

SUMMARY OF STUDIES OF CLOSED/OPEN MIRROR SYMMETRY FOR QUINTIC THREEFOLDS THROUGH LOG MIXED HODGE THEORY

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0. Introduction and Statements

This is a summary of [U14p].

We correct the definitions and descriptions of the integral structures in our previous paper [U14]. We use $\hat{\Gamma}$ -integral structure of Iritani in [I11] for A-model. Using the corrected version, we study open mirror symmetry for quintic threefolds through log mixed Hodge theory, especially the recent result on Néron models for admissible normal functions with non-torsion extensions in the joint work [KNU14] with K. Kato and C. Nakayama. We positively use integral structures of local systems with graded polarizations over the boundary points.

In a series of joint works with Kato and Nakayama, we are constructing a fundamental diagram which consists of various kind of partial compactifications of classifying space

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of mixed Hodge structures and their relations. We try to understand Hodge theoretic aspects of mirror symmetry in this framework of the fundamental diagram.

Fundamental Diagram

For a classifying space D of Hodge structures of specified type, we have

$$\begin{array}{ccccc}
 & & D_{\mathrm{SL}(2),\mathrm{val}} & \longrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & \downarrow & & \downarrow \\
 \Gamma \backslash D_{\Sigma,\mathrm{val}} & \longleftarrow & D_{\Sigma,\mathrm{val}}^{\sharp} & \longrightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
 \downarrow & & \downarrow & & & & \\
 \Gamma \backslash D_{\Sigma} & \longleftarrow & D_{\Sigma}^{\sharp} & & & &
 \end{array}$$

in pure case: [KU99], [KU02], [KU09]. For mixed case, we should extend to an amplified diagram: [KNU08], [KNU09], [KNU11], [KNU13], continuing.

Mirror symmetry for quintic threefolds

Mirror symmetry for the A-model of quintic threefold V and the B-model of its mirror V° was predicted in the famous paper [CDGP91]. We recall two styles of the theorem (1) and (2) below. Every statement in the present paper is near the large radius point q_0 of the complexified Kähler moduli $\mathcal{KM}(V)$ and the maximally unipotent monodromy point p_0 of the complex moduli $\mathcal{M}(V^{\circ})$.

Let $t := y_1/y_0$, $u := t/2\pi i$ be the canonical parameters and $q := e^t = e^{2\pi i u}$ be the canonical coordinate from 2.2 below and the respective ones in 2.3 below.

The following theorem is due to Lian-Liu-Yau [LLuY97].

(1) (*Potential*). The potentials of the two models coincide: $\Phi_{\mathrm{GW}}^V(t) = \Phi_{\mathrm{GM}}^{V^{\circ}}(t)$.

The following theorem is formulated by Morrison [M97] and proved by Iritani [I11].

(2) (*Variation of Hodge structure*). The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \xrightarrow{\sim} (p_0 \in \overline{\mathcal{M}}(V^{\circ}))$ of neighborhoods of the compactifications, by the canonical coordinate $q = \exp(2\pi i u)$, lifts to an isomorphism, over the punctured neighborhoods $\mathcal{KM}(V) \xrightarrow{\sim} \mathcal{M}(V^{\circ})$, of polarized $\mathbf{Z}$ -variations of Hodge structure with a specified section

$$(\mathcal{H}^V, S, \nabla^{\mathrm{even}}, \mathcal{H}_{\mathbf{Z}}^V, F; 1) \xrightarrow{\sim} (\mathcal{H}^{V^{\circ}}, Q, \nabla^{\mathrm{GM}}, \mathcal{H}_{\mathbf{Z}}^{V^{\circ}}, F; \tilde{\Omega}).$$

Our (3) below is equivalent to (1) and (2) by a log version [KU09, 2.5.14] of the nilpotent orbit theorem of Schmid [S73] (this part of [U14] is valid).

(3) (*Log Hodge structure, Log period map*). The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \xrightarrow{\sim} (p_0 \in \overline{\mathcal{M}}(V^{\circ}))$ of neighborhoods of the compactifications uniquely lifts to an isomorphism of B-model log variation of polarized Hodge structure with a specified section $\tilde{\Omega}$ for V° and A-model log variation of polarized Hodge structure with a specified section

1 for V , whose restriction over the punctured $\mathcal{KM}(V) \xleftarrow{\sim} \mathcal{M}(V^\circ)$ coincides with the isomorphism of variations of polarized Hodge structure with specified sections in (2).

This rephrases as follows. Let σ be the common monodromy cone, transformed by a level structure into End of a reference fiber of the local system, for the A-model and for the B-model. Then, we have a commutative diagram of horizontal log period maps

$$\begin{array}{ccc} (q_0 \in \overline{\mathcal{KM}}(V)) & \xleftarrow{\sim} & (p_0 \in \overline{\mathcal{M}}(V^\circ)) \\ & \searrow \quad \swarrow & \\ & ([\sigma, \exp(\sigma_{\mathbf{C}})F_0] \in \Gamma(\sigma)^{\text{gp}} \setminus D_\sigma) & \end{array}$$

with extensions of specified sections in (2), where $(\sigma, \exp(\sigma_{\mathbf{C}})F_0)$ is the nilpotent orbit, regarded as a boundary point, and $\Gamma(\sigma)^{\text{gp}} \setminus D_\sigma$ is the fine moduli of log Hodge structures of specified type. (For fine moduli $\Gamma(\sigma)^{\text{gp}} \setminus D_\sigma$, or more generally $\Gamma \setminus D_\Sigma$, see [KU09].)

Open mirror symmetry for quintic threefolds

The following theorem is due to Walcher [W07] and Morrison-Walcher [MW09].

(4) *(Inhomogenous solutions).*

Let $\mathcal{L}$ be the Picard-Fuchs differential operator for quintic mirror (cf. 2.2). Let

$$\mathcal{T}_A = \frac{u}{2} \pm \left(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2} \right)$$

be the A-model domainwall tension in [MW09], and

$$\mathcal{T}_B = \int_{C_-}^{C_+} \Omega$$

be the B-model domainwall tension, where $C_\pm \subset V^\circ$ are the disjoint smooth curves coming from the two conics in $\{x_1 + x_2 = x_3 + x_4 = 0\} \cap V_\psi \subset \mathbf{P}^4(\mathbf{C})$ [ibid].

Then

$$\mathcal{L}(y_0(z)\mathcal{T}_A(z)) = \mathcal{L}(\mathcal{T}_B(z)) \left(= \frac{15}{16\pi^2} \sqrt{z} \right) \quad \left(z = \frac{1}{(5\psi)^5} \right).$$

Concerning this, we have the following observations.

(5) *(Log mixed Hodge structure, Log normal function).* We describe for B-model. The same holds for A-model by (1)–(3) and the correspondence table in 2.5 below.

Put $\mathcal{H} := \mathcal{H}^{V^\circ}$ and $\mathcal{T} := \mathcal{T}_B$. We use $e^0 \in I^{0,0}$, $e^1 \in I^{1,1}$ which are a part of a basis of $\mathcal{H}_{\mathcal{O}^{\log}}$ respecting the Deligne decomposition at p_0 (see 2.5 (3B)) and a flat sections $s^0 = e^0$, $s^1 = e^1 - ue^0$ (see 2.5 (5B)). To make the local monodromy of $\mathcal{T}$ unipotent, we take a double cover $z^{1/2} \mapsto z$. Let $L_{\mathbf{Q}}$ be the translated local system from the trivial extension $\mathbf{Q} \oplus \mathcal{H}_{\mathbf{Q}}$ by $-(\mathcal{T}/y_0)s^0$ in $\mathcal{E}xt^1(\mathbf{Q}, \mathcal{H}_{\mathbf{Q}})$. Let $J_{L_{\mathbf{Q}}}$ be the Néron model on a neighborhood S of p_0 in the $z^{1/2}$ -plane which lies over $L_{\mathbf{Q}}$ in [KNU14]. Then,

$J_{L\mathbf{Q}} = \mathcal{E}xt_{\mathrm{LMH}/S}^1(\mathbf{Z}, \mathcal{H})$ (extension group of log mixed Hodge structures over S) in the present case ([KNU13, III, Corollary 6.1.6], cf. 1.4 below), and we have the following (5.1)–(5.3).

(5.1) The normalized tension $\mathcal{T}/y_0$ is understood as a truncated normal function by $(\mathcal{T}/y_0)s^0$. This extends as a truncated log normal function over the puncture. Then it lifts uniquely to a log normal function $S \rightarrow J_{L\mathbf{Q}}$ so that the corresponding exact sequence $0 \rightarrow \mathcal{H} \rightarrow H \rightarrow \mathbf{Z} \rightarrow 0$ of log mixed Hodge structures over S is given by the liftings $1_{\mathbf{Z}}$ and 1_F in H of $1 \in \mathbf{Z} \simeq (\mathrm{gr}^W)_{\mathbf{Z}}$ respecting the lattice and the Hodge filtration, respectively, which are defined as follows: $1_{\mathbf{Z}} := 1 - (\mathcal{T}/y_0)s^0$ with $(\mathcal{T}/y_0)s^0 \in \mathcal{H}_{\mathcal{O}^{\log}} = (\mathrm{gr}_3^W)_{\mathcal{O}^{\log}}$, and $1_F - 1_{\mathbf{Z}} := -(\theta(\mathcal{T}/y_0))e^1 + (\mathcal{T}/y_0)e^0$.

(5.2) A splitting of the weight filtration W of the local system $H_{\mathbf{Z}}$, i.e., a splitting compatible with the monodromy of the local system $H_{\mathbf{Z}}$, is given by $1_{\mathbf{Z}}^{\mathrm{spl}} = 1_{\mathbf{Z}} + s^1/2$, and the log normal function over it is given by $1_F^{\mathrm{spl}} - 1_{\mathbf{Z}}^{\mathrm{spl}} = -(\theta(\mathcal{T}/y_0))e^1 + (\mathcal{T}/y_0)e^0$.

(5.3) (4) says that the inverse of the truncated normal function in (5.1) from its image is given by $16\pi^2/15$ times the Picard-Fuchs differential operator $\mathcal{L}$.

Some geometric backgrounds of (5) are explained in Section 3.

We treat Tate twists case by case in this article.

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1. Log mixed Hodge theory

In this section, we recall some notions and results of log mixed Hodge theory from [KU09], [KNU13] and [KNU14] adapting to the present context.

1.1. Category $\mathcal{B}(\log)$

Let S be a subset of an analytic space Z . The *strong topology* of S in Z is the strongest one among those topologies on S in which, for any analytic space A and any morphism $f : A \rightarrow Z$ with $f(A) \subset S$ as sets, $f : A \rightarrow S$ is continuous. S is regarded as a local ringed space by the pullback sheaf of $\mathcal{O}_Z$.

Let $\mathcal{B}$ be the category of local ringed spaces S over $\mathbf{C}$ which have an open covering $(U_{\lambda})_{\lambda}$ satisfying the following condition: For each λ , there exist an analytic space Z_{λ} , and a subset S_{λ} of Z_{λ} such that, as local ringed space over $\mathbf{C}$, U_{λ} is isomorphic to S_{λ} which is endowed with the strong topology in Z_{λ} and the inverse image of $\mathcal{O}_{Z_{\lambda}}$.

A *log structure* on a local ringed space S is a sheaf of monoids M on S together with a homomorphism $\alpha : M \rightarrow \mathcal{O}_S$ such that $\alpha^{-1}\mathcal{O}_S^{\times} \xrightarrow{\sim} \mathcal{O}_S^{\times}$. A log structure means, locally on the underlying space, the log structure has a chart which is finitely generated, integral and saturated.

Let $\mathcal{B}(\log)$ be the category of objects of $\mathcal{B}$ endowed with an fs log structure (more precisely, cf. [KU09]).

1.2. Ringed space $(S^{\log}, \mathcal{O}_S^{\log})$

Let $S \in \mathcal{B}(\log)$. As a set define

$$S^{\log} := \{(s, h) \mid s \in S, h : M_s^{\text{gp}} \rightarrow \mathbf{S}^1 \text{ homomorphism s.t. } h(u) = u/|u| \ (u \in \mathcal{O}_{S,s}^\times)\}.$$

Endow $S^{\log}$ with the weakest topology such that the following two maps are continuous.

$$(1) \ \tau : S^{\log} \rightarrow S, (s, h) \mapsto s.$$

$$(2) \ \text{For any open set } U \subset S \text{ and any } f \in \Gamma(U, M^{\text{gp}}), \tau^{-1}(U) \rightarrow \mathbf{S}^1, (s, h) \mapsto h(f_s).$$

Then, τ is proper and surjective with fiber $\tau^{-1}(s) = (\mathbf{S}^1)^{r(s)}$, where $r(s)$ is the rank of $(M^{\text{gp}}/\mathcal{O}_S^\times)_s$ which varies with $s \in S$.

For $s \in S$ and $t \in S^{\log}$ lying over s , let $q_j \in M_s^{\text{gp}}$ ($1 \leq j \leq r(s)$) be elements such that their images in $(M^{\text{gp}}/\mathcal{O}_S^\times)_s$ form a basis. Let $t_j := \log(q_j)$ and define $\mathcal{O}_{S,t}^{\log}$ to be a polynomial ring $\mathcal{O}_{S,s}[t_j \ (1 \leq j \leq r(s))]$ over $\mathcal{O}_{S,s}$. Thus $\tau : (S^{\log}, \mathcal{O}_S^{\log}) \rightarrow (S, \mathcal{O}_S)$ is a morphism of ringed spaces over $\mathbf{C}$ (more precisely, cf. [KU09]).

1.3. Graded polarized log mixed Hodge structure

Let $S \in \mathcal{B}(\log)$. A *pre-graded polarized log mixed Hodge structure* on S is a tuple $H = (H_{\mathbf{Z}}, W, (\langle \cdot, \cdot \rangle_w)_w, H_{\mathcal{O}})$ consisting of a local system of $\mathbf{Z}$ -free modules $H_{\mathbf{Z}}$ of finite rank on $S^{\log}$, an increasing filtration W of $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, a nondegenerate $(-1)^w$ -symmetric $\mathbf{Q}$ -bilinear form $\langle \cdot, \cdot \rangle_w$ on gr_w^W , a locally free $\mathcal{O}_S$ -module $H_{\mathcal{O}}$ on S , a specified isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (*log Riemann-Hilbert correspondence*), and a specified decreasing filtration $FH_{\mathcal{O}}$ of $H_{\mathcal{O}}$ such that $F^p H_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^p H_{\mathcal{O}}$ are locally free. Put $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^p H_{\mathcal{O}}$. Then $\tau_* F^p = F^p H_{\mathcal{O}}$. For each integer w , the orthogonality condition $\langle F^p(\text{gr}_w^W), F^q(\text{gr}_w^W) \rangle_w = 0$ ($p + q > w$) is imposed.

A *pre-graded polarized log mixed Hodge structure* on S is a *graded polarized log mixed Hodge structure* on S if its pullback to each $s \in S$ is a graded polarized log mixed Hodge structure on s in the following sense.

Let $(H_{\mathbf{Z}}, W, (\langle \cdot, \cdot \rangle_w)_w, H_{\mathcal{O}})$ be a pre-graded polarized log mixed Hodge structure on a log point s . It is a *graded polarized log mixed Hodge structure* if it satisfies the following three conditions.

(1) (Admissibility). For each logarithm N of the local monodromy of the local system $(H_{\mathbf{R}}, W, (\langle \cdot, \cdot \rangle_w)_w)$, there exists a W -relative N -filtration $M(N, W)$.

(2) (Griffiths transversality). For any integer p , $\nabla F^p \subset \omega_s^{1,\log} \otimes F^{p-1}$ is satisfied, where $\omega_s^{1,\log}$ is the sheaf of $\mathcal{O}_S^{\log}$ -module of log differential 1-forms on $(s^{\log}, \mathcal{O}_s^{\log})$, and $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_S^{\log} \otimes H_{\mathbf{Z}} \rightarrow \omega_s^{1,\log} \otimes H_{\mathbf{Z}}$ is the log Gauss-Manin connection.

(3) (Positivity). For a point $t \in s^{\log}$ and a $\mathbf{C}$ -algebra homomorphism $a : \mathcal{O}_{s,t}^{\log} \rightarrow \mathbf{C}$, define a filtration $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ on $H_{\mathbf{C},t}$. Then, $(H_{\mathbf{Z},t}(\text{gr}_w^W), \langle \cdot, \cdot \rangle_w, F(a))$ is a polarized Hodge structure of weight w in the usual sense if a is sufficiently twisted, i.e., for $(q_j)_{1 \leq j \leq n} \subset M_s$ inducing generators of $M_s/\mathcal{O}_s^\times$, $|\exp(a(\log q_j))| \ll 1$ for any j .

1.4. Néron model for admissible normal function

We review some results from [KNU14, Theorem 1.3], [KNU13, III, Section 6.1] and [KNU10, Section 8] adapted to the situation (5) in Introduction.

For a pure case $h^{p,q} = 1$ ($p + q = 3$, $p, q \geq 0$) and $h^{p,q} = 0$ otherwise, a complete fan is constructed in [KU09, Section 12.3]. For a mixed case $h^{p,q} = 1$ (the above (p, q) , plus $(p, q) = (2, 2)$) and $h^{p,q} = 0$ otherwise, over the above fan, a weak fan of Néron model for given admissible normal function is constructed in [KNU14, Theorem 3.1], and we have a Néron model in the following sense.

Let $S \in \mathcal{B}(\log)$, $U := S_{\text{triv}} \subset S$ (consisting of those points with trivial log structure), $H_{(-1)}$ be a polarized variation of Hodge structure of weight -1 (Tate-twisted by 2 from $\mathcal{H}$ in Introduction (5)) on U and $L_{\mathbf{Q}}$ be a local system of $\mathbf{Q}$ -vector spaces which is an extension of $\mathbf{Q}$ by $H_{(-1), \mathbf{Q}}$. An admissible normal function over U for $H_{(-1)}$ underlain by the local system $L_{\mathbf{Q}}$ can be regarded as an admissible variation of mixed Hodge structure which is an extension of $\mathbf{Z}$ by $H_{(-1)}$ and lies over local system $L_{\mathbf{Q}}$.

For any given unipotent admissible normal function over U as above, $H_{(-1)}$ and $L_{\mathbf{Q}}$ extend to a polarized log mixed Hodge structure on S and a local system on $S^{\log}$, respectively, denoted by the same symbols, and there is a relative log manifold $J_{L_{\mathbf{Q}}}$ over S (cf. [KU09]) which is strict over S (i.e., endowed with the pullback log structure from S) and which represents the following functor on $\mathcal{B}/S^{\circ}$ ($S^{\circ} \in \mathcal{B}$ is the underlying space of S):

$S' \mapsto \{\text{LMH } H \text{ on } S' \text{ satisfying } H(\text{gr}_w^W) = H_{(w)}|_{S'} \text{ (} w = -1, 0 \text{) and } (*) \text{ below}\}/\text{isom.}$
 (*) Locally on S' , there is an isomorphism $H_{\mathbf{Q}} \simeq L_{\mathbf{Q}}$ on $(S')^{\log}$ preserving W .

Here $H_{(w)}|_{S'}$ is the pullback of $H_{(w)}$ by the structure morphism $S' \rightarrow S^{\circ}$, and S' is endowed with the pullback log structure from S .

Put $H' := H_{(-1)}$. In the present case, we have $J_{L_{\mathbf{Q}}} = \mathcal{E}xt_{\text{LMH}/S}^1(\mathbf{Z}, H')$ by [KNU13, Corollary 6.1.6]. This is the subgroup of $\tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}}))$ restricted by admissibility condition and log-point-wise Griffiths transversality condition ([KNU10, Section 8], cf. 1.3). Define $\bar{J}_{L_{\mathbf{Q}}}$ as the image of the composite map $J_{L_{\mathbf{Q}}} \rightarrow \tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}})) \rightarrow \tau_*(H'_{\mathcal{O}^{\log}}/(F^{-1} + \mathcal{H}_{\mathbf{Z}}))$. By using the polarization, we have a commutative diagram:

$$\begin{array}{ccccc}
 J_{L_{\mathbf{Q}}} & = & \mathcal{E}xt_{\text{LMH}/S}^1(\mathbf{Z}, H') & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/(F^0 + H'_{\mathbf{Z}})) & \xrightarrow[\sim]{\text{pol}} & \tau_*((F^0)^*/H'_{\mathbf{Z}}) \\
 \downarrow & & & & \downarrow & & \downarrow \\
 \bar{J}_{L_{\mathbf{Q}}} & & \subset & & \tau_*(H'_{\mathcal{O}^{\log}}/(F^{-1} + \mathcal{H}_{\mathbf{Z}})) & \xrightarrow[\sim]{\text{pol}} & \tau_*((F^1)^*/H'_{\mathbf{Z}}).
 \end{array}$$

2. Quintic threefolds

In this section, we give a correspondence table of A-model for quintic threefold and B-model for its mirror. This is a correction of our previous [U14, 3] by using $\hat{\Gamma}$ -integral structure of Iritani [I11].

2.1. Quintic threefold and its mirror

Let V be a general quintic threefold in $\mathbf{P}^4$.

Let $V_\psi : f := \frac{1}{5} \sum_{j=1}^5 x_j^5 - \psi \prod_{j=1}^5 x_j = 0$ ($\psi \in \mathbf{P}^1$) be a pencil of quintics in $\mathbf{P}^4$. Let μ_5 be the group consisting of the fifth roots of the unity in $\mathbf{C}$. Then the group $G := \{(a_j) \in (\mu_5)^5 \mid a_1 \dots a_5 = 1\}$ acts on V_ψ by $x_j \mapsto a_j x_j$. Let V_ψ° be a crepant resolution of quotient singularity of V_ψ/G (cf. [MW09]). Divide further by the action $(x_1, \dots, x_5) \mapsto (a^{-1}x_1, x_2, \dots, x_5)$ ($a \in \mu_5$).

2.2. Picard-Fuchs equation on the mirror V°

Let Ω be a 3-form on V_ψ° with a log pole over $\psi = \infty$ induced from

$$\left(\frac{5}{2\pi i}\right)^3 \text{Res}_{V_\psi} \left(\frac{\psi}{f} \sum_{j=1}^5 (-1)^{j-1} x_j dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_5 \right).$$

Let $z := 1/(5\psi)^5$ and $\theta := z d/dz$. Let

$$\mathcal{L} := \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4)$$

be the Picard-Fuchs differential operator for Ω , i.e., $\mathcal{L}\Omega = 0$ via the Gauss-Manin connection ∇ .

At $z = 0$, the Picard-Fuchs differential equation $\mathcal{L}y = 0$ has the indicial equation $\rho^4 = 0$ (ρ is indeterminate), i.e., maximally unipotent. By the Frobenius method, we have a basis of solutions $y_j(z)$ ($0 \leq j \leq 3$) as follows. Let

$$\tilde{y}(-z; \rho) := \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{5n} (5\rho + m)}{\prod_{m=1}^n (\rho + m)^5} (-z)^{n+\rho}$$

be a solution of $\mathcal{L}(\tilde{y}(-z; \rho)) = \rho^4(-z)^\rho$, and let

$$\tilde{y}(-z; \rho) = y_0(z) + y_1(z)\rho + y_2(z)\rho^2 + y_3(z)\rho^3 + \dots, \quad y_j(z) := \frac{1}{j!} \frac{\partial^j \tilde{y}(-z; \rho)}{\partial \rho^j} \Big|_{\rho=0}$$

be the Taylor expansion at $\rho = 0$. Then, y_j ($0 \leq j \leq 3$) form a basis of solutions for the equation $\mathcal{L}y = 0$. We have

$$y_0 = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n,$$

$$y_1 = y_0 \log z + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n.$$

Define the canonical parameters by $t := y_1/y_0$, $u := t/2\pi i$, and the canonical coordinate by $q := e^t = e^{2\pi i u}$ which is a specific chart of the log structure given by the divisor ($z = 0$) of $\mathbf{P}^1$ and gives a mirror map.

y_0 is holomorphic in z and invertible at $z = 0$. Write $z = z(q)$ which is holomorphic in q . Then we have

$$\log z = 2\pi i u - \frac{5}{y_0(z(q))} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j} \right) z(q)^n.$$

The Gauss-Manin potential of V_z° is

$$\Phi_{\text{GM}}^{V^\circ} = \frac{5}{2} \left(\frac{y_1}{y_0} \frac{y_2}{y_0} - \frac{y_3}{y_0} \right).$$

Let $\tilde{\Omega} := \Omega/y_0$. Then, the Yukawa coupling at $z = 0$ is

$$Y := - \int_{V^\circ} \tilde{\Omega} \wedge \nabla_\theta \nabla_\theta \nabla_\theta \tilde{\Omega} = \frac{5}{(1 + 5^5 z) y_0(z)^2}.$$

2.3. A-model of quintic V

Let V be a general quintic hypersurface in $\mathbf{P}^4$. Let $T^2 = H$ be the cohomology class of a hyperplane section of V in $\mathbf{P}^4$, $K(V) = \mathbf{R}_{>0} T^2$ be the Kähler cone of V , and u be the coordinate of $\mathbf{CT}^2$. Put $t := 2\pi i u$. A complexified Kähler moduli is defined as

$$\mathcal{KM}(V) := (H^2(V, \mathbf{R}) + iK(V))/H^2(V, \mathbf{Z}) \xrightarrow{\sim} \Delta^*, \quad uT^2 \mapsto q := e^{2\pi i u}.$$

Let $C \in H_2(V, \mathbf{Z})$ be the homology class of a line on V , and $T^1 \in H^4(V, \mathbf{Z})$ be the cohomology class Poincaré duality isomorphic to C .

For $\beta = dC \in H_2(V, \mathbf{Z})$, define $q^\beta := q^{\int_\beta T^1} = q^d$. The Gromov-Witten potential of V is defined as

$$\Phi_{\text{GW}}^V := \frac{1}{6} \int_V (tT^2)^3 + \sum_{0 \neq \beta \in H_2(V, \mathbf{Z})} N_d q^\beta = \frac{5t^3}{6} + \sum_{d>0} N_d q^d.$$

Here the Gromov-Witten invariant N_d is

$$\begin{aligned} \overline{M}_{0,0}(\mathbf{P}^4, d) &\xleftarrow{\pi_1} \overline{M}_{0,1}(\mathbf{P}^4, d) \xrightarrow{e_1} \mathbf{P}^4, \\ N_d &:= \int_{\overline{M}_{0,0}(\mathbf{P}^4, d)} c_{5d+1}(\pi_{1*} e_1^* \mathcal{O}_{\mathbf{P}^4}(5)). \end{aligned}$$

Note that $N_d = 0$ if $d \leq 0$. Let $N_d = \sum_{k|d} n_{d/k} k^{-3}$. Then $n_{d/k}$ is the instanton number.

2.4. Integral structure

Let S^* be $\mathcal{KM}(V)$ for A-model of V and $\mathcal{M}(V^\circ)$ for B-model for V° , and let S be $\overline{\mathcal{KM}}(V)$ for A-model and $\overline{\mathcal{M}}(V^\circ)$ for B-model (see 2.2, 2.3). Endow S with the log structure associated to the divisor $S \setminus S^*$.

The B-model variation of Hodge structure $\mathcal{H}^{V^\circ}$ is the usual variation of Hodge structure arising from the smooth projective family $f : X \rightarrow S^*$ of the quintic mirrors over a punctured neighborhood of the maximally unipotent monodromy point p_0 . Its integral structure is the usual one $\mathcal{H}_{\mathbf{Z}}^{V^\circ} = R^3 f_* \mathbf{Z}$. This is compatible with the monodromy weight filtration M around p_0 . Define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}_{\mathbf{Z}}^{V^\circ}$ for all k .

For the A-model $\mathcal{H}^V$ on S^* , the locally free sheaf on S^* , the Hodge filtration, and the monodromy weight filtration M around the large radius point q_0 are given by $\mathcal{H}_{\mathcal{O}}^V := \mathcal{O}_{S^*} \otimes (\bigoplus_{0 \leq p \leq 3} H^{2p}(V))$, $F^p := \mathcal{O}_{S^*} \otimes H^{\leq 2(3-p)}(V)$, and $M_{2p} := H^{\geq 2(3-p)}(V)$, respectively. Iritani defined $\hat{\Gamma}$ -integral structure in more general setting in [I11, Definition 3.6]. In the present case, it is characterized as follows. Let H and C be a hyperplane section and a line on V , respectively. Then, in the present case, a basis of the $\hat{\Gamma}$ -integral structure is given by $\{s(\mathcal{E}) \mid \mathcal{E} \text{ is } \mathcal{O}_V, \mathcal{O}_H, \mathcal{O}_C, \mathcal{O}_{\text{pt}}\}$ [ibid, Example 6.18], where $s(\mathcal{E})$ is a unique ∇^{even} -flat section satisfying

$$s(\mathcal{E}) \sim (2\pi i)^{-3} e^{-2\pi i u H} \cdot \hat{\Gamma}(T_V) \cdot (2\pi i)^{\deg/2} \text{ch}(\mathcal{E})$$

at the large radius point q_0 . Here, for the Chern roots $c(T_V) = \prod_{j=1}^3 (1 + \delta_j)$, the Gamma class $\hat{\Gamma}(T_V)$ is defined by

$$\begin{aligned} \hat{\Gamma}(T_V) &:= \prod_{j=1}^3 \Gamma(1 + \delta_j) = \exp(-\gamma c_1(V) + \sum_{k \geq 2} (-1)^k (k-1)! \zeta(k) \text{ch}_k(T_V)) \\ &= \exp(\zeta(2) \text{ch}_2(T_V) - 2\zeta(3) \text{ch}_3(T_V)) \end{aligned}$$

where γ is the Euler constant, and $\deg|_{H^{2p}(V)} := 2p$. The important point is that this class $\hat{\Gamma}(T_V)$ plays the role of a “square root” of the Todd class in Hirzebruch-Riemann-Roch ([I09, 1], [I11, 1, (13)]). Denote this $\hat{\Gamma}$ -integral structure by $\mathcal{H}_{\mathbf{Z}}^V$. This is compatible with the monodromy weight filtration M and we define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}_{\mathbf{Z}}^V$ for all k . For a direct definition of $\hat{\Gamma}$ -integral structure, see [I11, Definition 3.6].

In both A-model case and B-model case, the integral structures $\mathcal{H}_{\mathbf{Z}}^V$ and $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ on S^* extend to the local systems of $\mathbf{Z}$ -modules over $S^{\log}$ ([O03], [KU09, Proposition 2.3.5]), still denoted $\mathcal{H}_{\mathbf{Z}}^V$ and $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$, respectively.

Consider a diagram:

$$\begin{array}{ccc} \tilde{S}^{\log} := (\mathbf{R} \times i(0, \infty])^r & \supset & \tilde{S}^* := (\mathbf{R} \times i(0, \infty))^r \\ \downarrow & & \downarrow \\ S^{\log} & \supset & S^* \\ \tau \downarrow & & \\ S & & \end{array}$$

The coordinate u of $\tilde{S}^*$ extends over $\tilde{S}^{\log}$. Fix base points as $u_0 = 0 + i\infty \in \tilde{S}^{\log} \mapsto b := \bar{0} + i\infty \in S^{\log} \mapsto q = 0 \in S$, where $q = 0$ corresponds to q_0 for A-model and p_0

for B-model. Note that fixing a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fixing a base point b on $S^{\log}$ and also a branch of $(2\pi i)^{-1} \log q$.

Let $B := \mathcal{H}_{\mathbf{Z}}^V(u_0) = \mathcal{H}_{\mathbf{Z}}^V(b)$ for A-model and $B := \mathcal{H}_{\mathbf{Z}}^{V^\circ}(u_0) = \mathcal{H}_{\mathbf{Z}}^{V^\circ}(b)$ for B-model.

2.5. Correspondence table

In this section, we complete the approximation in the previous paper [U14]. These results will be used in Section 3.

We use (1) and (2) in Introduction. Put $\Phi := \Phi_{\text{GW}}^V = \Phi_{\text{GM}}^{V^\circ}$.

(1A) *Polarization of A-model of V .*

$$S(\alpha, \beta) := (-1)^p \int_V \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) *Polarization of B-model of V° .*

$$Q(\alpha, \beta) := (-1)^{3(3-1)/2} \int_{V^\circ} \alpha \cup \beta = - \int_{V^\circ} \alpha \cup \beta \quad (\alpha, \beta \in H^3(V^\circ)).$$

(2A) *$\mathbf{Z}$ -basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}_{\mathbf{Z}}^V(u_0) = \mathcal{H}_{\mathbf{Z}}^V(b)$. Then we have a basis b^0, b^1, b^2, b^3 of B compatible with the monodromy weight filtration M [I11, Example 6.18].

(2B) *$\mathbf{Z}$ -basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}_{\mathbf{Z}}^{V^\circ}(u_0) = \mathcal{H}_{\mathbf{Z}}^{V^\circ}(b)$. Then we have a basis b^0, b^1, b^2, b^3 of B compatible with the monodromy weight filtration M [ibid].

For both cases (2A) and (2B), we regard B as a constant sheaf endowed with M on $S^{\log}$ and also on S .

(3A) *Specified sections inducing $\mathbf{Z}$ -basis of gr^M for A-model of V .*

$$\begin{aligned} T^3 &:= 1 \in H^0(V, \mathbf{Z}), & T^2 &:= H \in H^2(V, \mathbf{Z}), \\ T^1 &:= C \in H^4(V, \mathbf{Z}), & T^0 &:= [\text{pt}] \in H^6(V, \mathbf{Z}), \end{aligned}$$

where H is a hyperplane section of V and C is a line on V . Then $S(T^3, T^0) = 1$ and $S(T^2, T^1) = -1$. Hence $T^3, T^2, -T^0, T^1$ form a symplectic base for S in (1A).

(3B) *Specified sections inducing $\mathbf{Z}$ -basis of gr^M for B-model of V° .*

We use Deligne decomposition [D97]. We consider B in (2B) as a constant sheaf on $S^{\log}$. We have locally free $\mathcal{O}_S$ -submodules $\mathcal{M}_{2p} := \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} M_{2p}B)$ and $\mathcal{F}^p$ of $\tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} B) = \mathcal{O}_S \otimes_{\mathbf{Z}} B$. The mixed Hodge structure of Hodge-Tate type $(\mathcal{M}, \mathcal{F})$ has decomposition:

$$\mathcal{O}_S \otimes_{\mathbf{Z}} B = \bigoplus_p I^{p,p}, \quad I^{p,p} := \mathcal{M}_{2p} \cap \mathcal{F}^p \xrightarrow{\sim} \text{gr}_{2p}^M.$$

Transporting the basis b^p ($0 \leq p \leq 3$) of B in (2B), regarded as sections of the constant sheaf B on $S^{\log}$, via isomorphism

$$I^{p,p} \xrightarrow{\sim} \mathcal{O}_S \otimes_{\mathbf{Z}} \mathrm{gr}_{2p}^M B$$

we define sections $e^p \in I^{p,p}$ ($0 \leq p \leq 3$). Then $e^3, e^2, -e^0, e^1$ form a symplectic basis for Q in (1B).

Note that $e^3 = \tilde{\Omega}$.

(4A) *A-model connection* $\nabla = \nabla^{\mathrm{even}}$ of V .

$$\begin{aligned} \nabla_{\theta} T^0 &:= 0, \quad \nabla_{\theta} T^1 := T^0, \\ \nabla_{\theta} T^2 &:= \frac{1}{(2\pi i)^3} \frac{d^3 \Phi}{du^3} T^1 = \left(5 + \frac{1}{(2\pi i)^3} \frac{d^3 \Phi_{\mathrm{hol}}}{du^3} \right) T^1, \\ \nabla_{\theta} T^3 &:= T^2. \end{aligned}$$

∇ is flat, i.e., $\nabla^2 = 0$.

(4B) *B-model connection* $\nabla = \nabla^{\mathrm{GM}}$ of V° .

$$\begin{aligned} \nabla_{\theta} e^0 &= 0, \quad \nabla_{\theta} e^1 = e^0, \\ \nabla_{\theta} e^2 &= \frac{1}{(2\pi i)^3} \frac{d^3 \Phi}{du^3} e^1 = Y e^1 = \frac{5}{(1 + 5^5) y_0(z)^2} \left(\frac{q}{z} \frac{dz}{dq} \right)^3 e^1, \\ \nabla_{\theta} e^3 &= e^2. \end{aligned}$$

(5A) *∇ -flat $\mathbf{Z}$ -basis for $\mathcal{H}_{\mathbf{Z}}^V$.*

$$\begin{aligned} s^0 &:= T^0, \\ s^1 &:= T^1 - u T^0, \\ s^2 &:= T^2 - \left(\frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) T^1 + \left(\frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u - \frac{25}{12} \right) T^0, \\ s^3 &:= T^3 - u T^2 + \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) - \frac{25}{12} \right) T^1 \\ &\quad - \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial \Phi}{\partial u} - 2\Phi \right) - \frac{25}{12} u - \frac{25i}{\pi^3} \zeta(3) \right) T^0. \end{aligned}$$

Then $s^3, s^2, -s^0, s^1$ form a symplectic basis for S in (1A).

(5B) *∇ -flat $\mathbf{Z}$ -basis for $\mathcal{H}_{\mathbf{Z}}^{V^{\circ}}$.*

$$\begin{aligned} s^0 &:= e^0, \\ s^1 &:= e^1 - u e^0, \\ s^2 &:= e^2 - \left(\frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) e^1 + \left(\frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u - \frac{25}{12} \right) e^0, \\ s^3 &:= e^3 - u e^2 + \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) - \frac{25}{12} \right) e^1 \\ &\quad - \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial \Phi}{\partial u} - 2\Phi \right) - \frac{25}{12} u - \frac{25i}{\pi^3} \zeta(3) \right) e^0. \end{aligned}$$

Then $s^3, s^2, -s^0, s^1$ form a symplectic basis for Q in (1B).

(6A) *Expression of the T^p by the s^p .*

It is computed that T^p are written by the ∇ -flat $\mathbf{Z}$ -basis s^p of $\mathcal{H}_{\mathbf{Z}}^V$ as follows.

$$\begin{aligned} T^0 &= s^0, \\ T^1 &= s^1 + us^0, \\ T^2 &:= s^2 + \left(\frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) s^1 + \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) + \frac{25}{12} \right) s^0, \\ T^3 &= s^3 + us^2 + \left(\frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u + \frac{25}{12} \right) s^1 \\ &\quad + \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial \Phi}{\partial u} - 2\Phi \right) + \frac{25}{12} u - \frac{25i}{\pi^3} \zeta(3) \right) s^0. \end{aligned}$$

Note that the section $1 = T^3$ varies with respect to the the lattice $\mathcal{H}_{\mathbf{Z}}^V$ as above while the section $[\text{pt}] = T^0 = s^0$ does not.

(6B) *Expression of the e^p by the s^p .*

It is computed that e^p are written by the ∇ -flat $\mathbf{Z}$ -basis s^p of $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ as follows.

$$\begin{aligned} e^0 &= s^0, \\ e^1 &= s^1 + us^0, \\ e^2 &:= s^2 + \left(\frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) s^1 + \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) + \frac{25}{12} \right) s^0, \\ e^3 &= s^3 + us^2 + \left(\frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u + \frac{25}{12} \right) s^1 \\ &\quad + \left(\frac{1}{(2\pi i)^3} \left(u \frac{\partial \Phi}{\partial u} - 2\Phi \right) + \frac{25}{12} u - \frac{25i}{\pi^3} \zeta(3) \right) s^0. \end{aligned}$$

Note that the normalized holomorphic 3-form $\tilde{\Omega} = \Omega/y_0 = e^3$ varies with respect to the lattice $\mathcal{H}_{\mathbf{Z}}^{V^\circ}$ as above, while the section $e^0 = s^0$ does not.

Idea of proof of (4A) and (4B). We prove (4B). (4A) follows by mirror symmetry theorems (1) and (2) in Introduction.

We improve the proof of [CoK99, Prop. 5.6.1] carefully by a log Hodge theoretic understanding of the relation among a constant sheaf and a local system on $S^{\log}$, of the canonical extension of Deligne on S , and of the Deligne decomposition.

Idea of proofs of (5A), (5B), (6A) and (6B). In [I11, Introduction] (cf. 2.4), the asymptotic condition in the large radius limit is stated for the flat integral section corresponding to $\mathcal{E} = \mathcal{O}_V \in K(V)$ in the situation (5A). Up to Tate twists, this condition coincides with the one in [CDGP91, (5.5)] stated in the situation (6A). By the mirror symmetry in [I11] (cf. (2) in Introduction), this condition is interpreted in the situation (6B). Our previous results in [U14, Sections 3.5–3.6] are insufficient (see Remark

below). In order to complete them, we compute here higher approximations in the situation (6B). The result in the situation (5B) is a linear algebraic solution of this.

Remark. The author was pointed out by Hiroshi Iritani that the definitions and the descriptions of integral structures in [U14, 3.5, 3.6] are insufficient. Actually, they were the first approximations of integral structures by means of gr^M , and the second proof in [ibid, 3.9] works well even in this approximation.

3. Discussions on geometries for (5) in Introduction

We discuss here the relation with geometries and local systems considered in [W07] and [MW09]. Forgetting Hodge structures, we consider only local systems corresponding to the monodromy of integral periods and tensions.

Let V_ψ and V_ψ° be a quintic threefold and its mirror from 2.1. Let S be a small neighborhood in the z -plane (z in 2.2) of the maximal unipotent monodromy point p_0 endowed with the log structure associated to the divisor p_0 .

We first consider B-model. Let the setting be as in [MW09, 4]. For $z \neq 0$ near 0, i.e., near p_0 , let V_z° be the mirror quintic and $C_{+,z} \cup C_{-,z}$ be the disjoint union of smooth rational curves on V_z° coming from the two conics contained in $V_\psi \cap \{x_1 + x_2 = x_3 + x_4 = 0\} \subset \mathbf{P}^4(\mathbf{C})$. From the relative homology sequence for $(V_z^\circ, (C_{+,z} \cup C_{-,z}))$, we have

$$(1) \quad 0 \rightarrow H_3(V_z^\circ; \mathbf{Z}) \rightarrow H_3(V_z^\circ, (C_{+,z} \cup C_{-,z}); \mathbf{Z}) \xrightarrow{\partial} \mathbf{Z}([C_{+,z}] - [C_{-,z}]) \rightarrow 0,$$

where $\mathbf{Z}([C_{+,z}] - [C_{-,z}])$ is $\text{Ker}(H_2(C_{+,z} \cup C_{-,z}); \mathbf{Z}) \rightarrow H_2(V_z^\circ; \mathbf{Z})$. The monodromy T_∞ around p_0 interchanges $C_{+,z}$ and $C_{-,z}$.

Respecting the sequence (1), we take a family of cycles Poincaré duality isomorphic to the flat integral basis s^p ($0 \leq p \leq 3$) in 2.5 (5B) and a family of chains joining from $C_{-,z}$ to $C_{+,z}$ (a choice up to integral cycles and up to half twists), and over them integrate the family of 3-forms $\Omega(z)$ with log pole over $z = 0$ (z in the punctured disc in the z -plane) in 2.2, then we have a family of vectors $(\eta_0, \eta_1, \eta_2, \eta_3, T)$ consisting of periods and a tension. This corresponds to the data in [W07], [MW09]. Since $T_\infty(T) = -(T + \eta_1 + \eta_0)$ by [W07, (3.14)], we find $T + \frac{1}{2}\eta_1 + \frac{1}{4}\eta_0 = \frac{15}{\pi^2}\tau$ is an eigenvector of the monodromy T_∞ with eigenvalue -1 .

The family of sequences (1) ($z \neq 0$) forms an exact sequence of local systems of $\mathbf{Z}$ -modules. To make the monodromy of this system unipotent, we take a double cover $z^{1/2} \mapsto z$. Let S be a neighborhood disc of p_0 in the $z^{1/2}$ -plane endowed with log structure associated to the divisor p_0 in S , and let $S^{\log}$ be as in 1.2. Let S^* be the punctured disc $S \setminus \{p_0\}$. Pull back the above local system to S^* and then extend it over $S^{\log}$.

Applying Tate twist (-3) and Poincaré duality isomorphism to the left and the right ends of this exact sequence, we have a local system L' over $S^{\log}$ which is an extension of $\mathbf{Z}(-2)$ by $\mathcal{H}_{\mathbf{Z}}$:

$$(2) \quad 0 \rightarrow \mathcal{H}_{\mathbf{Z}} \rightarrow L' \rightarrow \mathbf{Z}(-2) \rightarrow 0.$$

Let $1 \in \mathbf{Z} \simeq \mathrm{gr}_4^W \mathbf{Z}(-2)$, take a lifting $1_{\mathbf{Z}} := 1 - (T/\eta_0)s^0$ in L' of 1, and extend ∇ on $\mathcal{H}_{\mathbf{Z}}$ over L' by $\nabla(1_{\mathbf{Z}}) = 0$. We look for a T_{∞}^2 -invariant ∇ -flat element associated to $1_{\mathbf{Z}}$. This is computed as $1_{\mathbf{Z}}^{\mathrm{spl}} := 1_{\mathbf{Z}} - (s^1/2)$, and we know that L' coincides with $H_{\mathbf{Z}}$ in (5) in Introduction.

For the relative monodromy weight filtration $M = M(N, W)$, we see that $1_{\mathbf{Z}} \in M_4$ and $s^1 \in M_2$ are the smallest filters containing the elements in question. Taking the graded quotients by M of the sequence (2), we have

$$(3) \quad \begin{aligned} \mathrm{gr}_6^M \mathcal{H}_{\mathbf{Z}} &\xrightarrow{\sim} \mathrm{gr}_6^M L', \\ 0 \rightarrow \mathrm{gr}_4^M \mathcal{H}_{\mathbf{Z}} &\rightarrow \mathrm{gr}_4^M L' \rightarrow \mathbf{Z}(-2) \rightarrow 0, \\ 0 \rightarrow \mathrm{gr}_2^M \mathcal{H}_{\mathbf{Z}} &\rightarrow \mathrm{gr}_2^M L' \rightarrow (2\text{-torsion}) \rightarrow 0, \\ \mathrm{gr}_0^M \mathcal{H}_{\mathbf{Z}} &\xrightarrow{\sim} \mathrm{gr}_0^M L'. \end{aligned}$$

The 2-torsion in the third sequence of (3) corresponds to a half twist of chains from C_- to C_+ . Standing on a half integral point and looking at the integral points nearby, we have two orientations. These correspond to the two orientations of a half twist of the chains, and also correspond to $\mathcal{T}_{\pm} := \pm(\frac{15}{\pi^2}\tau - \frac{\eta_0}{4}) - \frac{\eta_1}{2}$ in [W07]. $\mathcal{T}_-$ is different from $-\mathcal{T}_+$ by the complementary half twist, i.e., $\mathcal{T}_+ + \mathcal{T}_- = -\eta_1$.

For A-model, we consider the setting in [W07, 2.1]. Let $V = V_{\psi}$ with $\psi = 0$ from 2.1 be a Fermat quintic threefold in $\mathbf{P}^4(\mathbf{C})$ and $Lg := V \cap \mathbf{P}^4(\mathbf{R})$ be a Lagrangian submanifold of its real locus. From the exact sequence of relative homology for (V, Lg) , we have

$$(4) \quad \begin{aligned} H_6(V; \mathbf{Z}) &\xrightarrow{\sim} H_6(V, Lg; \mathbf{Z}), \\ 0 \rightarrow H_4(V; \mathbf{Z}) &\rightarrow H_4(V, Lg; \mathbf{Z}) \rightarrow H_3(Lg; \mathbf{Z}) \rightarrow 0, \\ 0 \rightarrow H_2(V; \mathbf{Z}) &\rightarrow H_2(V, Lg; \mathbf{Z}) \rightarrow H_1(Lg; \mathbf{Z}) \rightarrow 0, \\ H_0(V; \mathbf{Z}) &\xrightarrow{\sim} H_0(V, Lg; \mathbf{Z}). \end{aligned}$$

Let $H' = H_{\bullet}(V)$, $H = H_{\bullet}(V, Lg)$ and $H'' = H_{\bullet}(Lg)$, and let

$$H_{\mathrm{even}}(V) := \bigoplus_{0 \leq p \leq 3} (H')_{2p}, \quad H_{\mathrm{even}}(V, Lg) := \bigoplus_{0 \leq p \leq 3} H_{2p}, \quad H_{\mathrm{odd}}(Lg) := \bigoplus_{0 \leq p \leq 1} (H'')_{2p+1}.$$

Then we have an exact sequence

$$(5) \quad 0 \rightarrow H_{\mathrm{even}}(V) \rightarrow H_{\mathrm{even}}(V, Lg) \rightarrow H_{\mathrm{odd}}(Lg) \rightarrow 0.$$

The weight filtration W is given by $W_3 H_{\mathrm{even}}(V, Lg) := H_{\mathrm{even}}(V)$, $W_4 H_{\mathrm{even}}(V, Lg) := H_{\mathrm{even}}(V, Lg)$, and the relative monodromy weight filtration $M = M(N, W)$ is given by $M_{2p} H_{\mathrm{even}}(V, Lg) = H_{\leq 2p}(V, Lg)$ ($0 \leq p \leq 3$).

In the above setting, the projection from $\mathbf{P}^4(\mathbf{R})$ to the real hyperplane $\{x_5 = 0\} = \mathbf{P}^3(\mathbf{R})$ with center $(0, 0, 0, 0, 1)$ induces a homeomorphism $Lg \simeq \mathbf{P}^3(\mathbf{R})$. Therefore there are two choices of flat $U(1)$ connections on Lg . Denote Lg endowed with these

structures by $Lg_{\pm}$. Morrison-Walcher [MW09, 3] explain the relation between $Lg_{\pm}$ for A-model of V and $C_{\pm}$ for B-model of V° .

After pulling back to the double cover $z^{1/2} \mapsto z$ ($z \neq 0$) and extending over $S^{\log}$, the sequence for A-model (5) and the sequence for B-model (2), and the set of sequences for A-model (4) and the set of sequences for B-model (3), respectively, seem to correspond in mirror symmetry. By Poincaré duality isomorphisms, $H^{\text{even}}(V) = H_{\text{even}}(V)(-3)$ and $H^{\text{even}}(Lg) \simeq H_{\text{odd}}(Lg)$.

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