

END-POINT MAXIMAL L^1 REGULARITY FOR A CAUCHY PROBLEM TO PARABOLIC EQUATIONS

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1. INTRODUCTION

In this summary, we consider maximal L^1 -regularity of the Cauchy problem for parabolic equations in the non-reflexive homogeneous Besov space.

Let X be a Banach space and A be a closed linear operator in X with a densely defined domain $\mathcal{D}(A)$. Given $f \in L^\rho(0, T; X)$ ($1 < \rho < \infty$), we consider the abstract Cauchy problem with $0 < t < T \leq \infty$:

$$\begin{cases} \frac{d}{dt}u + Au = f, & t > 0, \\ u(0) = 0, & t = 0 \end{cases} \quad (1.1)$$

Then it is called that A has maximal L^ρ regularity if there exists a unique solution $u \in W^{1,\rho}(0, T; X) \cap L^\rho(0, T; \mathcal{D}(A))$ to the abstract parabolic equation (1.1) and satisfies the estimate

$$\left\| \frac{d}{dt}u \right\|_{L^\rho(0, T; X)} + \|Au\|_{L^\rho(0, T; X)} \leq C\|f\|_{L^\rho(0, T; X)}, \quad (1.2)$$

where C is a positive constant independent of f . In a general theory, maximal regularity is well established for any Banach space X that satisfies “Unconditional Martingale Difference” (called as UMD). See for the details [2], [4], [8], [13], [14], [15], [20], [21], [26]. On the other hand, maximal regularity on non-UMD Banach spaces, for instance non-reflexive Banach space such as L^1 or L^∞ -like spaces, requires a different way to show it. When we consider the Cauchy problem for the linear parabolic equation the estimate for maximal regularity (1.2) reflects directly full regularity of the solution. Let u solve the Cauchy problem

$$\begin{cases} \partial_t u - \mathcal{L}_2 u = f, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where the operator $\mathcal{L}_2$ denotes the uniformly elliptic operator of second order, ∂_t denotes the partial derivative by t and u_0 and f are given initial and external data. Then general theory is stated avoiding the end point spaces such as L^1 or L^∞ in both space and time variables. In the case of $\mathcal{L}_2 = \Delta$, we explicitly proved maximal regularity on the homogeneous Banach spaces [22], [23]. To state the result precisely, we first recall the definition

of the Besov space. Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity satisfying that

$$\sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1$$

for all $\xi \neq 0$, where $\hat{\phi}$ is the Fourier transform of ϕ and $\text{supp } \hat{\phi}_j \subset \{\xi \in \mathbb{R}^n \mid 2^{j-1} < |\xi| < 2^{j+1}\}$. For $s \in \mathbb{R}$ and $1 \leq p, \sigma \leq \infty$, we define the homogeneous Besov space $\dot{B}_{p,\sigma}^s(\mathbb{R}^n)$ by

$$\dot{B}_{p,\sigma}^s(\mathbb{R}^n) = \{f \in \mathcal{S}^*/\mathcal{P}; \|f\|_{\dot{B}_{p,\sigma}^s} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}_{p,\sigma}^s} \equiv \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{js\sigma} \|\phi_j * f\|_p^\sigma \right)^{1/\sigma}, & 1 \leq \sigma < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\phi_j * f\|_p, & \sigma = \infty \end{cases}$$

and $\mathcal{P}$ denotes all polynomials. We also introduce the inhomogeneous Besov spaces $B_{p,\sigma}^s(\mathbb{R}^n)$ by

$$B_{p,\sigma}^s(\mathbb{R}^n) = \{f \in \mathcal{S}^*; \|f\|_{B_{p,\sigma}^s} < \infty\}$$

with the norm

$$\|f\|_{B_{p,\sigma}^s} \equiv \begin{cases} \left(\|\psi * f\|_p^\sigma + \sum_{j \geq 0} 2^{js\sigma} \|\phi_j * f\|_p^\sigma \right)^{1/\sigma}, & 1 \leq \sigma < \infty, \\ \|\psi * f\|_p + \sup_{j \geq 0} 2^{js} \|\phi_j * f\|_p, & \sigma = \infty \end{cases}$$

where ψ is a smooth cut off function with

$$\psi(\xi) + \sum_{j \geq 0} \hat{\phi}_j(\xi) \equiv 1$$

for all $\xi \in \mathbb{R}^n$ (cf. [5], [6], [25]).

One of a general result in the Besov spaces can be seen in [23]:

Proposition 1.1 (endpoint maximal regularity). *Let $\mathcal{L}_2 = \Delta$, $1 < \rho, \sigma \leq \infty$ and $I = [0, T)$ be an interval with $T \leq \infty$. For $f \in L^\rho(I; \dot{B}_{1,\rho}^0(\mathbb{R}^n))$ and $u_0 \in \dot{B}_{1,\rho}^{2(1-1/\rho)}(\mathbb{R}^n)$, let u be a solution of the Cauchy problem of the heat equation (1.3). Then there exists a constant $C_M > 0$ such that*

$$\|\partial_t u\|_{L^\rho(I; \dot{B}_{1,\rho}^0)} + \|\nabla^2 u\|_{L^\rho(I; \dot{B}_{1,\rho}^0)} \leq C_M \left(\|u_0\|_{\dot{B}_{1,\rho}^{2(1-1/\rho)}} + \|f\|_{L^\rho(I; \dot{B}_{1,\rho}^0)} \right).$$

Proposition 1.1 does not cover the end-point case $\rho = 1$, partially because the argument in the proof in [23] involves a duality structure and it is not clear if maximal L^1 -regularity holds by applying the method utilized there. On the other hand, Danchin [10], [11] (see also Haspot [17]) obtained maximal regularity in the homogeneous Besov space for the case $\rho = 1$. In this paper, we reconsider maximal L^1 -regularity in the Besov space and its optimality in the homogeneous Besov spaces.

2. RESULTS FOR A CONSTANT COEFFICIENT CASE

Our main statement for the Cauchy problem for the heat equation (1.3) is the following:

Theorem 2.1 (optimal maximal L^1 regularity). *Let $\mathcal{L}_2 = \Delta$, $1 \leq p \leq \infty$. For $f \in L^1(\mathbb{R}_+; \dot{B}_{p,1}^0(\mathbb{R}^n))$ and $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$ there exists a unique solution u to (1.3) which satisfies the estimate: There exists a positive constant $C_M > 0$ only depending on n, p such that*

$$\|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} \leq C_M \left(\|u_0\|_{\dot{B}_{p,1}^0} + \|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} \right). \quad (2.1)$$

Besides if $f \equiv 0$, then the regularity condition for the initial data is optimal. Namely there exists a constant $C_m = C_m(n, p) > 0$ such that for all $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$

$$C_m \|u_0\|_{\dot{B}_{p,1}^0} \leq \|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)}. \quad (2.2)$$

The upper estimate of (2.1) was obtained by Danchin [9], [10], [11] and Haspot [17] with $1 < p < \infty$ (see also Danchin-Mucha [12]). However our method to obtaining the estimates (2.1) seems very different from those existing arguments. In fact, our method admits the fractional order elliptic operator such as $\mathcal{L}_\alpha = (-\Delta)^{\alpha/2}$ for $\alpha > 0$ and an analogous estimate in Theorem 2.1 also holds. We state this version precisely in below (Theorem 2.9).

If we replace $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$ into $u_0 \in \dot{B}_{p,\sigma}^0(\mathbb{R}^n)$ or $\dot{F}_{p,\sigma}^0(\mathbb{R}^n)$ for $1 < \sigma \leq \infty$, then maximal regularity in $L^1(\mathbb{R}_+; \dot{B}_{p,\sigma}^0(\mathbb{R}^n))$ or $L^1(\mathbb{R}_+; \dot{F}_{p,\sigma}^0(\mathbb{R}^n))$ fails since the lower bound by the initial data and the strict inclusion result for the sub-suffix σ such as $\dot{B}_{p,1}^0(\mathbb{R}^n) \subsetneq \dot{B}_{p,\sigma}^0(\mathbb{R}^n)$. In particular the estimate in $L^1(\mathbb{R}_+; L^p(\mathbb{R}^n))$;

$$\int_0^\infty \|\Delta e^{t\Delta} u_0\|_p dt \leq C \|u_0\|_p \quad (2.3)$$

generally fails. If $1 < p \leq 2$, then $\dot{B}_{p,1}^0 \subsetneq L^p = \dot{F}_{p,2}^0 \subset \dot{B}_{p,2}^0$, and if $2 \leq p < \infty$ then $\dot{B}_{p,1}^0 \subsetneq \dot{B}_{p,2}^0 \subset \dot{F}_{p,2}^0 = L^p$ so that the estimate (2.3) contradicts the result (2.2) for general data u_0 . The equivalence between the homogeneous Besov norm and the expression of the heat kernel is also pointed out in Bahouri-Chemin-Danchin [3] by the following form:

$$\int_0^\infty \|e^{t\Delta} u_0\|_p dt \simeq \|u_0\|_{\dot{B}_{p,1}^{-2}}.$$

See for the application of this expression to the initial boundary value problem for the incompressible Navier-Stokes equation, Cannone-Planchon-Schonbek [7].

Giga-Saal [16], proved maximal L^1 -regularity over the class of Fourier transformed finite Radon measures $\mathcal{FM}(\mathbb{R}^n)$. Let $\mathcal{M}(\mathbb{R}^n)$ be a class of signed finite Radon measures and let

$$\mathcal{FM}(\mathbb{R}^n) \equiv \{f = \hat{\mu}, \mu \in \mathcal{M}(\mathbb{R}^n)\}$$

with the norm $\|f\|_{\mathcal{FM}} \equiv \|\mu\|_{\mathcal{M}}$, where $\|\mu\|_{\mathcal{M}}$ denotes the total variation of $\mu \in \mathcal{M}(\mathbb{R}^n)$.

Proposition 2.2 (Giga-Saal). *Let u be a solution to the Cauchy problem of the heat equation (1.3) with $\mathcal{L}_2 = \Delta$. Then there exists a constant $C > 0$ such that Then for*

$u_0 \in \mathcal{FM}(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}_+; \mathcal{FM}(\mathbb{R}^n))$ maximal L^1 -regularity for the heat equation holds:

$$\|\partial_t u\|_{L^1(I; \mathcal{FM})} + \|\nabla^2 u\|_{L^1(I; \mathcal{FM})} \leq C_M(\|u_0\|_{\mathcal{FM}} + \|f\|_{L^1(I; \mathcal{FM})}). \quad (2.4)$$

They applied this estimate for solving the Cauchy problem of the incompressible Navier-Stokes equations with the Coriolis force. Our result is a version of improvement of the Giga-Saal estimate (2.4) since the following embedding holds.

$$\mathcal{FM}(\mathbb{R}^n)/\{\text{constant}\} \hookrightarrow \dot{B}_{\infty,1}^0(\mathbb{R}^n).$$

In particular the embedding is continuous. For the case initial data is constant, then maximal regularity is trivial. If $f = 1$ and $u_0 = 0$ then $u(t, x) = t$ is a unique solution and again maximal regularity holds in $\mathcal{FM}$. The homogeneous Besov space can not include this case however the estimate itself is trivial.

As a corollary of Theorem 2.1, we obtain the lower estimate for $f \neq 0$ case.

Corollary 2.3. *Let $\mathcal{L}_2 = \Delta$, $1 \leq p \leq \infty$ and the constants C_M and C_m represents the upper bound of (2.1) and the lower bound of (2.2), respectively. If $u_0 \in \dot{B}_{p,1}^0$ and $f \in L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)$ satisfy*

$$C_M \|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} < C_m \|u_0\|_{\dot{B}_{p,1}^0}$$

or

$$C_M \|u_0\|_{\dot{B}_{p,1}^0} < \|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)},$$

then there exists a constant $C(n, p) > 0$ such that the solution to the heat equation (1.3) satisfies

$$C(\|u_0\|_{\dot{B}_{p,1}^0} + \|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)}) \leq \|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)}.$$

For the case $u_0 = 0$, the lower estimate holds for the sum of the norm for $\partial_t u$ and $\nabla^2 u$ as

$$\|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} \leq \|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)}.$$

On the other hand, for the case that $f = 0$, the lower estimate (2.2) holds for the each term of the right-hand side as

$$\begin{aligned} C^{-1} \|u_0\|_{\dot{B}_{p,1}^0(\mathbb{R}^n)} &\leq \|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0(\mathbb{R}^n))}, \\ C^{-1} \|u_0\|_{\dot{B}_{p,1}^0(\mathbb{R}^n)} &\leq \|\nabla^2 u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0(\mathbb{R}^n))}, \end{aligned}$$

which are derived from the following proposition.

Proposition 2.4. *For $1 \leq p \leq \infty$, let $u_0 \in \dot{B}_{p,1}^0$.*

(1) *Then there exists a constant $C > 0$ such that for any $k \in \mathbb{Z}$ it holds*

$$C^{-1} \sum_{\ell \leq k} \|\phi_\ell * u_0\|_p \leq \sum_{\ell \leq k} \int_{2^{-2\ell}}^{2^{-2\ell+2}} \|\Delta e^{s\Delta} u_0\|_p ds \leq C \sum_{j \in \mathbb{Z}} \min(1, e^{-2^{j-k}}) \|\phi_j * u_0\|_p. \quad (2.5)$$

(2) For $I = [0, T]$, there exists an integer $\tilde{\ell} = \left\lceil -\frac{\log T}{2 \log 2} \right\rceil$ and a constant $C \geq \tilde{C} > 0$ only depending on n, p and $\|\phi\|_1$ such that

$$\tilde{C} \sum_{j \geq \tilde{\ell}} \|\phi_j * u_0\|_p \leq \int_0^T \|\Delta e^{s\Delta} u_0\|_p ds \leq C \sum_{j \in \mathbb{Z}} \min(2^{2(j-\tilde{\ell})}, 1) \|\phi_j * u_0\|_p. \quad (2.6)$$

When we consider a time local problem to (1.3), then the initial data can be chosen in the inhomogeneous Besov space $B_{p,1}^0$. Indeed, we have the following:

Theorem 2.5. *Let $1 \leq p \leq \infty$ and for $T < \infty$ let $I = [0, T)$. For $u_0 \in B_{p,1}^0$, there exists $C_0 > 0$ and $C_T > 0$*

$$C_0 \|u_0\|_{B_{p,1}^0} \leq \int_0^T \|\Delta e^{s\Delta} u_0\|_p ds \leq C_T \|u_0\|_{B_{p,1}^0},$$

where $C_0 \simeq C_T = O(\log T)$. In particular maximal L^1 regularity in the local interval holds for $I = [0, T)$. For the solution of the heat equation (1.3), there exists a constant $C_T > 0$ such that

$$\|\partial_t u\|_{L^1(I; B_{p,1}^0)} + \|\nabla^2 u\|_{L^1(I; B_{p,1}^0)} \leq C_T \left(\|u_0\|_{B_{p,1}^0} + \|f\|_{L^1(I; B_{p,1}^0)} \right), \quad (2.7)$$

where $C_T = O(\log T)$ as $T \rightarrow \infty$. The estimate can be uniform in T if we exchange into the homogeneous Besov space $\dot{B}_{p,1}^0$.

Now we shall show the results for the Cauchy problem of the heat equation with constant coefficients in a slightly general setting. We consider the Cauchy problem of the parabolic equation with the fractional Laplacian $\mathcal{L}_\alpha = -(-\Delta)^{\alpha/2}$ with $\alpha > 0$:

$$\begin{cases} \partial_t u - \mathcal{L}_\alpha u = f, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (2.8)$$

Theorem 2.6 (optimal maximal L^1 regularity). *Let $\alpha > 0$ and $1 \leq p \leq \infty$. For $f \in L^1(\mathbb{R}_+; \dot{B}_{p,1}^0(\mathbb{R}^n))$ and $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$ there exists a unique solution u to (2.8) which satisfies the estimate: There exists a positive constant $C_M > 0$ only depending on α, n, p such that*

$$\|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} + \|\mathcal{L}_\alpha u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} \leq C_M \left(\|u_0\|_{\dot{B}_{p,1}^0} + \|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} \right). \quad (2.9)$$

Besides if $f \equiv 0$, then the regularity condition for the initial data is optimal. Namely there exists a constant $C_m = C_m(n, p) > 0$ such that for all $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$

$$C_m \|u_0\|_{\dot{B}_{p,1}^0} \leq \|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)} + \|\mathcal{L}_\alpha u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^0)}. \quad (2.10)$$

Theorem 2.1 is a direct consequence from Theorem 2.6 with $\alpha = 2$ and the boundedness of the singular integral operator from $\dot{B}_{p,1}^0$ to itself. This general form has some applications. See for instance Iwabuchi [18].

3. RESULTS FOR A VARIABLE COEFFICIENT CASE

We consider the case where a coefficient is variable.

$$\begin{cases} \partial_t u - a(t, x) \Delta u = f, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (3.1)$$

We assume that $a(t, x)$ satisfies the following:

- (1) $a(t, x) = 1 + b(t, x)$,
- (2) there exists $\underline{b} > -1$ s.t. $b(t, x) \geq \underline{b}$ a.e x ,
- (3) $b \in L^\infty(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n)) \cap C(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n))$ for $1 \leq q < \infty$.

Theorem 3.1. *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and a variable coefficients $a(t, x)$ satisfies the assumption (1), (2), (3). For $T > 0$ we set $I = [0, T]$ and $\underline{\nu} := \inf_{t \in I, x \in \mathbb{R}^n} (1 + b(t, x))$. For $b \in L^\infty(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n)) \cap C(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n))$, $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$ and $f \in L^1(0, T; \dot{B}_{p,1}^0(\mathbb{R}^n))$, there exists $C_M > 0$ the solution u to (3.1) satisfies the estimate:*

$$\begin{aligned} & \|\partial_t u\|_{L^1(0,T;\dot{B}_{p,1}^0)} + \underline{\nu} \|\nabla^2 u\|_{L^1(0,T;\dot{B}_{p,1}^0)} \\ & \leq C_M \left\{ 1 + \|b\|_{L^\infty(I;\dot{B}_{q,1}^{n/q})} \exp \left(\mu T (1 + \|b\|_{L^\infty(I;\dot{B}_{q,1}^{n/q})})^2 \right) \right\} \|u_0\|_{\dot{B}_{p,1}^0} \\ & \quad + C_M \int_0^T \exp \left(\mu \int_s^T (1 + \|b(r)\|_{\dot{B}_{q,1}^{n/q}})^2 dr \right) \|f(s)\|_{\dot{B}_{p,1}^0} ds, \end{aligned}$$

where $\mu = (CC_1 \underline{\nu})^2 \log(1 + C_M)$.

Theorem 3.2. *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and a variable coefficients $a(t, x)$ satisfies the assumption (1), (2), (3). For $I = [0, T]$, we set $k = \lfloor -\frac{\log T}{2 \log 2} \rfloor$. For $b \in L^\infty(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n)) \cap C(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n))$, $u_0 \in \dot{B}_{p,1}^0(\mathbb{R}^n)$, (3.1) with $f \equiv 0$ admits a unique solution u which satisfies*

$$\frac{C}{(1 + \|b\|_{L^\infty(I;\dot{B}_{q,1}^{n/q})})} \sum_{\ell \geq k} \|\phi_\ell * u_0\|_p \leq \left(\|\partial_t u\|_{L^1(I;\dot{B}_{p,1}^0)} + \|\nabla^2 u\|_{L^1(I;\dot{B}_{p,1}^0)} \right).$$

Theorem 3.2 shows that for $b \in L^\infty(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n)) \cap C(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n))$, the class $\dot{B}_{p,1}^0(\mathbb{R}^n)$ of u_0 could not be replaced by $L^p(\mathbb{R}^n)$, $\dot{B}_{p,\sigma}^0(\mathbb{R}^n)$, $\dot{F}_{p,\sigma}^0(\mathbb{R}^n)$ ($1 < \sigma \leq \infty$) for maximal L^1 -regularity.

Danchin [9] and Haspot [17] obtained an analogous estimate for the variable coefficient case by an elegant usage of L^p type energy estimate and the Chemin-Laners spaces. In this case, the Chemin-Laners space coincides with the Bochner space as

$$\widetilde{L^1(I; \dot{B}_{p,1}^0)} \equiv \ell^1(\{L^1(I; L_j^p)\}_{j \in \mathbb{Z}}) = L^1(I; \dot{B}_{p,1}^0),$$

thanks to the fact that the time L^1 norm and Littlewood-Paley sequence ℓ^1 norm can be interchanged, where L_j^p denotes the Littlewood-Paley decomposed L^p space given by $\|f\|_{L_j^p} \equiv \|\phi_j * f\|_p$. As in the constant coefficient case, our method is very much different from theirs. We use the estimate for the constant coefficient case (Theorem 2.1) and employ a freezing argument in space-time variables and then time variable to obtain the above result for variable coefficient. Our theorems Theorem 3.1 and 3.2 can be generalized

for more general parabolic type equation with a second order uniformly elliptic operator $\mathcal{L}$:

(1) a parabolic system

$$\begin{cases} \partial_t u - \sum_{i,j=1}^n a_{ij}(t, x) \partial_i \partial_j u = f, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $a_{ij}(t, x)$ satisfies

(a) $a_{ij}(t, x) \in L^\infty(0, T; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n)) \cap C(I; \dot{B}_{q,1}^{n/q}(\mathbb{R}^n))$, $1 \leq p, q \leq \infty$,

(b) $a_{ij}(t, x) = \delta_{ij} + b_{ij}(t, x)$, $1 \leq i, j \leq \infty$,

(c) $b_{ij}(t, x) = b_{ji}(t, x)$, $1 \leq i, j \leq \infty$,

(d) there exists $\lambda \geq 0$ such that $\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$ for all $\xi \in \mathbb{R}^n$.

(2) the vector valued system such as the Stokes equation or the Lamé equation:

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi = f, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

$$\begin{cases} \partial_t u - (\mu + \lambda) \Delta u + \lambda \nabla (\operatorname{div} u) = f, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

To treat the variable coefficients, we remark that the estimate in the Besov space such as

$$\|af\|_{\dot{B}_{p,1}^0} \leq C \|a\|_\infty \|f\|_{\dot{B}_{p,1}^0}$$

fails in general. This is the reason why we adapt the space $\dot{B}_{q,1}^{n/q}(\mathbb{R}^n)$ for the variable coefficient which plays a role instead of L^∞ space.

Proposition 3.3. *Let $1 \leq p \leq \infty$ and $1 \leq q < \infty$. For $f \in \dot{B}_{q,1}^{\frac{n}{q}}$ and $g \in \dot{B}_{p,1}^0$ there exists $C > 0$ such that*

$$\|fg\|_{\dot{B}_{p,1}^0} \leq C \|f\|_{\dot{B}_{q,1}^{\frac{n}{q}}} \|g\|_{\dot{B}_{p,1}^0}. \quad (3.2)$$

For the proof, we refer to Abidi-Paicu [1].

The space $\dot{B}_{q,1}^{n/q}(\mathbb{R}^n)$ has nice embedding property. Let

$$C_v(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) \mid |f(x)| \rightarrow 0 \text{ as } |x| \rightarrow \infty\}.$$

Proposition 3.4. *Let $1 \leq q < \infty$ and $\mathcal{S}(\mathbb{R}^n)$ be the rapidly decreasing smooth functions. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{B}_{q,1}^{n/q}(\mathbb{R}^n) \hookrightarrow C_v(\mathbb{R}^n). \quad (3.3)$$

In particular, the embedding of the left-hand side is dense.

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