1}{2} \log \left[\frac{4|\wp'(-i \log \zeta)|^2}{(1 + |\wp(-i \log \zeta)|^2)^2 (\alpha - \cos \theta)^2} \right] + \log \left[\frac{(\alpha - \cos \theta_k)}{|v_k|} |\zeta - v_k| \right] \right) \\ &= \lim_{\zeta \rightarrow v_k} \left(-\frac{1}{2} \log \left[\frac{4|v_k \wp'(-i \log \zeta)|^2}{(1 + |\wp(-i \log \zeta)|^2)^2 |\zeta - v_k|^2} \right] \right) + 2 \log(\alpha - \cos \theta_k). \end{aligned}$$

To proceed, we split into two cases, considering first the point vortices at $\zeta = v_k$, $k = 1, 2, 3$. Owing to $\wp'(\omega_k) = 0$ for $\omega_k = -i \log v_k$, $k = 1, 2, 3$, we have the local behavior

$$\wp'(-i \log \zeta) - \wp'(-i \log v_k) = -\frac{i}{v_k} \wp''(-i \log v_k)(\zeta - v_k) + o(\zeta - v_k), \quad \zeta \rightarrow v_k,$$

from which we can calculate the limit

$$\lim_{\zeta \rightarrow v_k} \frac{|v_k \wp'(-i \log \zeta)|^2}{(1 + |\wp(-i \log \zeta)|^2)^2 |\zeta - v_k|^2} = \frac{|\wp''(-i \log v_k)|^2}{(1 + |\wp(-i \log v_k)|^2)^2}.$$

After a little algebra, this gives us the modified stream function

$$\tilde{\psi}(v_k, \bar{v}_k) = -\log 2 - \frac{1}{2} \log |\wp''(-i \log v_k)|^2 + \log(1 + |\wp(-i \log v_k)|^2) + 2 \log(\alpha - \cos \theta_k).$$

The velocity field at $\zeta = v_k$ is, then, given by

$$v_k \frac{\partial}{\partial v_k} \tilde{\psi}(v_k, \bar{v}_k) = \frac{i}{2} \frac{\wp'''(-i \log v_k)}{\wp''(-i \log v_k)} - i \frac{\wp'(-i \log v_k) \overline{\wp(-i \log v_k)}}{1 + |\wp(-i \log v_k)|^2} - \sin \theta_k = 0,$$

where we have used (A2) and the fact that $\theta_k = 0$ or π for $k = 1, 2, 3$.

In the case $k = 0$ ($z = 0$, $v_0 = 1$, and $\theta_0 = 0$), it follows from (A3) and (A4) that

$$\frac{|\wp'(z)|^2}{(1 + |\wp(z)|^2)^2} = \frac{\frac{4}{|z|^6}(1 + o(|z|^3))}{\frac{1}{|z|^8}(1 + o(|z|^3))} = 4|z|^2(1 + o(|z|^3)), \quad z \rightarrow 0.$$

Accordingly, we have

$$\log \left[\frac{|\wp'(-i \log \zeta)|^2}{(1 + |\wp(-i \log \zeta)|^2)^2 |\zeta - 1|^2} \right] = \log \left[\left| \frac{2 \log \zeta}{\zeta - 1} \right|^2 (1 + o(|\zeta - 1|^3)) \right] \rightarrow \log 4, \quad \zeta \rightarrow 1,$$

which yields

$$v_0 \frac{\partial}{\partial v_0} \tilde{\psi}(v_0, \bar{v}_0) = -\sin \theta_0 = 0.$$

Thus, the four point vortices at the antipodal locations of the toroidal surface are stationary and the stream function (25) provides a steady solution of the incompressible Euler equation.

Figure 2 shows some example solutions for selected values of α , and we see the topology of the flow change as α is varied. There are four point vortices with strength 2π embedded in the smooth background vorticity $\Omega_{\text{sm}} = -e^{-2\psi} + \kappa(\mathbb{T}_\alpha)$ at $(\theta, \phi) \in \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$. A ring of smooth vorticity around the point vortex at $(0, 0)$ is seen in Figs. 2(a) and 2(b). As α increases, we see the predominant flow direction shift from along ϕ to along θ . The background vorticity becomes smoother with increasing α in the sense that the difference between the minima and maxima of Ω_{sm} decreases.

B. N-ring configuration

We construct a steady solution with more quantized point vortices. Instead of using the stereographic projection (7), we introduce a map

$$\zeta = \zeta_N(\theta, \phi) = \exp(r_c(\theta))e^{iN\phi}$$

for a fixed positive integer N . For each $k = 0, \dots, N-1$, the map ζ_N defines a bijection from T_k^N to D_ζ , where

$$T_k^N = (\theta, \phi) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \left[\frac{2\pi}{N}k, \frac{2\pi}{N}(k+1) \right)$$

for $k = 0, \dots, N-1$. Since the toroidal surface is divided into N domains with T_k^N , the map $z = -i \log \zeta$ gives rise to a many-to-one map from $\mathbb{T}_\alpha$ to D_z . Hence, the pre-image of a point D_z with respect to this map consists of N points on $\mathbb{T}_\alpha$ that are equally spaced along the line of a latitude of the toroidal surface, i.e., an N -ring configuration.

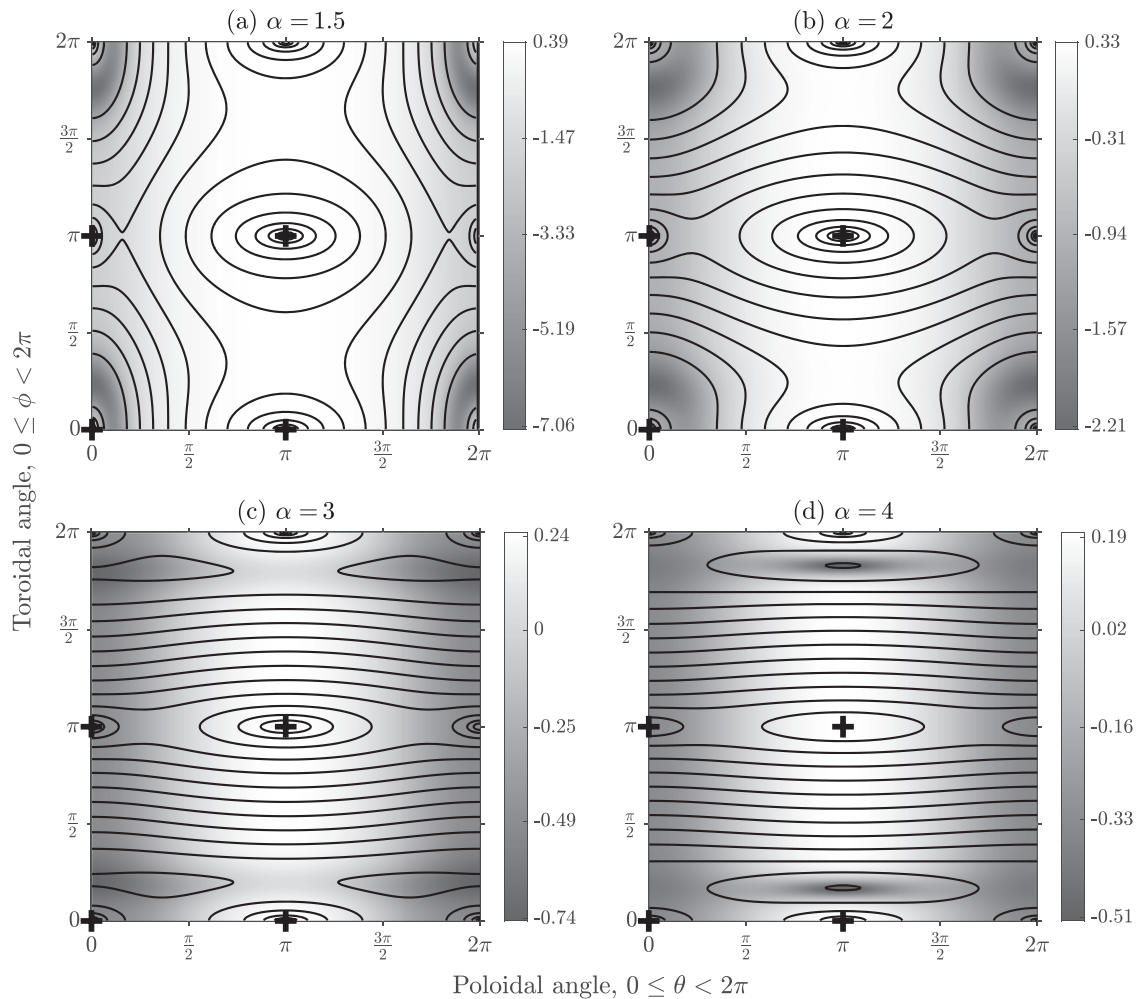


FIG. 2. Streamline contour plots, vorticity heatmaps, and embedded point vortices (+) corresponding to the stream function (25).

The choice of the analytic function is, then, given by

$$f(\zeta) = f(\zeta_N(\theta, \phi)) = \wp(-i \log \zeta_N(\theta, \phi)).$$

Since

$$f'(\zeta) = \left(\frac{1}{2i\zeta} \frac{\partial}{\partial \phi} - \frac{\alpha - \cos \theta}{2\zeta} \frac{\partial}{\partial \theta} \right) \wp(-i \log \zeta_N(\theta, \phi)) = -\frac{i(N+1)}{2\zeta} \wp'(-i \log \zeta_N(\theta, \phi)),$$

we obtain the following new analytic solution:

$$\psi(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{(N+1)^2 |\wp'(-i \log \zeta_N(\theta, \phi))|}{(1 + |\wp(-i \log \zeta_N(\theta, \phi))|^2)^2 (\alpha - \cos \theta)^2} \right]. \quad (27)$$

The analytic solution is apparently doubly periodic on D_ζ owing to the definition of $f(\zeta)$. Since there exist three zeros of $\wp'(z)$ at $z = \omega_1, \omega_2, \omega_3$ and double zeros at $z = 0 (= \omega_0)$ in D_z and they are pulled back to $4N$ points on $\mathbb{T}_\alpha$, this solution contains a $4N$ quantized point vortices with the strength 2π . The configuration of the $4N$ point vortices is two $2N$ -rings where $2N$ point vortices are equally arranged along the line of latitudes $\theta = 0, \pi$, namely, $(\theta_k, \phi_k) = (0, \frac{\pi k}{N})$ and $(\pi, \frac{\pi k}{N})$ for $k = 0, \dots, 2N-1$. The solution satisfies the Gauss constraint (18) since the integration of $e^{-2\psi}$ over the toroidal surface is divided into N identical integrations over D_z , for each of which the Gauss constraint is satisfied. It is also easy to show that the $4N$ point vortices are in a fixed equilibrium state since the calculation of (24) proceeds in the same way as in the antipodal case on D_z , and additionally, their locations are $\theta_k = 0$ and π .

Let us finally note another choice of the function

$$f(\zeta) = \wp(-i \log \zeta^N) \quad (28)$$

for a fixed positive integer N . Owing to $\zeta^N = \exp(Nr_c(\theta))e^{iN\phi}$, it gives rise to a bijection from T_{k_1, k_2}^N from D_z , in which

$$T_{k_1, k_2}^N = (\theta, \phi) \in \left[\frac{2\pi}{N}k_1, \frac{2\pi}{N}(k_1 + 1) \right) \times \left[\frac{2\pi}{N}k_2, \frac{2\pi}{N}(k_2 + 1) \right)$$

for $k_1, k_2 = 0, 1, \dots, N-1$. Since the toroidal surface is divided into N^2 subdomains, the map $z = -i \log \zeta^N(\theta, \phi)$ becomes a many-to-one map from $\mathbb{T}_\alpha$ to D_z . Substitution of (28) into (15) yields an analytic solution formally. It is easy to see that the solution is doubly periodic and satisfies the Gauss constraint. The zeros of $\wp'(z)$ and the double pole of $\wp(z)$ in the fundamental domain D_z are pulled back on the $4N^2$ point vortices with the strength 2π consisting of $2N$ $2N$ -rings on the toroidal surface. However, this configuration is not a fixed equilibrium since they do not satisfy the stationary condition.

IV. STEADY SOLUTIONS IN TERMS OF LOXODROMIC FUNCTIONS

In this section, we consider solutions to the modified Liouville equation (5), given in terms of loxodromic functions. The special functions used here are the Schottky–Klein prime function (Crowdy, 2020) and its derivatives defined on the annulus $D_\zeta = \{\zeta \in \mathbb{C} \mid \rho < |\zeta| \leq 1\}$ shown in Fig. 1. We call any function $G(\zeta)$ defined on D_ζ a loxodromic function if it is invariant under $\zeta \mapsto \rho\zeta$, i.e., $G(\zeta) = G(\rho\zeta)$. Basic properties of the prime function and its derivatives are reviewed in Appendix B and will be referred to throughout this section. We consider the function

$$f(\zeta) = W_0(\zeta) \equiv \sum_{k=1}^{\widehat{n}} (K(\zeta/a_k, \sqrt{\rho}) - K(\zeta/b_k, \sqrt{\rho})), \quad \widehat{n} \in \mathbb{N}, \quad (29)$$

on D_ζ , where $a_k, b_k \in D_\zeta$ with $a_k \neq b_k$, $k = 1, \dots, \widehat{n}$, are parameters. Here, the K -function is defined by (B2). It can be proven using (B3) that $W_0(\zeta)$ is a loxodromic function satisfying $W_0(\rho\zeta) = W_0(\zeta)$. Note that it follows from (B4) that $W_0(\zeta)$ has $2\widehat{n}$ simple poles, located at $\zeta = a_k, b_k$ for $k = 1, \dots, \widehat{n}$.

We now introduce a second function

$$W_1(\zeta) \equiv \zeta f'(\zeta) = \sum_{k=1}^{\widehat{n}} (L(\zeta/a_k, \sqrt{\rho}) - L(\zeta/b_k, \sqrt{\rho})), \quad (30)$$

where the L -function is defined by (B5). We see using (B10) that $W_1(\zeta)$ is also a loxodromic function, and from (B9), we see that it has $2\widehat{n}$ double poles, one each at $\zeta = a_k$ and $\zeta = b_k$, for $k = 1, \dots, \widehat{n}$. It, thus, has $4\widehat{n}$ zeros, located at, say, $\zeta = v_j$, $j = 1, \dots, 4\widehat{n}$, with multiplicity in the annulus D_ζ according to the general theory of elliptic functions (Hurwitz and Courant, 1944). We can now rewrite the stream function (15) as

$$\psi(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4|W_1(\zeta)|^2}{(1 + |W_0(\zeta)|^2)^2 (\alpha - \cos \theta)^2} \right]. \quad (31)$$

The function $W_0(\zeta)$ consists of $\widehat{n}$ simple poles at a_k and $\widehat{n}$ simple poles at b_k . Following the discussion in Sec. II, we see that the stream function (31) is regular at the $2\widehat{n}$ simple poles $\zeta = a_k, b_k$, $k = 1, \dots, \widehat{n}$. However, the stream function has $4\widehat{n}$ point vortices (logarithmic singularities) at the zeros v_j of $W_1(\zeta)$. If these point vortices are stationary and if the Gauss constraint (21) is further satisfied, then (31) provides a steady solution of the modified Liouville equation (5).

Let us next confirm whether the stream function (31) satisfies the Gauss condition (21). Since the exact solution contains $n = 4\widehat{n}$ point vortices with the identical strength $\Gamma_j = 2\pi$, the total circulation owing to the point vortices is given by $\sum_{j=1}^n \Gamma_j = 8\pi\widehat{n}$. In the meantime, we need to evaluate the integral in the Gauss constraint (21) with $f(\zeta)$ given by (29). To do this, note that the only singularities of the quantity $W_0'(\zeta)\overline{W_0(\zeta)}/(1 + |W_0(\zeta)|^2)$ are at the poles of $W_0(\zeta)$, so we investigate its behavior in the neighborhood of $\zeta = a_k$ and $\zeta = b_k$. We find from (29) and (B4) that near these singularities,

$$\frac{W_0'(\zeta)\overline{W_0(\zeta)}}{1 + |W_0(\zeta)|^2} \sim \frac{W_0'(\zeta)}{W_0(\zeta)} = \frac{d}{d\zeta} \log W_0(\zeta) = \begin{cases} -\frac{1}{\zeta - a_k} + O(1) & \text{as } \zeta \rightarrow a_k, \\ -\frac{1}{\zeta - b_k} + O(1) & \text{as } \zeta \rightarrow b_k \end{cases}$$

for $k = 1, \dots, \widehat{n}$. This indicates that this function has a simple pole with residue -1 at each $\zeta = a_k, b_k$, $k = 1, \dots, \widehat{n}$. Note that the residues at $\zeta = a_k$ and $\zeta = b_k$ are both equal. Accordingly, the first term in (21) gives

$$\begin{aligned} 2i \oint_{\partial D_\epsilon} \frac{W'_0(\zeta) \overline{W_0(\zeta)}}{1 + |W_0(\zeta)|^2} d\zeta &= 2i \cdot \lim_{\epsilon \rightarrow 0} \sum_{k=1}^{\widehat{n}} \left[\oint_{|\zeta - a_k|=\epsilon} \frac{W'_0(\zeta) \overline{W_0(\zeta)}}{1 + |W_0(\zeta)|^2} d\zeta + \oint_{|\zeta - b_k|=\epsilon} \frac{W'_0(\zeta) \overline{W_0(\zeta)}}{1 + |W_0(\zeta)|^2} d\zeta \right] \\ &= 2i \cdot 2\pi i \cdot \sum_{k=1}^{\widehat{n}} [(-1) + (-1)] \\ &= 8\pi \widehat{n}, \end{aligned}$$

showing that the Gauss condition (21) is satisfied by the stream function (31).

Finally, we derive the stationary condition (24) for the point vortex at $\zeta = v_j, j = 1, \dots, 4\widehat{n}$. Since this point vortex (of strength 2π) appears as a simple zero of $W_1(\zeta)$, we have $m = 1$ in (23). By definition, let us first compute from (31),

$$\begin{aligned} \psi(\zeta, \bar{\zeta}) + \log \left[\frac{\alpha - \cos \theta}{|v_j|} |\zeta - v_j| \right] &= -\frac{1}{2} \log \left| \frac{v_j (W_1(\zeta) - W_1(v_j))}{\zeta - v_j} \right|^2 + \log(1 + |W_0(\zeta)|^2) \\ &\quad + \log(\alpha - \cos \theta) + \log(\alpha - \cos \theta_j) - \log 2, \end{aligned}$$

where we have used the fact that $W_1(v_j) = 0$. The modified stream function is, then, obtained by taking the limit $\zeta \rightarrow v_j$,

$$\widetilde{\psi}(v_j, \bar{v}_j) = -\frac{1}{2} \log |W_2(v_j)|^2 + \log(1 + |W_0(v_j)|^2) + 2 \log(\alpha - \cos \theta_j) - \log 2. \quad (32)$$

Here, we have defined the new function

$$W_2(\zeta) \equiv \zeta W'_1(\zeta) = \sum_{k=1}^{\widehat{n}} (M(\zeta/a_k, \sqrt{\rho}) - M(\zeta/b_k, \sqrt{\rho})) \quad (33)$$

in terms of the M -function in (B6). Applying $v_j \frac{\partial}{\partial v_j}$ to the modified stream function (32), we get

$$v_j \frac{\partial \widetilde{\psi}}{\partial v_j} = -\frac{1}{2} \frac{W_3(v_j)}{W_2(v_j)} + \frac{W'_0(v_j) \overline{W_0(v_j)}}{1 + |W_0(v_j)|^2} - \sin \theta_j = -\frac{1}{2} \frac{W_3(v_j)}{W_2(v_j)} - \sin \theta_j,$$

where we have defined the new functions

$$W_3(\zeta) \equiv \zeta W'_2(\zeta) = \sum_{k=1}^{\widehat{n}} (N(\zeta/a_k, \sqrt{\rho}) - N(\zeta/b_k, \sqrt{\rho})) \quad (34)$$

in terms of the N -function in (B7). We have also used $W_1(v_j) = v_j W'_0(v_j) = 0$ and calculated using (12) that

$$2v_j \frac{\partial}{\partial v_j} \log(\alpha - \cos \theta_j) = -\sin \theta_j.$$

In summary, the stationary condition (24) is reduced to

$$\frac{W_3(v_j)}{W_2(v_j)} + 2 \sin \theta_j = 0 \quad \text{for } j = 1, \dots, 4\widehat{n}. \quad (35)$$

In (35), the angle θ_j is related to v_j via $\log|v_j| = r_c(\theta_j)$, where $r_c(\theta_j)$ is given by (8). The stationary condition can be further simplified by considering symmetric solutions, which have $\theta_j = 0$ or π so that $\sin \theta_j = 0$ for all j . This provides a restriction on the location of the zeros v_j since from (8), we get $\log|v_j| = r_c(0) = 0$ or $\log|v_j| = r_c(\pi) = -\pi\mathcal{A} = \log \sqrt{\rho}$. Thus, for such symmetric solutions, we must have $|v_j| = 1$ or $|v_j| = \sqrt{\rho}$, leading to the stationary point vortex condition in the form

$$W_3(v_j) = 0, \quad |v_j| = 1 \text{ or } \sqrt{\rho} \quad \text{for } j = 1, \dots, 4\widehat{n}. \quad (36)$$

We note that $W_2(v_j) \neq 0$ for all j . The M and N -functions, and hence W_2 and W_3 , are all loxodromic functions. Properties (B10) and (B12) satisfied by the L and N -functions are identical, while the corresponding property in (B11) differs by a sign. Condition (36) means that the four point vortices of the stationary solution are necessarily on the innermost and the outermost antipodal locations of the toroidal surface. We construct symmetric families of solutions to the modified Liouville equation, continuously dependent on a parameter, which satisfy (36), using these properties for any $\widehat{n}$.

A. Examples with four point vortices

We consider the case when $\hat{n} = 1$, in which $W_1(\zeta)$ has double poles at $a_1, b_1 \in D_\zeta$, $a_1 \neq b_1$, and four zeros. We now solve $W_1(\zeta) = 0$ to obtain the locations v_1, v_2, v_3, v_4 of point vortices contained in the exact solution (31). It is easy to verify that $\pm\sqrt{a_1 b_1}$ and $\pm\sqrt{a_1 b_1/\rho}$ are four zeros of $W_1(\zeta)$. Indeed, using properties (B10), we have

$$\begin{aligned} W_1(\pm\sqrt{a_1 b_1}) &= L(\pm\sqrt{a_1 b_1}/a_1, \sqrt{\rho}) - L(\pm\sqrt{a_1 b_1}/b_1, \sqrt{\rho}) \\ &= L(\pm\sqrt{b_1/a_1}, \sqrt{\rho}) - L(\pm\sqrt{a_1/b_1}, \sqrt{\rho}) \\ &= 0. \end{aligned}$$

We can similarly show that $W_1(\pm\sqrt{a_1 b_1/\rho}) = 0$,

$$\begin{aligned} W_1(\pm\sqrt{a_1 b_1/\rho}) &= L(\pm\sqrt{a_1 b_1/\rho}/a_1, \sqrt{\rho}) - L(\pm\sqrt{a_1 b_1/\rho}/b_1, \sqrt{\rho}) \\ &= L(\pm\sqrt{b_1/\rho a_1}, \sqrt{\rho}) - L(\pm\sqrt{a_1/\rho b_1}, \sqrt{\rho}) \\ &= L(\pm\sqrt{\rho a_1/b_1}, \sqrt{\rho}) - L(\pm\sqrt{a_1/\rho b_1}, \sqrt{\rho}) \\ &= 0, \end{aligned}$$

where we have used (B10). Since $W_1(\zeta)$ is loxodromic, it follows that any ρ -multiple of these zeros are further zeros of $W_1(\zeta)$, for example, $W_1(\pm\sqrt{\rho a_1 b_1}) = 0$. Depending on the values of a_1 and b_1 , we can locate the four zeros of $W_1(\zeta)$ in D_ζ .

The N -function satisfies the same properties as the L -function [see (B12)], and since we have used only these properties to find the zeros of $W_1(\zeta)$, we see that the function $W_3(\zeta)$ must also vanish at the zeros of $W_1(\zeta)$. The point vortex stationary condition (36) is, thus, automatically satisfied if we choose v_j to be these zeros with $|v_j| = 1$ or $\sqrt{\rho}$. This last requirement is met if we choose a_1 and b_1 such that $|a_1 b_1| = \rho$ so that the positions of the point vortices are $v_1 = 1$, $v_2 = -1$, $v_3 = \sqrt{\rho}$, and $v_4 = -\sqrt{\rho}$ [see Fig. 3(a)]. We, thus, have a family of solutions parameterized by a_1 (if we choose $b_1 = \rho/a_1$) in which the four stationary point vortices are fixed at $\pm 1, \pm\sqrt{\rho}$ as $|a_1|$ varies in the range $\sqrt{\rho} < |a_1| < 1$. We can restrict $|a_1|$ to be in this range without loss of generality because of the symmetry between a_1 and b_1 ; we disallow the value $a_1 = 1$ since, then, $b_1 = \rho$, and so $W_1(\zeta)$ is identically zero. Although setting $|a_1 b_1| = 1$ would also satisfy (36), this does not lead to a parameterized family of solutions since we need $a_1, b_1 \in D_\zeta$. However, we obtain solutions when $|a_1| = |b_1| = 1$, for example, $a_1 = i$, $b_1 = -i$ leads to four point vortices at $\pm 1, \pm\sqrt{\rho}$.

Figure 4 shows the chosen example solutions for $\hat{n} = 1$, corresponding to the schematic shown in Fig. 3(a). There are four point vortices embedded in the smooth background vorticity $\Omega_{\text{sm}} = -e^{-2\psi} + \kappa(\mathbb{T}_\alpha)$. These point vortices remain fixed at $(\theta, \phi) \in \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$ when a_1 is changed, as shown in Figs. 4(a)–4(d). However, the centers of the smooth vorticity move around and the values of the vorticity minima change as a_1 is varied. The solution behaves similarly in the parameter range $\pi < \arg(a_1) < 2\pi$. Figure 5 shows the same plots for $\alpha = 3$, in which we observe the changing of the flow topology, the motion of the smooth vorticity centers, and the values of the vorticity minima (compare Figs. 4 and 5).

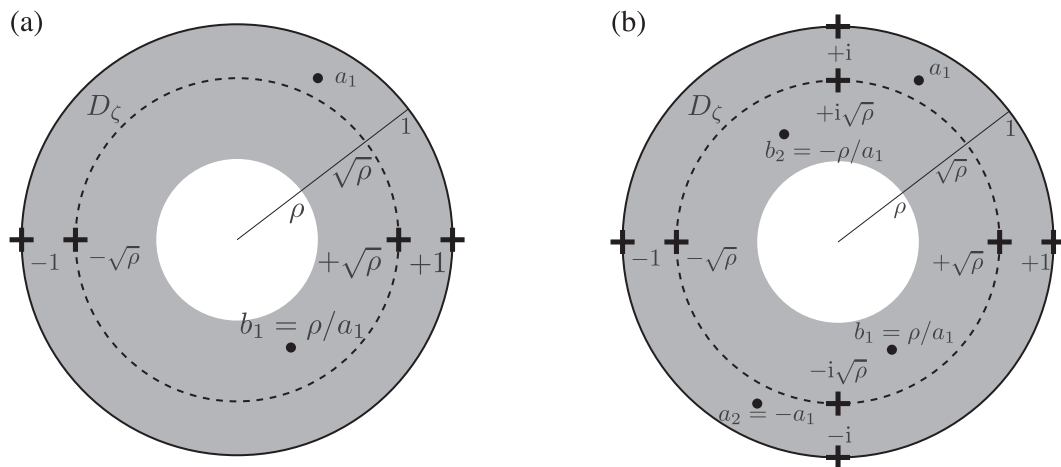


FIG. 3. Schematic showing the location of poles a_k, b_k ($\bullet$) and point vortices ($+$), in the annulus D_ζ , for the stream function (31). (a) We have taken $\hat{n} = 1$ and set $b_1 = \rho/a_1$. The stream function is, then, a function of the arbitrary parameter a_1 , with the four point vortices in the solution fixed at $\pm 1, \pm\sqrt{\rho}$. (b) We have taken $\hat{n} = 2$ and set $b_1 = \rho/a_1$, $a_2 = -a_1$, and $b_2 = -\rho/a_1$. The stream function is again a function of a_1 with eight point vortices fixed at $\pm 1, \pm\sqrt{\rho}, \pm i, \pm i\sqrt{\rho}$. See Sec. IV A for details. Some examples of vorticity and streamline plots for different choices of a_1 are shown in Figs. 4 and 5 for $\hat{n} = 1$ and in Fig. 6 for $\hat{n} = 2$.

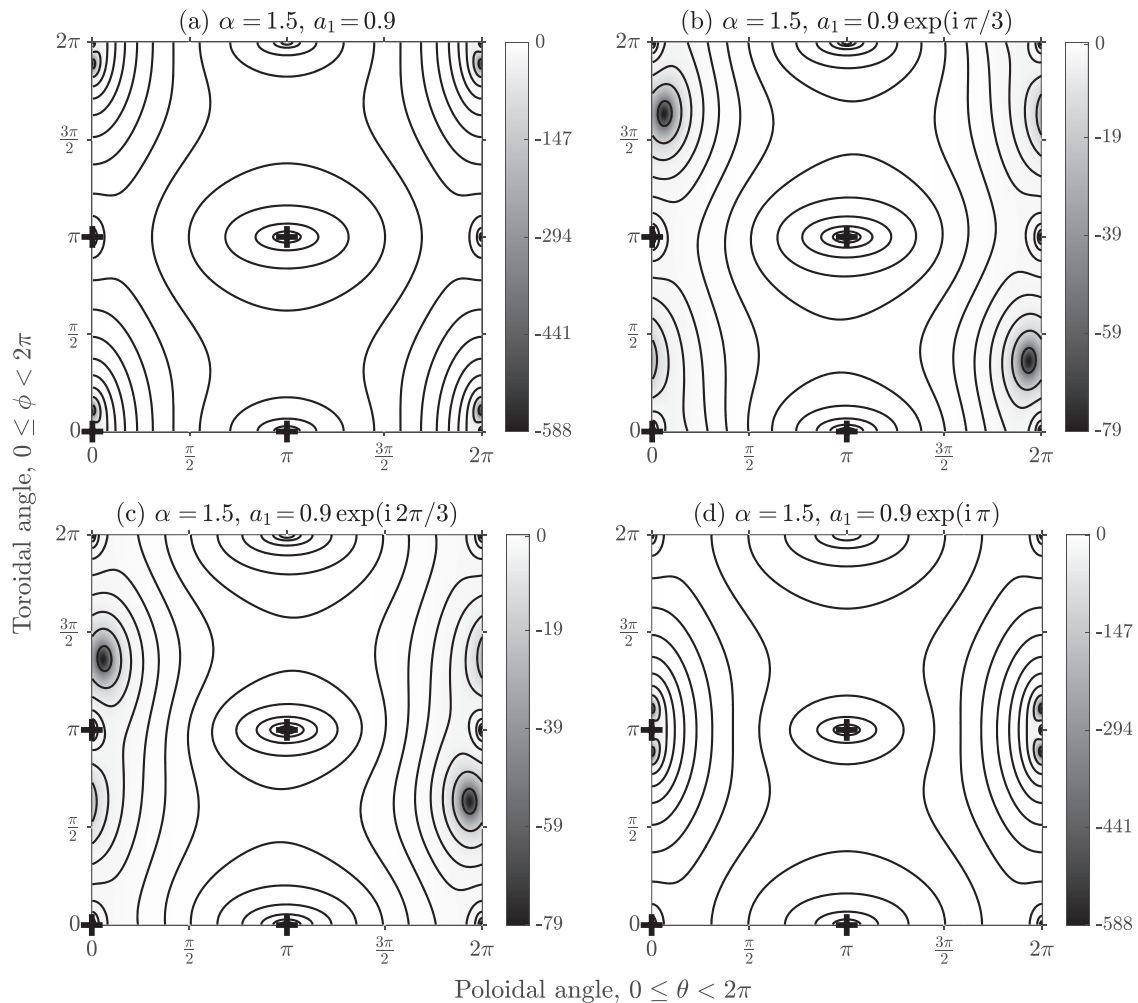


FIG. 4. Streamline contour plots, vorticity heatmaps, and embedded point vortices (+) in the θ - ϕ plane corresponding to the stream function (31) for $\hat{n} = 1$ and $\alpha = 1.5$.

B. Example with eight point vortices

We can generalize the preceding arguments for any $\hat{n}$. Let us assume that a_k and b_k , $k = 1, \dots, \hat{n}$, satisfy the following condition:

$$a_k b_k = \rho, \quad a_k = a_1 e^{i(k-1)2\pi/\hat{n}}, \quad k = 1, \dots, \hat{n}. \quad (37)$$

Then, it is straightforward to show that

$$v_j = \begin{cases} e^{i(j-1)\pi/\hat{n}} & \text{for } j = 1, \dots, 2\hat{n}, \\ \sqrt{\rho} e^{i(j-1)\pi/\hat{n}} & \text{for } j = 2\hat{n} + 1, \dots, 4\hat{n} \end{cases} \quad (38)$$

are the zeros of $W_1(\zeta)$. Namely, we have using (B10) that

$$W_1(v_j) = \sum_{k=1}^{\hat{n}} L(1/a_1 e^{i(-j+2k-1)\pi/\hat{n}}, \sqrt{\rho}) - L(a_1 e^{i(j+2k-3)\pi/\hat{n}}, \sqrt{\rho}) = 0,$$

for $j = 1, \dots, 2\hat{n}$ since $-j + 2k - 1$ and $j + 2k - 3$ run from 0 to $\hat{n} - 1$ for fixed j and $k = 1, \dots, \hat{n}$ with the periodicity $e^{i(\hat{n})2\pi/\hat{n}} = 1$. We also have using (B10) that

$$W_1(v_j) = \sum_{k=1}^{\hat{n}} L(\sqrt{\rho}/a_1 e^{i(-j+2k-1)\pi/\hat{n}}, \sqrt{\rho}) - L(\sqrt{\rho} a_1 e^{i(j+2k-3)\pi/\hat{n}}, \sqrt{\rho}) = 0,$$

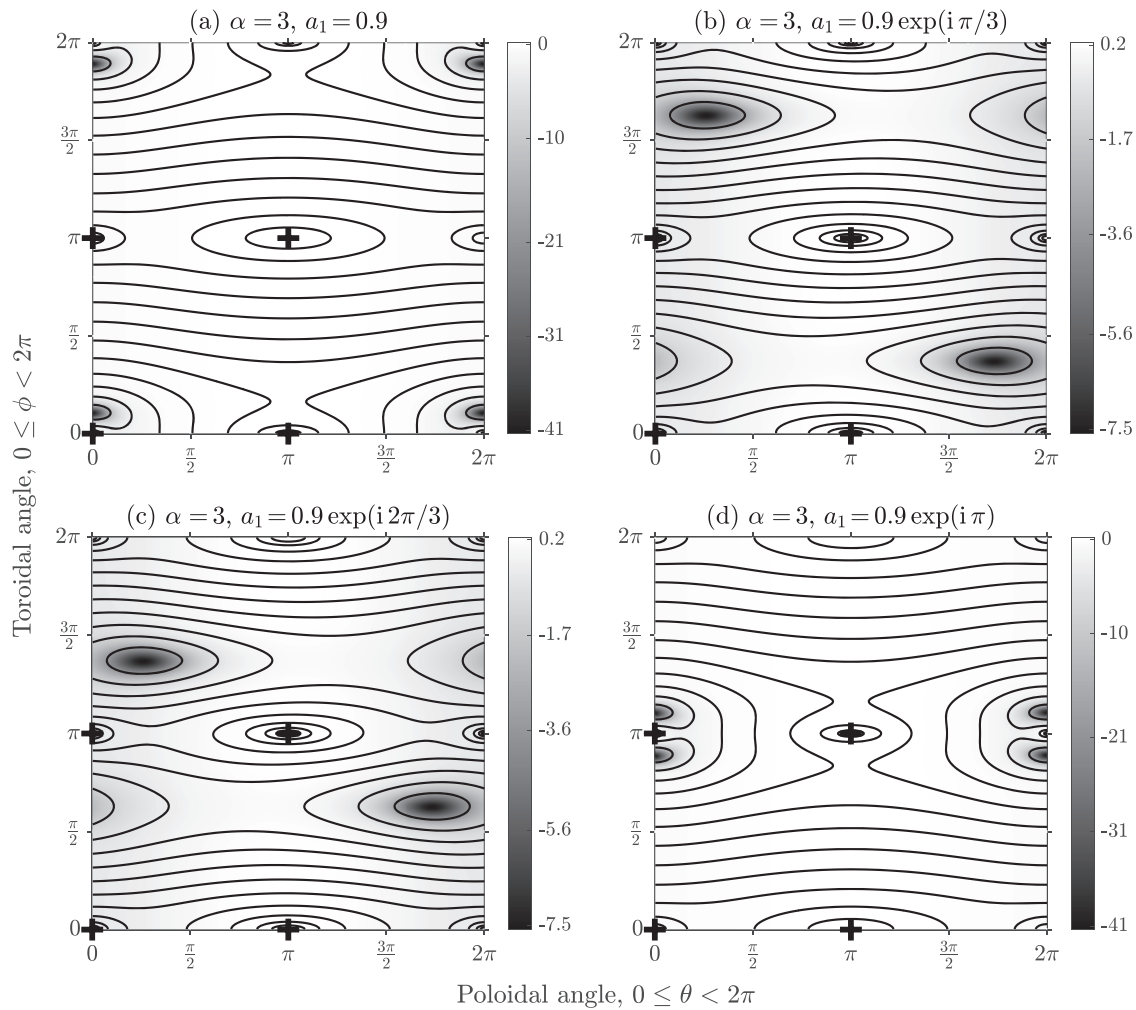


FIG. 5. Streamline contour plots, vorticity heatmaps, and embedded point vortices (+) in the θ - ϕ plane corresponding to the stream function (31) for $\hat{h} = 1$ and $\alpha = 3$.

for $j = 2\hat{n} + 1, \dots, 4\hat{n}$. Since the function $N(\zeta)$ satisfies the same properties as $L(\zeta)$, we have $W_3(v_j) = 0$ for $j = 1, \dots, 4\hat{n}$. Moreover, these zeros correspond to point vortices at the innermost and outermost locations of the torus, i.e., $\theta_j = 0$ or π (in other words, $|v_j| = 1$ or $\sqrt{\rho}$) for any j . Hence, the stationary condition (36) is satisfied.

The situation in the annulus for $\hat{h} = 2$ and solutions parameterized by a_1 is shown as a schematic in Fig. 3(b). The streamline contours are projected onto the torus in Fig. 6 for (a) $\alpha = 2$ and (b) $\alpha = 3$. We choose $a_1 = 0.9 e^{i2\pi/3}$, $b_1 = \rho/a_1$, $a_2 = -a_1$, and $b_2 = \rho/a_1$. Four of the eight point vortices and one of the four smooth centers are visible from the viewing angle in Fig. 6.

V. LIMIT SOLUTIONS

We can construct classes of “limit solutions” to the Liouville-type equation [Eq. (5)] by substituting $f(\zeta) = A(h(\zeta) + C)$, where $A \in \mathbb{R}$, $A > 0$, and $C \in \mathbb{C}$ are constants, into the stream function (15),

$$\begin{aligned} \psi(\zeta, \bar{\zeta}) &= -\frac{1}{2} \log \left[\frac{A|h'(\zeta)|}{1 + A^2|h(\zeta) + C|^2} \right]^2 - \frac{1}{2} \log \left[\frac{4|\zeta|^2}{(\alpha - \cos \theta)^2} \right] \\ &= + \log \left[\frac{1}{A|h'(\zeta)|} + A \frac{|h(\zeta) + C|^2}{|h'(\zeta)|} \right] - \frac{1}{2} \log \left[\frac{4|\zeta|^2}{(\alpha - \cos \theta)^2} \right]. \end{aligned}$$

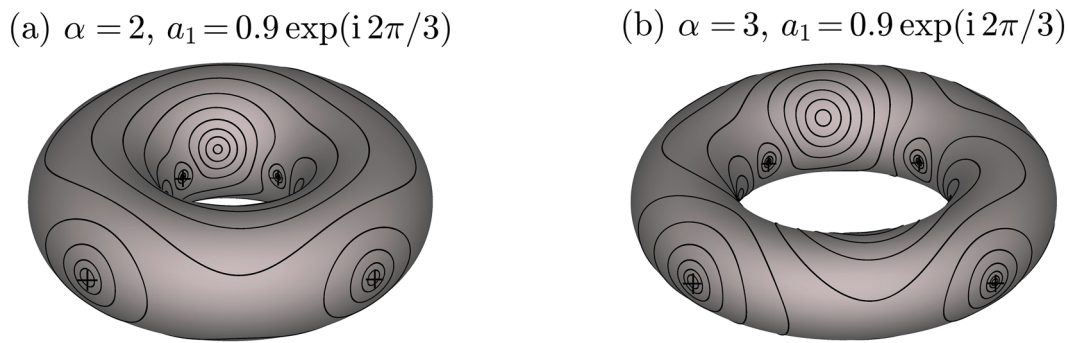


FIG. 6. Streamline contours projected on the torus, along with embedded point vortices (+), in the case $\hat{n} = 2$ (vorticity heatmaps are not shown here).

Note that since only $|A|$ would appear in the stream function above, we can take A to be real and non-negative without any loss of generality. By adding and subtracting the constant term $\log A$ and taking the limits $A \rightarrow 0$ and $A \rightarrow \infty$, we obtain the following two *limit* solutions:

$$\psi_0(\zeta, \bar{\zeta}) = \lim_{A \rightarrow 0} [\psi(\zeta, \bar{\zeta}) + \log A] = -\log |h'(\zeta)| - \frac{1}{2} \log \left[\frac{4|\zeta|^2}{(\alpha - \cos \theta)^2} \right], \quad (39)$$

$$\psi_\infty(\zeta, \bar{\zeta}) = \lim_{A \rightarrow \infty} [\psi(\zeta, \bar{\zeta}) - \log A] = -\log \left[\frac{|h'(\zeta)|}{|h(\zeta) + C|^2} \right] - \frac{1}{2} \log \left[\frac{4|\zeta|^2}{(\alpha - \cos \theta)^2} \right]. \quad (40)$$

When we act the Laplace–Beltrami operator $\nabla_{\mathbb{T}_\alpha}^2$ on these limit solutions, the first terms vanish and the second terms become $-\kappa(\mathbb{T}_\alpha)$. Moreover, they contain pole vortices with the strength quantized by $2m\pi$, $m \in \mathbb{Z}$, at poles and zeros of the functions $h'(\zeta)$ and $h(\zeta) + C$ of degree m . Hence, they are solutions of

$$\nabla_{\mathbb{T}_\alpha}^2 \psi = -\kappa(\mathbb{T}_\alpha) - \sum_{j=1}^n \Gamma_j \delta_{v_j}, \quad (41)$$

where n point vortices with the strength Γ_j exist at v_j for $j = 1, \dots, n$. The vorticity distribution on the toroidal surface is, thus, given by

$$-\Omega = -\kappa(\mathbb{T}_\alpha) - \sum_{j=1}^n \Gamma_j \delta_{v_j}. \quad (42)$$

It indicates that the limit solutions represent flows of n point vortices embedded in the background vorticity distribution $\kappa(\mathbb{T}_\alpha)$. The Gauss condition is reduced to

$$\iint_{\mathbb{T}_\alpha} \Omega \, d\sigma = \sum_{j=1}^n \Gamma_j = 0, \quad (43)$$

in which the contribution from the nonlinear term $e^{-2\psi}$ is not included.

In what follows, as in Sec. III, we consider two functions, i.e., the Weierstrass $\wp$ function $h(\zeta) = \wp(-i \log \zeta)$ and a loxodromic function $h(\zeta) = K(\zeta/a) - K(\zeta/b)$ for $a, b \in D_\zeta$ as examples of limit solutions.

A. Weierstrass $\wp$ -function

Choosing $h(\zeta) = \wp(-i \log \zeta)$, the limit solution $\psi_0(\zeta, \bar{\zeta})$ given by (39) becomes

$$\psi_0(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4|\wp'(-i \log \zeta)|^2}{(\alpha - \cos \theta)^2} \right]. \quad (44)$$

The derivative of the Weierstrass $\wp$ -function, $\wp'(z)$, has three simple zeros at $z = \omega_1, \omega_2, \omega_3$, and a pole of degree three at $\omega_0 = 0$ (see Sec. III). The stream function (44), thus, contains three point vortices with strength $\Gamma_k = +2\pi$ each at $v_k = \exp(i\omega_k)$ for $k = 1, 2, 3$ and a point vortex with strength $\Gamma_0 = -6\pi$ at $v_0 = \exp(i\omega_0) = 1$. Then, the total point vortex circulation vanishes, namely, $\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. Hence, the Gauss condition (43) is satisfied.

The stationary condition on the point vortices is checked for $\zeta = v_k$, $k = 1, 2, 3$, and $\zeta = v_0$ separately since the strengths of the point vortices are different. For the point vortices with strengths $\Gamma_k = +2\pi$ at $\zeta = v_k$, $k = 1, 2, 3$, we get from (23) (where $m = 1$)

$$\begin{aligned}\tilde{\psi}_0(v_k, \bar{v}_k) &= \lim_{\zeta \rightarrow v_k} \left(-\frac{1}{2} \log \left[\frac{4|\wp'(-i \log \zeta)|^2}{(\alpha - \cos \theta)^2} \right] + \log \left[\frac{(\alpha - \cos \theta_k)}{|v_k|} |\zeta - v_k| \right] \right) \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_k) - \frac{1}{2} \log |v_k|^2 - \frac{1}{2} \lim_{\zeta \rightarrow v_k} \log \left| \frac{\wp'(-i \log \zeta) - \wp'(-i \log v_k)}{\zeta - v_k} \right|^2 \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_k) - \frac{1}{2} \log |\wp''(-i \log v_k)|^2.\end{aligned}$$

Hence, it follows from (12) that

$$v_k \frac{\partial}{\partial v_k} \tilde{\psi}_0(v_k, \bar{v}_k) = -\sin \theta_k + \frac{i}{2} \frac{\wp'''(-i \log v_k)}{\wp''(-i \log v_k)} = 0, \quad k = 1, 2, 3,$$

owing to (A2) and $\theta_1 = \theta_2 = \pi$, $\theta_3 = 0$. For the point vortex at $\zeta = v_0 = 1$ with strength $\Gamma_0 = -6\pi$, we get using (23) (with $m = -3$)

$$\begin{aligned}\tilde{\psi}_0(v_0, \bar{v}_0) &= \lim_{\zeta \rightarrow v_0} \left(-\frac{1}{2} \log \left[\frac{4|\wp'(-i \log \zeta)|^2}{(\alpha - \cos \theta)^2} \right] - 3 \log \left[\frac{(\alpha - \cos \theta_0)}{|v_0|} |\zeta - v_0| \right] \right) \\ &= -\log 2 - 2 \log(\alpha - \cos \theta_0) + 3 \log |v_0| - \lim_{\zeta \rightarrow v_0} \log |(\zeta - v_0)^3 \wp'(-i \log \zeta)| \\ &= -2 \log 2 - 2 \log(\alpha - \cos \theta_0).\end{aligned}$$

Here, we have used (A4) to calculate

$$\lim_{\zeta \rightarrow v_0} \log |(\zeta - v_0)^3 \wp'(-i \log \zeta)| = \lim_{\zeta \rightarrow v_0} \log \left| -\frac{2v_0^3(\zeta/v_0 - 1)^3}{(-i \log(\zeta/v_0))^3} (1 + o(-i \log \zeta)^3) \right| = \log |2v_0^3|.$$

Hence, we obtain

$$v_0 \frac{\partial}{\partial v_0} \tilde{\psi}_0(v_0, \bar{v}_0) = \sin \theta_0 = 0.$$

Consequently, the stream function $\psi_0(v_0, \bar{v}_0)$ gives rise to a stationary solution of the Euler equation. It is important to note that it contains the quantized point vortices with different strengths at the innermost and the outermost locations.

The other limit solution $\psi_\infty(\zeta, \bar{\zeta})$ is given by

$$\psi_\infty(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \theta)^2} \frac{|\wp'(-i \log \zeta)|^2}{|\wp(-i \log \zeta) + C|^4} \right]. \quad (45)$$

Since $\wp(z)$ has a double pole at $z = \omega_0 = 0$, the equation $\wp(z) + C = 0$ has two zeros, say, at $z = \widehat{\omega}_1$ and $\widehat{\omega}_2$. Hence, the solution contains four point vortices at $v_k = \exp(i\omega_k)$ with strengths $\Gamma_k = +2\pi$ for $k = 0, 1, 2, 3$. There are two additional point vortices at $\bar{v}_k = \exp(i\widehat{\omega}_k)$ with strengths $\widehat{\Gamma}_k = -4\pi$ for $k = 1, 2$. Hence, the total circulation satisfies $\sum_{k=1}^4 \Gamma_k + \sum_{k=1}^2 \widehat{\Gamma}_k = 0$. Hence, the Gauss condition (19) holds true.

We confirm if the stationary condition is satisfied by these six point vortices. First, consider the point vortices with strength $\Gamma_k = +2\pi$ at $\zeta = v_k = \exp(i\omega_k)$, $k = 1, 2, 3$. These point vortices are located at the zeros of $\wp'(-i \log \zeta)$, and the modified stream function (23) in this case is (with $m = 1$)

$$\begin{aligned}\tilde{\psi}_\infty(v_k, \bar{v}_k) &= \lim_{\zeta \rightarrow v_k} \left(-\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \theta)^2} \frac{|\wp'(-i \log \zeta)|^2}{|\wp(-i \log \zeta) + C|^4} \right] + \log \left[\frac{(\alpha - \cos \theta_k)}{|v_k|} |\zeta - v_k| \right] \right) \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_k) + \log |\wp(-i \log v_k) + C|^2 \\ &\quad - \frac{1}{2} \lim_{\zeta \rightarrow v_k} \log \left| \frac{v_k(\wp'(-i \log \zeta) - \wp'(-i \log v_k))}{\zeta - v_k} \right|^2 \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_k) + \log |\wp(-i \log v_k) + C|^2 - \frac{1}{2} \log |\wp''(-i \log v_k)|^2.\end{aligned}$$

We can see that the stationary condition is satisfied,

$$v_k \frac{\partial}{\partial v_k} \tilde{\psi}_\infty(v_k, \bar{v}_k) = -\sin \theta_k - i \frac{\wp'(-i \log v_k)}{\wp(-i \log v_k) + C} + \frac{i}{2} \frac{\wp'''(-i \log v_k)}{\wp''(-i \log v_k)} = 0$$

using (A2) and $\theta_k = 0$ or π for $k = 1, 2, 3$. Second, for the point vortex with the strength $\Gamma_0 = +2\pi$ at $\zeta = v_0 = 1$, we have (again with $m = 1$)

$$\begin{aligned}\tilde{\psi}_\infty(v_0, \bar{v}_0) &= \lim_{\zeta \rightarrow v_0} \left(-\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \theta)^2} \frac{|\wp'(-i \log \zeta)|^2}{|\wp(-i \log \zeta) + C|^4} \right] + \log \left[\frac{(\alpha - \cos \theta_0)}{|v_0|} |\zeta - v_0| \right] \right) \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_0) - \frac{1}{2} \lim_{\zeta \rightarrow v_0} \log \left| \frac{v_0 \wp'(-i \log \zeta)}{(\zeta - v_0)(\wp(-i \log \zeta) + C)^2} \right|^2 \\ &= -2 \log 2 + 2 \log(\alpha - \cos \theta_0)\end{aligned}$$

since using (A3) and (A4), we find

$$\lim_{\zeta \rightarrow v_0} \log \left| \frac{v_0 \wp'(-i \log \zeta)}{(\zeta - v_0)(\wp(-i \log \zeta) + C)^2} \right|^2 = \lim_{\zeta \rightarrow v_0} \log \left| \left(\frac{2i \log(\zeta/v_0)}{\zeta/v_0 - 1} \right)^2 (1 + o(-i \log \zeta)) \right| = \log 4.$$

From this, we find that the stationary condition is satisfied: $v_0 \frac{\partial}{\partial v_0} \tilde{\psi}_\infty(v_0, \bar{v}_0) = -\sin \theta_0 = 0$.

Finally, the point vortices at $\zeta = \widehat{v}_k = \exp(i\widehat{\omega}_k)$ with $\wp(\widehat{\omega}_k) + C = 0$ and $|\widehat{v}_k| = \exp(r_c(\widehat{\theta}_k))$ have the same strength $\widehat{\Gamma}_k = -4\pi$ for $k = 1, 2$. The modified stream function (23) in this case is given by (with $m = -2$)

$$\begin{aligned}\tilde{\psi}_\infty(\widehat{v}_k, \bar{\widehat{v}}_k) &= \lim_{\zeta \rightarrow \widehat{v}_k} \left(-\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \widehat{\theta}_k)^2} \frac{|\wp'(-i \log \zeta)|^2}{|\wp(-i \log \zeta) + C|^4} \right] - 2 \log \left[\frac{(\alpha - \cos \widehat{\theta}_k)}{|\widehat{v}_k|} |\zeta - \widehat{v}_k| \right] \right) \\ &= -\log 2 - \log(\alpha - \cos \widehat{\theta}_k) - \log |\wp'(-i \log \widehat{v}_k)| + \lim_{\zeta \rightarrow \widehat{v}_k} \log \left| \frac{\widehat{v}_k (\wp(-i \log \zeta) + C)}{\zeta - \widehat{v}_k} \right|^2 \\ &= -\log 2 - \log(\alpha - \cos \widehat{\theta}_k) + \log |\wp'(-i \log \widehat{v}_k)|.\end{aligned}$$

Here, we have used

$$\begin{aligned}\lim_{\zeta \rightarrow \widehat{v}_k} \log \left| \frac{\widehat{v}_k (\wp(-i \log \zeta) + C)}{\zeta - \widehat{v}_k} \right|^2 &= \lim_{\zeta \rightarrow \widehat{v}_k} \log \left| \frac{\widehat{v}_k (\wp(-i \log \zeta) - \wp(-i \log \widehat{v}_k))}{\zeta - \widehat{v}_k} \right|^2 \\ &= \log |\wp'(-i \log \widehat{v}_k)|^2.\end{aligned}$$

Hence, the stationary condition is reduced to

$$H(\widehat{v}_k) \equiv \widehat{v}_k \frac{\partial}{\partial \widehat{v}_k} \tilde{\psi}_\infty(\widehat{v}_k, \bar{\widehat{v}}_k) = \frac{\sin \widehat{\theta}_k}{2} - \frac{i}{2} \frac{\wp''(-i \log \widehat{v}_k)}{\wp'(-i \log \widehat{v}_k)} = 0, \quad k = 1, 2.$$

Let us now suppose that C is real. Since $\wp$ is even, i.e., $\wp(z) = \wp(-z)$, it follows that $0 = \wp(\widehat{\omega}_1) + C = \wp(-\widehat{\omega}_1) + C$, and so $\widehat{\omega}_2 = -\widehat{\omega}_1$. We, then, have $H(\widehat{v}_1) = -H(\widehat{v}_2)$ since using $\widehat{\omega}_2 = -\widehat{\omega}_1$, we first get

$$\wp'(\widehat{\omega}_2) = \wp'(-\widehat{\omega}_1) = -\wp'(\widehat{\omega}_1) \quad \Rightarrow \quad \wp''(\widehat{\omega}_2) = \wp''(-\widehat{\omega}_1) = \wp''(\widehat{\omega}_1)$$

and also $\widehat{\theta}_2 = 2\pi - \widehat{\theta}_1$. It is, thus, sufficient to confirm that $H(\widehat{v}_1) = 0$. We first show that $\widehat{\theta}_1 = \pi$. The double periodicity of $\wp(z)$ gives a formula for $\widehat{\omega}_2$,

$$\wp(\widehat{\omega}_2) = \wp(-\widehat{\omega}_1) = \wp(2\omega_1 + 2\omega_3 - \widehat{\omega}_1) \quad \Rightarrow \quad \widehat{\omega}_2 = 2\omega_1 + 2\omega_3 - \widehat{\omega}_1 \in D_z, \quad (46)$$

for $\widehat{\omega}_1 \in D_z$. The series expansion for $\wp(\omega_1 + z)$ near $z = 0$ is given by

$$\wp(\omega_1 + z) = \wp(\omega_1) + \frac{z^2}{2} \wp''(\omega_1) + \frac{z^4}{4!} \wp^{(4)}(\omega_1) + \dots,$$

where the odd derivative terms must be zero because of the $z \mapsto -z$ symmetry $\wp(\omega_1 + z) = \wp(-\omega_1 - z) = \wp(2\omega_1 - \omega_1 - z) = \wp(\omega_1 - z)$. In addition, since all the even derivatives $\wp^{(2k)}(\omega_1)$ are real, we have $\wp(\omega_1 - \bar{z}) = \wp(\omega_1 + \bar{z})$. Choosing the point $\bar{z} = \omega_1 - \widehat{\omega}_1$, we obtain another formula for $\widehat{\omega}_2$,

$$0 = \wp(\widehat{\omega}_1) + C = \overline{\wp(2\omega_1 - \widehat{\omega}_1)} + C = \overline{\wp(2\omega_1 - \widehat{\omega}_1)} + C \quad \Rightarrow \quad \widehat{\omega}_2 = 2\omega_1 - \overline{\widehat{\omega}_1}. \quad (47)$$

Equating two formulas (46) and (47) for $\widehat{\omega}_2$, we obtain $2\omega_1 - \overline{\widehat{\omega}_1} = 2\omega_1 + 2\omega_3 - \widehat{\omega}_1$, which reduces to $\text{Im } \widehat{\omega}_1 = -i\omega_3 = -\frac{1}{2} \log \rho$. It follows from $\text{Im } \widehat{\omega}_1 = -r_c(\widehat{\theta}_1) = -\log \sqrt{\rho}$ that $\widehat{\theta}_1 = \pi$. For the second term in $H(\widehat{v}_1)$, we use formula (A5) and $\wp(\widehat{\omega}_1) + C = 0$ to get

$$|\wp'(\widehat{\omega}_k)|^2 = |4(C + e_1)(C + e_2)(C + e_3)| \neq 0 \quad \text{and} \quad \wp''(\widehat{\omega}_1) = -6C - \frac{1}{2}(e_1^2 + e_2^2 + e_3^2).$$

Accordingly, if we choose $C = -\frac{1}{12}(e_1^2 + e_2^2 + e_3^2)$, then $\wp''(\widehat{\omega}_1) = 0$, and we get a stationary solution.

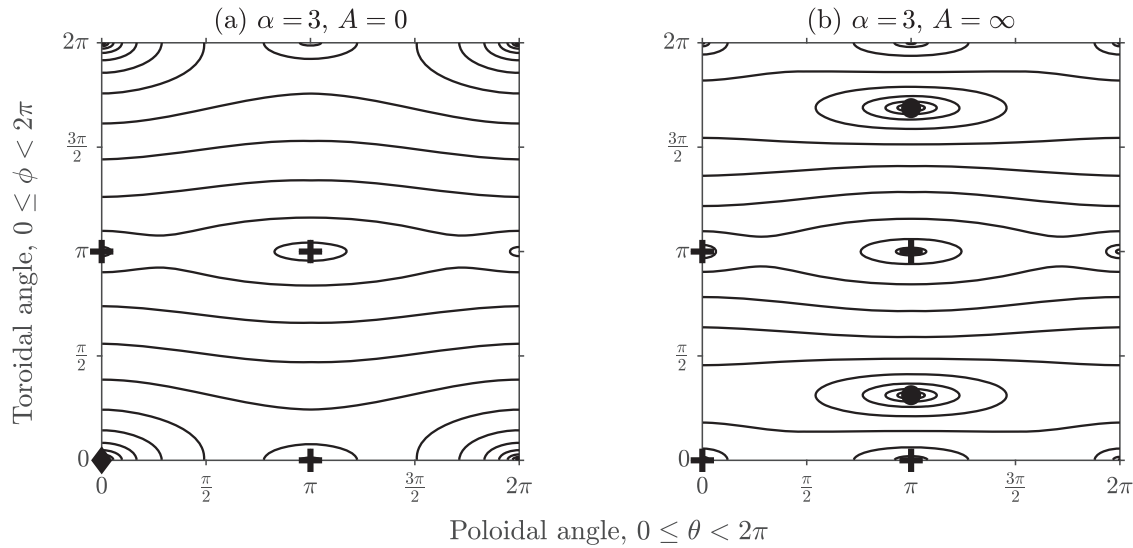


FIG. 7. Streamline contour plots with embedded point vortices (+, ♦, •) with inhomogeneous strengths in (a) the $A = 0$ limit (44) and (b) the $A = \infty$ limit (45).

Figure 7 shows the streamline patterns in the two limiting cases. The vorticity in the limiting case $A = 0$ (a) consists of three positive point vortices (+) with circulations $+2\pi$ each, together with one negative point vortex (♦) with circulation -6π , so that the total circulation is zero. In the limiting case $A = \infty$ (b), there are four point vortices (+) with circulations $+2\pi$ each and two negative vortices (♦) with circulations -4π each. The limiting solutions in Figs. 7(a) and 7(b), thus, contain point vortices of inhomogeneous strengths. The locations of the negative point vortices in Fig. 7(b) show rotational asymmetry in the toroidal direction. The background vorticity is due solely to the curvature term $\kappa(\mathbb{T}_\alpha)$ and does not contribute to the total circulation. This background vorticity is not shown in Fig. 7.

B. Loxodromic function

We can also obtain limiting solutions by substituting loxodromic functions for $h(\zeta)$ in (39) and (40). Taking $h(\zeta) = K(\zeta/a, \sqrt{\rho}) - K(\zeta/b, \sqrt{\rho})$ for $a, b \in D_\zeta$ in (39), the limiting solution ψ_0 is

$$\psi_0(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4|L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})|^2}{(\alpha - \cos \theta)^2} \right]. \quad (48)$$

The four solutions of $L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho}) = 0$ are $\zeta = v_k = \pm \sqrt{ab}, \pm \sqrt{ab/\rho}$, $k = 1, 2, 3, 4$ (see Sec. IV A) so that there are four point vortices at these points, each with strength $+2\pi$. In addition, it follows from (B9) that the function $L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})$ has double poles at $\zeta = a$ and $\zeta = b$. Hence, the stream function (48) also contains two point vortices at $\zeta = a, b$, each with strength -4π , and the total circulation of these six point vortices, therefore, vanishes. That is to say, the Gauss condition (43) is satisfied.

We confirm the stationary condition separately for the point vortices at $\zeta = v_k$ and at $\zeta = a, b$. For the four point vortices with strength $+2\pi$ at $\zeta = v_k$, $k = 1, 2, 3, 4$, the modified stream function is [using $m = 1$ in (23)] given by

$$\begin{aligned} \tilde{\psi}_0(v_k, \bar{v}_k) &= \lim_{\zeta \rightarrow v_k} \left(-\frac{1}{2} \log \left[\frac{4|L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})|^2}{(\alpha - \cos \theta)^2} \right] + \log \left[\frac{(\alpha - \cos \theta_k)}{|v_k|} |\zeta - v_k| \right] \right) \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_k) - \frac{1}{2} \log |M(v_k/a, \sqrt{\rho}) - M(v_k/b, \sqrt{\rho})|^2, \end{aligned}$$

where $r_c(\theta_k) = \log |v_k|$. We, thus, have the stationary condition

$$v_k \frac{\partial}{\partial v_k} \tilde{\psi}_0(v_k, \bar{v}_k) = -\sin \theta_k - \frac{1}{2} \frac{N(v_k/a, \sqrt{\rho}) - N(v_k/b, \sqrt{\rho})}{M(v_k/a, \sqrt{\rho}) - M(v_k/b, \sqrt{\rho})} = -\sin \theta_k = 0$$

since the zeros v_k also satisfy $N(v_k/a, \sqrt{\rho}) - N(v_k/b, \sqrt{\rho}) = 0$ due to (B10) and (B12). Hence, the stationary condition is reduced to $\theta_k = 0$ or π . For the point vortex at $\zeta = a$ with strength -4π , the modified stream function (23) is given by (taking $m = -2$)

$$\begin{aligned}\tilde{\psi}_0(a, \bar{a}) &= \lim_{\zeta \rightarrow a} \left(-\frac{1}{2} \log \left[\frac{4|L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})|^2}{(\alpha - \cos \theta)^2} \right] - 2 \log \left[\frac{(\alpha - \cos \theta_a)}{|a|} |\zeta - a| \right] \right) \\ &= -\log 2 - \log(\alpha - \cos \theta_a) + 2 \log |a| - \lim_{\zeta \rightarrow a} \log |(\zeta - a)^2 (L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho}))| \\ &= -\log 2 - \log(\alpha - \cos \theta_a).\end{aligned}$$

Here, $\log |a| = r_c(\theta_a)$, and we have used $(\zeta - a)^2 (L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})) \rightarrow -a^2$ as $\zeta \rightarrow a$, which follows from (B9). Hence, we get the stationary condition

$$a \frac{\partial}{\partial a} \tilde{\psi}_0(a, \bar{a}) = \frac{1}{2} \sin \theta_a = 0,$$

which shows that $\theta_a = 0$ or π . In the same way, we can also show that for the point vortex at b , we must have $\theta_b = 0$ or π [θ_b is defined by $\log |b| = r_c(\theta_b)$].

Thus, in order for the stream function to be a steady solution, we need all six point vortices to be on the circles $|\zeta| = 1$ or $\sqrt{\rho}$. If we choose $|a| = |b| = 1$ or $|a| = |b| = \sqrt{\rho}$, then we also have $|v_k| = 1$ or $\sqrt{\rho}$ and the necessary conditions for the vortices to be stationary are satisfied. An example is the choice $a = i$, $b = -i$, which leads to $v_1 = 1$, $v_2 = -1$, $v_3 = \sqrt{\rho}$, and $v_4 = -\sqrt{\rho}$. Note that the situation for the stream function ψ_0 is somewhat different from the case of $\hat{n} = 1$ discussed in Sec. IV A, as the point vortex stationary conditions impose further conditions on a and b .

Figure 8 shows example solutions in the limit $A = 0$. In (a), we choose $a_1 = i$ and $b_1 = -i$, and in (b), we choose $a_1 = e^{i\pi/3}$ and $b_1 = e^{i2\pi/3}$. There are four positive point vortices (+) with circulations $+2\pi$ each and two negative point vortices (•) of circulations -4π each so that the total circulation is zero. Only the curvature term contributes to the background vorticity and is not shown in Fig. 8.

We now turn to a description of the other limiting solution ψ_∞ given by (40). Taking $h(\zeta) = K(\zeta/a, \sqrt{\rho}) - K(\zeta/b, \sqrt{\rho})$ for $a, b \in D_\zeta$ with $a \neq b$, we get

$$\psi_\infty(\zeta, \bar{\zeta}) = -\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \theta)^2} \frac{|L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})|^2}{|K(\zeta/a, \sqrt{\rho}) - K(\zeta/b, \sqrt{\rho}) + C|^4} \right]. \quad (49)$$

The stream function ψ_∞ contains point vortices with strength $+2\pi$ at the four zeros $v_k = \pm\sqrt{ab}$, $\pm\sqrt{ab/\rho}$ of $L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})$, $k = 1, 2, 3, 4$. It also contains two point vortices with strength -4π at the two zeros, say $\bar{v}_k$, $k = 1, 2$, of $h(\zeta) + C$. Note that the poles of $h(\zeta)$, at $\zeta = a, b$, are removable. Since the total circulation vanishes, the Gauss condition (21) is satisfied.

Turning to the stationary condition for the six point vortices, we first consider the four point vortices with strength $+2\pi$ at v_k , $k = 1, 2, 3, 4$. The modified stream function (23) is given by (with $m = 1$)

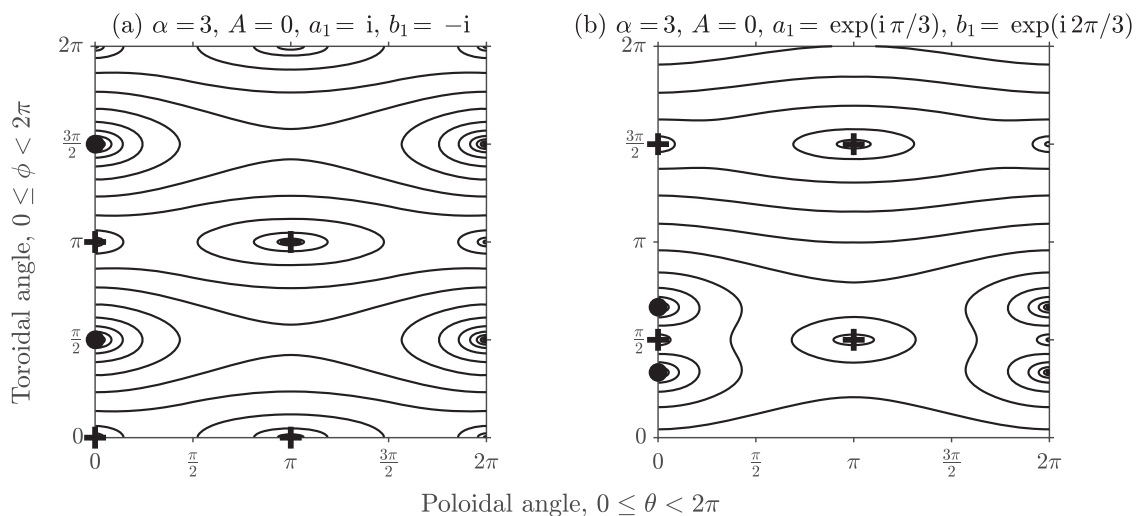


FIG. 8. Streamline contour plots with embedded point vortices (+, •) for the limiting stream function (48).

$$\begin{aligned}\tilde{\psi}_{\infty}(v_k, \bar{v}_k) &= \lim_{\zeta \rightarrow v_k} \left(-\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \theta)^2} \frac{|L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})|^2}{|K(\zeta/a, \sqrt{\rho}) - K(\zeta/b, \sqrt{\rho}) + C|^4} \right] + \log \frac{(\alpha - \cos \theta_k)}{|v_k|} |\zeta - v_k| \right) \\ &= -\log 2 + 2 \log(\alpha - \cos \theta_k) + \log |K(v_k/a, \sqrt{\rho}) - K(v_k/b, \sqrt{\rho}) + C|^2 \\ &\quad - \frac{1}{2} \log |M(v_k/a, \sqrt{\rho}) - M(v_k/b, \sqrt{\rho})|^2,\end{aligned}$$

where $r_c(\theta_k) = \log|v_k|$. Hence, the stationary condition is given by

$$\begin{aligned}v_k \frac{\partial}{\partial v_k} \tilde{\psi}_{\infty}(v_k, \bar{v}_k) &= -\sin \theta_k + \frac{L(v_k/a, \sqrt{\rho}) - L(v_k/b, \sqrt{\rho})}{K(v_k/a, \sqrt{\rho}) - K(v_k/b, \sqrt{\rho}) + C} \\ &\quad - \frac{1}{2} \frac{N(v_k/a, \sqrt{\rho}) - N(v_k/b, \sqrt{\rho})}{M(v_k/a, \sqrt{\rho}) - M(v_k/b, \sqrt{\rho})} \\ &= -\sin \theta_k = 0,\end{aligned}$$

where we have used (B10) and (B12). The stationary condition is once again reduced to $\theta_k = 0$ or π , which is equivalent to $|v_k| = 1$ or $\sqrt{\rho}$.

Next, for the point vortices with strength -4π at the zeros $\zeta = \widehat{v}_k$ satisfying

$$h(\zeta) + C = K(\widehat{v}_k/a, \sqrt{\rho}) - K(\widehat{v}_k/b, \sqrt{\rho}) + C = 0, \quad (50)$$

we have [using $m = -2$ in (23)]

$$\begin{aligned}\tilde{\psi}_{\infty}(\widehat{v}_k, \widehat{\bar{v}}_k) &= \lim_{\zeta \rightarrow \widehat{v}_k} \left(-\frac{1}{2} \log \left[\frac{4}{(\alpha - \cos \theta)^2} \frac{|L(\zeta/a, \sqrt{\rho}) - L(\zeta/b, \sqrt{\rho})|^2}{|K(\zeta/a, \sqrt{\rho}) - K(\zeta/b, \sqrt{\rho}) + C|^4} \right] - 2 \log \frac{(\alpha - \cos \widehat{\theta}_k)}{|\widehat{v}_k|} |\zeta - \widehat{v}_k| \right) \\ &= -\log 2 - \log(\alpha - \cos \widehat{\theta}_k) + \frac{1}{2} \log |L(\widehat{v}_k/a, \sqrt{\rho}) - L(\widehat{v}_k/b, \sqrt{\rho})|^2,\end{aligned}$$

where $\log|\widehat{v}_k| = r_c(\widehat{\theta}_k)$. This gives rise to the following stationary condition:

$$\widehat{v}_k \frac{\partial}{\partial \widehat{v}_k} \tilde{\psi}_{\infty}(\widehat{v}_k, \widehat{\bar{v}}_k) = \frac{1}{2} \sin \widehat{\theta}_k + \frac{1}{2} \frac{M(\widehat{v}_k/a, \sqrt{\rho}) - M(\widehat{v}_k/b, \sqrt{\rho})}{L(\widehat{v}_k/a, \sqrt{\rho}) - L(\widehat{v}_k/b, \sqrt{\rho})}.$$

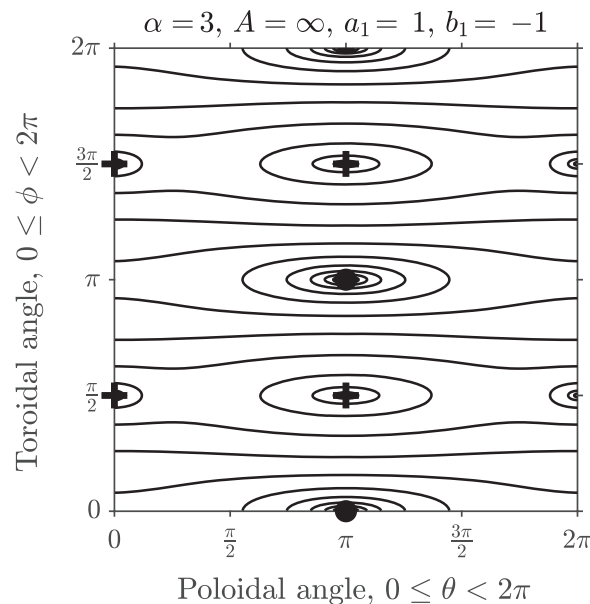


FIG. 9. Streamline contour plots for the limiting stream function (49). In this case, there are four point vortices with circulations $+2\pi$ each (+) and two point vortices with circulations -4π each (•), making the total circulation zero. The background vorticity due to $\kappa(\mathbb{T}_\alpha)$ is not shown.

Hence, we need to check if

$$\sin \widehat{\theta}_k + \frac{M(\widehat{v}_k/a, \sqrt{\rho}) - M(\widehat{v}_k/b, \sqrt{\rho})}{L(\widehat{v}_k/a, \sqrt{\rho}) - L(\widehat{v}_k/b, \sqrt{\rho})} = 0 \quad (51)$$

for given a , b , and C .

When we take $a = 1$, $b = -1$, and $C = 0$, for instance, we have six point vortices at $v_k = \pm i$, $\pm i\sqrt{\rho}$, and $\widehat{v}_k = \pm\sqrt{\rho}$. The zeros $\widehat{v}_k$ can be obtained by noting that $K(\pm\sqrt{\rho}, \sqrt{\rho}) = 0$ due to (B3). Since $|ab| = 1$, the point vortices at v_k with strengths $+2\pi$ are stationary. The point vortices at $\widehat{v}_k$ with the strength -4π satisfy (51) since $\widehat{\theta}_k = \pi$ and $M(\pm\sqrt{\rho}, \sqrt{\rho}) = 0$ due to (B11). This example is shown in Fig. 9. Note that it is not easy to obtain point vortex equilibria for other choices of a , b , and C since (50) and (51) give rise to three complex equations for two complex numbers, which are over-determined, in general.

VI. SUMMARY AND DISCUSSION

We have constructed a class of vortex solutions to the modified Liouville equation (5) on the surface of a curved torus, in which point vortices are in equilibrium with fixed background vorticity. The non-trivial part of the background vorticity is exponentially related to the stream function, and this allows us to construct stationary point vortex configurations with the strengths of the embedded point vortices being quantized. The Liouville solutions are regarded as equilibria of the point-vortex system (22) in the sense of the one-way interaction model discussed in Sec. I. They are not stationary solutions to the incompressible Euler equation since the background Liouville-type vorticity (5) is not a functional of the stream function ψ . This is in contrast to the solutions of the Liouville-type equation in the plane (Krishnamurthy *et al.*, 2019) and on the sphere (Crowdy, 2004), providing steady solutions of the Euler equation.

Various solutions can be constructed by making different choices for the analytic function in the Liouville solution (15). We have chosen such functions in terms of the Weierstrass $\wp$ -function and in terms of the Schottky–Klein prime function on an annulus. We also find limiting solutions in which the background Liouville-type vorticity can be turned off, leaving a set of quantized point vortices in equilibrium with only the background vorticity due to the curvature of the torus. The examples discussed show several interesting features of the solutions. The point vortex strengths are all equal and of the same sign for solutions containing both point vortices and the background Liouville-type vorticity. There are also regions of concentrated but smooth vorticity, which can move around smoothly as some parameter is varied, while the point vortex positions are independent of the parameter. In the limiting cases, we get point vortices with quantized but inhomogeneous strengths with some of the solutions lacking rotational symmetry. We find that the curvature of the torus restricts the poloidal locations of the point vortices to the innermost ($\theta = 0$) and outermost ($\theta = \pi$) circles of the torus. This restriction originating from the curvature shows up in the point vortex stationary condition as the $\sin \theta$ term, for example, in (35).

In the planar case, point vortex equilibria with no background vorticity have been shown to be closely related to the Liouville-type equation (3) (Krishnamurthy *et al.*, 2020). These “pure” point vortex equilibria were obtained as limiting solutions. The limiting solutions that we find in Sec. V result in quantized point vortex equilibria on the torus but are embedded in background vorticity due to the curvature of the torus. More general families of solutions called “Liouville chains” have been found in the planar case (Krishnamurthy *et al.*, 2021); finding such iterated solutions on the torus is an open problem. One important difference between the planar case and the torus is the absence of the Gauss condition in the former. We can also consider the problem on other manifolds such as a sphere. Since the Euler characteristic of a sphere is non-zero, the curvature term affects the Gauss condition differently. The point vortex stationary condition is, however, simpler in the case of a sphere due to its constant curvature.

A two-way interaction model has been considered by Newton and Sakajo (2007) in the numerical computation of point vortices embedded in continuous background vorticity corresponding to solid-body rotation on a sphere. They discretize the background vorticity distribution into a collection of strips of constant vorticity and tracked the evolutions of these strips and the point vortices simultaneously. For the case of an initially rigidly rotating ring of point vortices embedded in the background, they observe the formation of Rossby waves in the solution, which destroy the ring structure of the point vortex equilibria.

We can consider a similar two-way interaction model on the torus with a set of quantized point vortices embedded in a Liouville-type background vorticity distribution. It must be noted here that the numerical solution of the Euler equation on a torus is itself a challenging problem. Since the two-way interaction model is a useful discretization of Euler flow on a rotating sphere (Newton and Sakajo, 2007), it is interesting to ask whether such a model can be considered on the torus. The solutions obtained here can be used as computationally convenient initial vorticity distributions for a two-way interaction model on the surface of a torus. Such investigations will be reported in the future.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: THE WEIERSTRASS $\wp$ FUNCTION

In this appendix, we provide a short review of the doubly periodic Weierstrass $\wp$ function using formulas provided in the handbook of Abramowitz and Stegun (1992). Consider the fundamental domain D_z , in a complex z -plane, over which $\wp(z)$ is defined,

$$D_z = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) < 2\pi, -\log \rho < \operatorname{Im}(z) \leq 0\}. \quad (\text{A1})$$

Here, $\rho \in \mathbb{R}$, $0 < \rho < 1$ is a parameter, and the half periods of $\wp(z)$, ω_1 , and ω_3 are given by $\omega_1 = \pi$ and $\omega_3 = -\frac{i}{2} \log \rho$. We first note that the derivative of $|\wp(z)|$ is also doubly periodic since $\wp'(z)^2 = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ for some constants $e_1, e_2, e_3 \in \mathbb{C}$. Second, $\wp(z)$ has a double pole at $z = 0 \equiv \omega_0$ and its derivative $\wp'(z)$ has three simple zeros, one each at the points of half periods $z = \omega_1, \omega_3$, and one at the center of the fundamental domain $z = \omega_2 = \omega_1 + \omega_3 = \pi - \frac{i}{2} \log \rho$. The following useful formula shows that the third derivative $\wp'''(z)$ also vanishes at the points ω_k :

$$\wp'''(\omega_k) = 12\wp(\omega_k)\wp'(\omega_k) = 0, \quad k = 1, 2, 3. \quad (\text{A2})$$

Finally, the $\wp$ -function admits a Laurent series expansion in the neighborhood of the origin,

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_3 z^4 + \cdots = \frac{1}{z^2} (1 + o(z^3)), \quad z \rightarrow 0, \quad (\text{A3})$$

for some non-zero constants c_2 and c_4 . Thereby, the derivative is represented by

$$\wp'(z) = -\frac{2}{z^3} + 2c_2 z + 4c_3 z^3 + \cdots = -\frac{2}{z^3} (1 + o(z^3)), \quad z \rightarrow 0. \quad (\text{A4})$$

We have the standard definition $\wp(\omega_k) = e_k \in \mathbb{R}$, $k = 1, 2, 3$, with $e_1 > 0 > e_3$ and $e_1 > e_2 > e_3$. We recall the formulas

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \quad \wp''(z) = 6\wp(z) - \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \quad (\text{A5})$$

APPENDIX B: SCHOTTKY-KLEIN PRIME FUNCTIONS

The Schottky-Klein prime function is a special function defined on multiply-connected circular domains (Crowdy, 2020). The prime function defined on the annulus $D_\zeta = \{\zeta \in \mathbb{C} \mid \rho < |\zeta| \leq 1\}$ is essentially the P -function given by the infinite product

$$P(\zeta, \sqrt{\rho}) = (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^k \zeta)(1 - \rho^k / \zeta). \quad (\text{B1})$$

Note that $P(\zeta, \sqrt{\rho})$ has a simple zero in D_ζ at $\zeta = 1$. The K -function defined in terms of the logarithmic derivative of $P(\zeta, \sqrt{\rho})$ is given by

$$K(\zeta, \sqrt{\rho}) = \frac{\zeta P'(\zeta, \sqrt{\rho})}{P(\zeta, \sqrt{\rho})}. \quad (\text{B2})$$

Here, the prime denotes derivatives with respect to the first argument, and thus, $P'(\zeta, \sqrt{\rho}) = \frac{dP(\zeta, \sqrt{\rho})}{d\zeta}$. It can be verified from these definitions that the K -function satisfies the useful identity

$$K(\rho\zeta, \sqrt{\rho}) = K(\zeta, \sqrt{\rho}) - 1 = -K(1/\zeta, \sqrt{\rho}). \quad (\text{B3})$$

Furthermore, we can deduce the infinite series formula

$$K(\zeta, \sqrt{\rho}) = \frac{\zeta}{\zeta - 1} + \sum_{k=1}^{\infty} \left(\frac{-\rho^k \zeta}{1 - \rho^k \zeta} + \frac{\rho^k / \zeta}{1 - \rho^k / \zeta} \right) = \frac{1}{\zeta - 1} + O(1) \quad \text{as } \zeta \rightarrow 1, \quad (\text{B4})$$

showing that $K(\zeta, \sqrt{\rho})$ has a simple pole singularity at $\zeta = 1$.

We can define further functions via derivatives. The L , M , and N -functions are defined by

$$L(\zeta, \sqrt{\rho}) = \zeta K'(\zeta, \sqrt{\rho}), \quad (\text{B5})$$

$$M(\zeta, \sqrt{\rho}) = \zeta L'(\zeta, \sqrt{\rho}), \quad (\text{B6})$$

$$N(\zeta, \sqrt{\rho}) = \zeta M'(\zeta, \sqrt{\rho}). \quad (\text{B7})$$

The infinite series expansion for the L -function can be worked out using (B4) and (B5) to be

$$L(\zeta, \sqrt{\rho}) = -\frac{\zeta}{(\zeta - 1)^2} - \sum_{k=1}^{\infty} \left(\frac{\rho^k \zeta}{(1 - \rho^k \zeta)^2} + \frac{\rho^k / \zeta}{(1 - \rho^k / \zeta)^2} \right), \quad (\text{B8})$$

showing that it has a double pole at $\zeta = 1$ since

$$L(\zeta, \sqrt{\rho}) = -\frac{1}{(\zeta - 1)^2} - \frac{1}{\zeta - 1} + O(1) \quad \text{as } \zeta \rightarrow 1. \quad (\text{B9})$$

The following important properties of the L -function can be proven using (B8):

$$L(\zeta, \sqrt{\rho}) = L(\rho\zeta, \sqrt{\rho}) = L(1/\zeta, \sqrt{\rho}). \quad (\text{B10})$$

The properties of the M -function can be found by taking derivatives of (B10). We find

$$M(\zeta, \sqrt{\rho}) = M(\rho\zeta, \sqrt{\rho}) = -M(1/\zeta, \sqrt{\rho}). \quad (\text{B11})$$

Then, from (B11), we can find the properties of the N -function,

$$N(\zeta, \sqrt{\rho}) = N(\rho\zeta, \sqrt{\rho}) = N(1/\zeta, \sqrt{\rho}). \quad (\text{B12})$$

These properties show that the L , M , and N -functions are all loxodromic functions, i.e., they are unchanged under $\zeta \mapsto \rho\zeta$. Furthermore, the L and N -functions are unchanged under $\zeta \mapsto 1/\zeta$, whereas the M -function picks up a minus sign under this mapping.

We finally mention that for the purpose of numerical computations, the above special functions are all evaluated using a rapidly convergent Laurent series for the prime function in the annulus; see the work of Crowdy (2010; 2020) for details.

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