数理解析研究所講究録
第 1640 巻 2009 年 67-84

Abstract approach to the Dirac equation

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Abstract

A new existence and uniqueness theorem is established for linear evolution equations in a separable Hilbert space. The result is applied to the Dirac equation with time-dependent potential.

1. Introduction and statement of the result

In this paper we consider the Cauchy problem for the Dirac equation in $L^2(\mathbb{R}^3)^4$:

$$i\frac{\partial u}{\partial t} + H_D u + V(x)u + q(x,t)u = f(x,t),$$

with $u(\cdot,0) = u_0 \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$, where H_D is the free Dirac operator, $H^1(\mathbb{R}^3)$ is the usual Sobolev space and $H_1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3); (1+|x|^2)^{1/2}u \in L^2(\mathbb{R}^3)\}$. We shall show the existence of a unique strong solution under some conditions on potentials V, q and inhomogeneous term f. To do so we employ an abstract approach.

Let $\{A(t); 0 < t < T\}$ be a family of closed linear operators in a *separable* complex Hilbert space X. Then the Dirac equation is regarded as one of linear evolution equations of the form

$$\text{(E)} \qquad \frac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \text{on} \quad (0,T).$$

So we first establish the existence of a unique strong solution to the Cauchy problem of (E) with initial condition. Now let S be a selfadjoint operator in X, satisfying

$$\text{(1.1)} \qquad (u, Su) \geq \|u\|^2 \ \text{ for } u \in D(S).$$

Then the square root $S^{1/2}$ is well-defined and $Y := D(S^{1/2})$ is also a separable Hilbert space, with inner product $(u,v)_Y := (S^{1/2}u, S^{1/2}v)$, embedded continuously and densely in X.

Let $B(Y,X)$ be the space of all bounded linear operators on a Banach space Y to another X, with norm $\|\cdot\|_{Y\to X}$. We shall also use the following abbreviation. Namely, $B(X) := B(X,X)$ and $B(Y) := B(Y,Y)$. We use the subscript $_*$ to refer the strong operator topology in $B(Y,X)$. For instance, $F(\cdot) \in L^p_*(0,T,B(Y,X))$ for $1 \leq p \leq \infty$ means that $F(t) \in B(Y,X)$ is defined for a.a. $t \in (0,T)$, is strongly measurable, and there exists $\gamma_F \in L^p(0,T)$ such that $\|F(t)\|_{Y\to X} \leq \gamma_F(t)$ for a.a. $t \in (0,T)$ (for this notation see Kato [**8**] and Tanaka [**16**]).

The first purpose of this paper is to prove

Theorem 1.1. *Let $\{A(t)\}$ be a family of closed linear operators in a separable Hilbert space X, S a selfadjoint operator in X, satisfying* (1.1). *Assume that $A(t)$ satisfies following four conditions.*

(I) *There exists $\alpha \in L^1(0,T)$, $\alpha \geq 0$, such that*

$$|\operatorname{Re}(A(t)v, v)| \leq \alpha(t)\,\|v\|^2, \quad v \in D(A(t)), \text{ a.a. } t \in (0,T). \tag{1.2}$$

(II) $Y = D(S^{1/2}) \subset D(A(t))$, a.a. $t \in (0,T)$.

(III) *There exists $\beta \in L^1(0,T)$, $\beta \geq \alpha$, such that*

$$|\operatorname{Re}(A(t)u, Su)| \leq \beta(t)\,\left\|S^{1/2}u\right\|^2, \quad u \in D(S), \text{ a.a. } t \in (0,T). \tag{1.3}$$

(IV) $A(\cdot) \in L^1_*(0,T; B(Y,X))$, *i.e., there exists $\gamma \in L^1(0,T)$ such that*

$$\|A(t)\|_{Y\to X} \leq \gamma(t), \quad \text{a.a. } t \in (0,T). \tag{1.4}$$

Then there exists a unique evolution operator $\{U(t,s); (t,s) \in \Delta\}$, where $\Delta := \{(t,s); 0 \leq s \leq t \leq T\}$, having the following properties.

(i) *$U(\cdot,\cdot)$ is strongly continuous on Δ to $B(X)$, with*

$$\|U(t,s)\|_{B(X)} \leq \exp\Bigl(\int_s^t \alpha(r)\,dr\Bigr), \quad (t,s) \in \Delta. \tag{1.5}$$

(ii) *$U(t,r)U(r,s) = U(t,s)$ on Δ and $U(s,s) = 1$ (the identity).*

(iii) *$U(t,s)Y \subset Y$ and $U(\cdot,\cdot)$ is strongly continuous on Δ to $B(Y)$, with*

$$\|U(t,s)\|_{B(Y)} \leq \exp\Bigl(\int_s^t \beta(r)\,dr\Bigr), \quad (t,s) \in \Delta. \tag{1.6}$$

Furthermore, let $v \in Y$, Then $U(\cdot,\cdot)v \in W^{1,1}(\Delta; X)$, with

(iv) $(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v$, $\quad (t,s) \in \Delta$, a.a. $t \in (s,T)$, *and*

(v) $(\partial/\partial s)U(t,s)v = U(t,s)A(s)v$, $\quad (t,s) \in \Delta$, a.a. $s \in (0,t)$.

In particular, if $A(\cdot) \in C([0,T]; B(Y,X))$, then Theorem 1.1 has already been proved in Mori [**9**] (unpublished). For lack of the continuity to the contrary we cannot approximate the family $\{A(\cdot)\}$ by a sequence $\{A_n(\cdot)\}$ of piecewise constant families. Therefore, we should consider some other approximation (see Definition 2.2 below).

Here we note that (III) is a consequence of conditions (I), (II) and the commutator type condition

(K) There exists $B(\cdot) \in L^1_*(0,T; B(X))$ such that

$$S^{1/2}A(t)S^{-1/2} = A(t) + B(t), \quad \text{a.a. } t \in (0,T),$$

in which the domain relation is exact. Under condition (K) and the so-called stability condition, a similar theorem as in Theorem 1.1 was first established by Kato [**4**] and [**5**].

Under conditions (I)–(III) with $t = t_0$ fixed both $\alpha(t_0) \pm A(t_0)$ become m-accretive in X (see Lemma 2.1). Thus $A(t_0)$ together with $-A(t_0)$ is not in general the negative generator of an analytic C_0-semigroup on X. That is, (E) is definitely an equation of hyperbolic type. In other words, "hyperbolic" may be replaced with "non-parabolic".

In order to state the main theorem we need the notion of a strong solution. We say that $u(\cdot)$ is a *strong* solution of (E) if

(i) $u(\cdot) \in W^{1,1}(0,T;X)$,

(ii) $u(t) \in Y$ $(0 \le t \le T)$, and

(iii) $u(\cdot)$ satisfies (E) almost everywhere.

Note that $A(t)u(t)$ is meaningful. Under this definition we have

Theorem 1.2. *Let $u_0 \in Y$ and $f(\cdot) \in L^1(0,T;Y)$. If $u(\cdot)$ is defined by*

$$u(t) := U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds,$$

then $u(\cdot) \in W^{1,1}(0,T;X) \cap C([0,T];Y)$ and $u(\cdot)$ is a unique strong solution of (E) *with $u(0) = u_0$.*

In Section 2 we prepare some lemmas. Then we shall prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. In Section 5 we show the selfadjointness of some operators for applications. Last, in Section 6 we apply Theorem 1.1 to the Dirac equation.

2. Preliminaries

Let X be a separable Hilbert space.

Lemma 2.1. *Let A be a closed linear operator in X, satisfying*

$$\operatorname{Re}(Av,v) \ge -\alpha\|v\|^2, \quad v \in D(A),$$

where $\alpha \ge 0$ is a constant. Let S be a selfadjoint operator in X, with $D(S) \subset D(A)$, satisfying (1.1). *Assume that there exist nonnegative constants β and γ such that for all $u \in D(S)$,*

$$\operatorname{Re}(Au,Su) \ge -\gamma\|u\|^2 - \beta\|u\|\cdot\|Su\|.$$

Then

(a) *$A + \alpha$ is m-accretive in X.*

(b) *$D(S)$ is a core for A.*

This lemma was obtained by Kato [**6**]. For a complete proof see Okazawa [**11**].

Definition 2.2 (Ishii [**3**]). Let $\{A(t)\}$ be a family as above, satisfying (1.2)–(1.4). Put

$$A_n(t) := A(t)\Big(1 + \frac{1}{\nu_n(t)}A(t)\Big)^{-1} = \nu_n(t)\Big[1 - \Big(1 + \frac{1}{\nu_n(t)}A(t)\Big)^{-1}\Big],$$
$$\nu_n(t) := n\big(1+\gamma(t)\big) + 2\beta(t), \quad n \in \mathbb{N}, \quad \text{a.a. } t \in (0,T).$$

Then $\{A_n(t)\}_{n\in\mathbb{N}}$ is called a *modified Yosida approximation* of $\{A(t)\}$.

If $t_0 \in (0,T)$ is fixed, then $\alpha(t_0)$, $\beta(t_0)$, $\gamma(t_0)$ and $A_n(t_0)$ are considered as nonnegative constants α, β, γ and the usual Yosida approximation of $A(t_0)$ (provided $\nu_n(t_0) > 2\beta$), respectively. Therefore the following lemmas are proved in the same way as in [**12**].

Lemma 2.3. *Let $A(t)$ be as in* Definition 2.2. *Then*

(a) $\left\|\left(1+\frac{1}{\nu_n(t)}A(t)\right)^{-1}\right\|_{B(X)} \le \left(1-\frac{\alpha(t)}{\nu_n(t)}\right)^{-1}, \quad n \in \mathbb{N}, \quad \text{a.a. } t \in (0,T).$

(b) $\operatorname{Re}(A_n(t)w, w) \ge -\alpha(t)\left(1-\frac{\alpha(t)}{\nu_n(t)}\right)^{-1}\|w\|^2, \quad w \in X, \quad \text{a.a. } t \in (0,T).$

(c) $\|A_n(t)\|_{B(X)} \le \nu_n(t), \quad n \in \mathbb{N}, \quad \text{a.a. } t \in (0,T).$

Lemma 2.4. *Let $A(t)$ be as in* Lemma 2.3. *Assume that there exist $\beta \in L^1(0,T)$ and $\gamma \in \mathbb{R}$ such that $\beta \ge \alpha \ge 0$ and*

$$\operatorname{Re}(A(t)u, Su) \ge -\gamma\,\|u\|^2 - \beta(t)(u, Su) \quad \forall\, u \in D(S), \quad \text{a.a. } t \in (0,T), \tag{2.1}$$

where S is a selfadjoint operator in X satisfying (1.1). *Then, for $S_\varepsilon := S(1+\varepsilon S)^{-1}$,*

$$\operatorname{Re}(A(t)u, S_\varepsilon u) \ge -\gamma\,\|u\|^2 - \beta(t)(u, S_\varepsilon u) \quad \forall\, u \in D(A(t)), \quad \text{a.a. } t \in (0,T).$$

Lemma 2.5. *Let $A(\cdot)$ and S be as in* Lemma 2.4. *Assume that* (2.1) *with $\gamma = 0$ is satisfied. Then*

(a) $\left(1+\frac{1}{\nu_n(t)}A(t)\right)^{-1}D(S^{1/2}) \subset D(S^{1/2}), \quad \text{a.a. } t \in (0,T),$ *with*

$$\left\|S^{1/2}\left(1+\frac{1}{\nu_n(t)}A(t)\right)^{-1}v\right\| \le \left(1-\frac{\beta(t)}{\nu_n(t)}\right)^{-1}\|S^{1/2}v\|, \quad v \in D(S^{1/2}), \quad \text{a.a. } t \in (0,T).$$

(b) $\operatorname{Re}(A_n(t)w, S_\varepsilon w) \ge -\beta(t)\left(1-\frac{\beta(t)}{\nu_n(t)}\right)^{-1}(w, S_\varepsilon w), \quad w \in X, \quad \text{a.a. } t \in (0,T).$

Lemma 2.6. *Let $\{A_n\}$ be the Yosida approximation of a linear m-accretive operator A in X. Let $\{w_n\}$ be a sequence in X such that $w_n \to u$ $(n \to \infty)$ weakly in X. If $\{A_n w_n\}$ is bounded, then $u \in D(A)$ and $A_n w_n \to Au$ $(n \to \infty)$ weakly in X.*

3. Construction of evolution operators

In this section we shall prove Theorem 1.1. Let $\{A(t)\}$ be a family of closed linear operators in a separable Hilbert space X. Let S be a selfadjoint operator in X, satisfying (1.1). Since we need conditions (I) and (III) as a whole only in the last step of the proof (see Lemmas 3.9 and 3.11 below), we may introduce weaker conditions $(\mathrm{I})_+$ and $(\mathrm{III})_+$. Namely assume that

$(\mathrm{I})_+$ There exists $\alpha \in L^1(0,T)$, $\alpha \ge 0$ such that

$$\operatorname{Re}(A(t)v, v) \ge -\alpha(t)\,\|v\|^2, \quad v \in D(A(t)), \text{ a.a. } t \in (0,T).$$

(II) $Y = D(S^{1/2}) \subset D(A(t))$, a.a. $t \in (0,T)$.

(III)$_+$ There exists $\beta \in L^1(0,T)$, $\beta \geq \alpha$ such that

$$\mathrm{Re}(A(t)u, Su) \geq -\beta(t) \left\| S^{1/2}u \right\|^2, \quad u \in D(S), \text{ a.a. } t \in (0,T).$$

(IV) $A(\cdot) \in L^1_*(0,T; B(Y,X))$ with $\|A(t)\|_{Y\to X} \leq \gamma(t)$, a.a. $t \in (0,T)$.

Under these conditions we shall construct a two parameter family $\{U(t,s); (t,s) \in \Delta\}$ in $B(X)$, satisfying among others (i), (ii), (iv) and (v) of Theorem 1.1.

First of all, by virtue of conditions (I)$_+$, (II) and (III)$_+$ we see from Lemma 2.1 (a) that $A(t) + \alpha(t)$ is m-accretive in X for almost all $t \in (0,T)$.

Lemma 3.1. *Let $\{A_n(t)\}$ and $\{\nu_n(t)\}$ be as in* Definition 2.2. *Then*

(a) $A_n(\cdot) \in L^1_*(0,T; B(X))$ *with* $\|A_n(t)\|_{B(X)} \leq \nu_n(t)$, a.a. $t \in (0,T)$.

(b) $\|A(t)v - A_n(t)v\| \to 0, \quad \forall\, v \in D(A(t))$, a.a. $t \in (0,T)$.

Proof. (a) follows from Lemma 2.3 (c).

(b) is well-known as a property of the Yosida approximation. □

Proposition 3.2. *Let $s \in [0,T)$. Then the approximate problem:*

$$\begin{cases} (d/dt)u_n(t) + A_n(t)u_n(t) = 0, & \text{a.a. } t \in (s,T), \\ u_n(s) = w \end{cases} \tag{3.1}$$

has a unique strong solution $u_n \in W^{1,1}(s,T;X)$.

In particular, if $A_n(\cdot) \in C([0,T]; B(Y,X))$, then the assertion is found in Pazy [**15**, Section 5.1]. The proof is standard (see e.g. Brézis [**1**, Theorem VII.3]).

We define the "solution operator" of the approximate problem by

$$U_n(t,s)w := u_n(t) \quad \text{for } (t,s) \in \Delta$$

where u_n is the solution of (3.1). The main properties of $U_n(t,s)$ are given in the next lemma (cf. [**15**, Section 5.1]).

Lemma 3.3. *For every $n \in \mathbb{N}$, let $\{A_n(t)\}$ and $\{U_n(t,s)\}$ be as defined above. Then $\{U_n(t,s)\}$ is a sequence of bounded linear operators on X, with*

(a) $\|U_n(t,s)\|_{B(X)} \leq \exp\Big(\int_s^t \nu_n(r)\,dr\Big)$ *on* Δ.

(b) $U_n(t,r)U_n(r,s) = U_n(t,s)$ *on* Δ *and* $U_n(s,s) = 1$.

(c) $U_n(\cdot,\cdot)$ *is uniformly continuous on* Δ.

(d) $(\partial/\partial t)U_n(t,s)w = -A_n(t)U_n(t,s)w$, $w \in X$, $(t,s) \in \Delta$, a.a. $t \in (s,T)$.

(e) $(\partial/\partial s)U_n(t,s)w = U_n(t,s)A_n(s)w$, $w \in X$, $(t,s) \in \Delta$, a.a. $s \in (0,t)$.

For the limiting procedure we need the following

Lemma 3.4. *Let* $\{U_n(t,s)\}$ *and* $\nu_n(t)$ *be as in* Lemma 3.3. *Then*

(a) $\|U_n(t,s)\|_{B(X)} \le \exp\Big[\int_s^t \alpha(r)\Big(1-\frac{\alpha(r)}{\nu_n(r)}\Big)^{-1} dr\Big] \le \exp\Big(2\int_s^t \alpha(r)\,dr\Big)$ *on* Δ.

(b) $U_n(t,s)Y \subset Y$ *and*

$$\|U_n(t,s)\|_{B(Y)} \le \exp\Big[\int_s^t \beta(r)\Big(1-\frac{\beta(r)}{\nu_n(r)}\Big)^{-1} dr\Big] \le \exp\Big(2\int_s^t \beta(r)\,dr\Big) \text{ on } \Delta.$$

(c) *For* $v \in Y$, $\|A_n(t)U_n(t,s)v\| \le 2\gamma(t)\exp\Big(2\int_s^t \beta(r)\,dr\Big)\|v\|_Y$, a.a. $(t,s) \in \Delta$.

Proof. First we prove (b). Let $\{S_\varepsilon\}$ be the Yosida approximation of S. Since S_ε is a bounded linear operator on X, we see from Lemma 3.3 (d) and Lemma 2.5 (b) that for $v \in Y$, a.a. $r \in (s,T)$,

$$\begin{aligned} (\partial/\partial r)\left\|S_\varepsilon^{1/2}U_n(r,s)v\right\|^2 &= -2\,\mathrm{Re}(A_n(r)U_n(r,s)v, S_\varepsilon U_n(r,s)v) \\ &\le 2\beta(r)\Big(1-\frac{\beta(r)}{\nu_n(r)}\Big)^{-1}\left\|S_\varepsilon^{1/2}U_n(r,s)v\right\|^2. \end{aligned} \tag{3.2}$$

Integrating this inequality on $[s,t]$. By the Gronwall inequality we have

$$\begin{aligned} \left\|S_\varepsilon^{1/2}U_n(r,s)v\right\|^2 &\le \exp\Big[2\int_s^t \beta(r)\Big(1-\frac{\beta(r)}{\nu_n(r)}\Big)^{-1} dr\Big]\left\|S_\varepsilon^{1/2}v\right\|^2 \\ &\le \exp\Big[2\int_s^t \beta(r)\Big(1-\frac{\beta(r)}{\nu_n(r)}\Big)^{-1} dr\Big]\left\|S^{1/2}v\right\|^2. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we can obtain the first inequality of (b). The second inequality is trivial because $\nu_n(t) \ge 2\beta(t)$ a.a. $t \in (0,T)$.

(a) is proved similarly by Lemma 2.3 (b), starting with

$$(\partial/\partial r)\,\|U_n(r,s)w\|^2 = -2\,\mathrm{Re}(A_n(r)U_n(r,s)w, U_n(r,s)w).$$

(c) follows from (b). In fact, we see from conditions (II), (IV) and Lemma 2.3 (a) that

$$\|A_n(t)v\| \le \Big(1-\frac{\alpha(t)}{\nu_n(t)}\Big)^{-1}\|A(t)v\| \le 2\gamma(t)\|v\|_Y, \quad \text{a.a. } t \in (0,T). \tag{3.3}$$

The assertion follows from (b). □

Lemma 3.5. *Let* $\{U_n(t,s)\}$ *be as in* Lemma 3.3. *Then there is a family* $\{U(t,s); (t,s) \in \Delta\}$ *in* $B(X)$ *such that*

(a) $U(t,s) :=$ s-$\lim_{n\to\infty} U_n(t,s)$, *where the convergence is uniform on* Δ, *and hence* $U(\cdot,\cdot)$ *is strongly continuous on* Δ *to* $B(X)$, *with*

$$\|U(t,s)v - U_n(t,s)v\|^2 \le \frac{2}{n}\|\gamma\|_{L^1(s,t)}\exp\Big(4\int_s^t \beta(r)\,dr\Big)\|v\|_Y^2, \quad v \in Y \tag{3.4}$$

and $\|U(t,s)\|_{B(X)} \le \exp\Big(\int_s^t \alpha(r)\,dr\Big)$ *on* Δ.

(b) $U(t,r)U(r,s) = U(t,s)$ *on* Δ *and* $U(s,s) = 1$.

(c) $U(t,s)Y \subset Y$ *and* $S^{1/2}U(t,s)v = \text{w-lim}_{n\to\infty} S^{1/2}U_n(t,s)v$, *with*

$$\|S^{1/2}U(t,s)v\| \le \exp\Big(\int_s^t \beta(r)\,dr\Big)\,\|S^{1/2}v\|, \quad v \in Y, \quad (t,s) \in \Delta. \tag{3.5}$$

Proof. (a) Let $v \in Y$. Then we shall show that

$$\|U_n(t,s)v - U_m(t,s)v\|^2 \le 2\Big|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\Big|^2 \|\gamma\|_{L^1(s,t)} \exp\Big(4\int_s^t \beta(r)\,dr\Big)\,\|v\|_Y^2. \tag{3.6}$$

The computation is similar as in [**12**]. Put

$$\begin{aligned} u_{nm}(r,s) &:= U_n(r,s)v - U_m(r,s)v,\\ w_{nm}(r,s) &:= J_n(r)U_n(r,s)v - J_m(r)U_m(r,s)v, \end{aligned}$$

where $J_n(r) := (1+\nu_n(r)^{-1}A(r))^{-1} = 1-\nu_n(r)^{-1}A_n(r)$. Then by Lemma 3.3 (d) we have

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial r}\|u_{nm}(r,s)\|^2\\ &= -\operatorname{Re}\big(A_n(r)U_n(r,s)v - A_m(r)U_m(r,s)v, u_{nm}(r,s) - w_{nm}(r,s)\big)\\ &\quad -\operatorname{Re}\big(A(r)w_{nm}(r,s), w_{nm}(r,s)\big). \end{aligned}$$

Noting that

$$u_{nm}(r,s) - w_{nm}(r,s) = \nu_n(r)^{-1}A_n(r)U_n(r,s)v - \nu_m(r)^{-1}A_m(r)U_m(r,s)v, \tag{3.7}$$

we see that

$$\begin{aligned} &-\operatorname{Re}\big(A_n(r)U_n(r,s)v - A_m(r)U_m(r,s)v, u_{nm}(r,s) - w_{nm}(r,s)\big)\\ &= (\nu_n(r)^{-1} + \nu_m(r)^{-1})\operatorname{Re}\big(A_n(r)U_n(r,s)v, A_m(r)U_m(r,s)v\big)\\ &\quad -\nu_n(r)^{-1}\|A_n(r)U_n(r,s)v\|^2 - \nu_m(r)^{-1}\|A_m(r)U_m(r,s)v\|^2. \end{aligned}$$

On the other hand, it follows from condition $(\mathrm{I})_+$ that

$$\begin{aligned} -\operatorname{Re}\big(A(r)w_{nm}(r,s), w_{nm}(r,s)\big) &\le \alpha(r)\,\|w_{nm}(r,s)\|^2\\ &\le \beta(r)\,\|w_{nm}(r,s)\|^2. \end{aligned}$$

We see from (3.7) that $\|w_{nm}(r,s)\|^2$ is estimated as follows:

$$\begin{aligned} &\frac{1}{2}\|w_{nm}(r,s)\|^2 - \|u_{nm}(r,s)\|^2\\ &\le \big\|\nu_n(r)^{-1}A_n(r)U_n(r,s)v - \nu_m(r)^{-1}A_m(r)U_m(r,s)v\big\|^2\\ &= \nu_n(r)^{-2}\|A_n(r)U_n(r,s)v\|^2 + \nu_m(r)^{-2}\|A_m(r)U_m(r,s)v\|^2\\ &\quad - 2\nu_n(r)^{-1}\nu_m(r)^{-1}\operatorname{Re}\left(A_n(r)U_n(r,s)v, A_m(r)U_m(r,s)v\right). \end{aligned}$$

Combining these estimates and using Lemma 3.4 (c), we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial r}\|u_{nm}(r,s)\|^2 - 2\beta(r)\|u_{nm}(r,s)\|^2\\ &\leq \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}\right|^2 \gamma(r)\exp\Big(4\int_s^r \beta(\tau)\,d\tau\Big)\|v\|_Y^2.\end{aligned}$$

Integrating this inequality on $[s,t]$, we obtain (3.6). Since Y is dense in X, we see from Lemma 3.4 (a) that the family $\{U(t,s);(t,s)\in\Delta\}$ in $B(X)$ is defined: for $w\in X$,

$$U_n(\cdot,\cdot)w \longrightarrow U(\cdot,\cdot)w \quad \text{in} \quad C(\Delta;X) \quad \text{as} \quad n\longrightarrow\infty.$$

(b) follows from Lemma 3.3 (b).

(c) is a consequence of (a) and Lemma 3.4 (b). □

Lemma 3.6. *Let* $\{U(t,s)\}$ *be as in* Lemma 3.5. *Let* $v\in Y$ *and* $(t,s)\in\Delta$. *Then*
(a) $U(t,s)v\in D(A(t))$, *and*

$$\|A(t)U(t,s)v\| \leq \gamma(t)\exp\Big[\int_s^t \beta(r)dr\Big]\,\|v\|_Y \quad \text{a.a. } t\in(s,T)$$

with

$$A(t)U(t,s)v = \operatorname*{w-lim}_{n\to\infty} A_n(t)U_n(t,s)v \quad \text{a.a. } t\in(s,T). \tag{3.8}$$

(b) $\displaystyle\int_s^t U(t,r)A(r)v\,dr = \operatorname*{s-lim}_{n\to\infty}\int_s^t U_n(t,r)A_n(r)v\,dr$ *in* X.

(c) $(\partial/\partial s)U(t,s)v = U(t,s)A(s)v$ a.a. $s\in(0,t)$.

Proof. (a) $A(\cdot)U(\cdot,s)v\in L^1(s,t;X)$ follows from condition (IV) and (3.5). By virtue of Lemma 2.6, (3.8) follows from Lemmas 3.4 (c) and 3.5 (c).

(b) For a.a. $r\in(s,t)$, it follows from Lemmas 3.1 (b), 3.4 (a) and 3.5 (a) that

$$U(t,r)A(r)v = \operatorname*{s-lim}_{n\to\infty} U_n(t,r)A_n(r)v \quad \text{in } X.$$

On the other hand, Lemma 3.4 (a) and (3.3) yield that

$$\|U_n(t,r)A_n(r)v\| \leq 2\gamma(r)\exp\Big(2\int_0^T \alpha(\tau)\,d\tau\Big)\|v\|_Y \in L^1(s,t).$$

Therefore we obtain the assertion by the Lebesgue convergence theorem.

(c) By Lemma 3.3 (e) we have

$$v - U_n(t,s)v = \int_s^t U_n(t,r)A_n(r)v\,dr, \quad v\in Y.$$

Letting $n\longrightarrow\infty$, we see from (3.4) and (b) that

$$v - U(t,s)v = \int_s^t U(t,r)A(r)v\,dr, \quad v\in Y. \tag{3.9}$$

Since condition (IV) and Lemma 3.5 (a), $U(t,\cdot)A(\cdot)v\in L^1(0,t;X)$. Therefore (3.9) is strongly differentiable on a.a. $s\in(0,t)$ and we obtain the assertion. □

Lemma 3.7. *Let* $\{U(t,s)\}$ *be as in* Lemma 3.5. *Let* $v \in Y$. *Then*
(a) *For each* $s \in [0,T]$, $A(\cdot)U(\cdot,s)v$ *is Bochner integrable on* $[s,T]$, *with*

$$U(t,s)v = v - \int_s^t A(r)U(r,s)v\,dr, \quad t \in [s,T], \tag{3.10}$$

and hence $U(\cdot,s)$ *is absolutely continuous on* $[s,T]$*:*

$$\|U(t,s)v - U(t',s)v\| \le \left|\int_{t'}^t \gamma(r)\,dr\right| \exp\left[\int_0^T \beta(r)\,dr\right] \|v\|_Y. \tag{3.11}$$

(b) $(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v$, a.a. $t \in (s,T)$.

Proof. (a) It follows from Lemma 3.6 (a) that $A(\cdot)U(\cdot,s)v$ is Bochner integrable on [s,T]. Now Lemma 3.3 (d) implies that for each $w \in X$,

$$(U_n(t,s)v,w) = (v,w) - \int_s^t (A_n(r)U_n(r,s)v,w)\,dr.$$

Letting $n \to \infty$, we see from (3.4) and (3.8) that

$$(U(t,s)v,w) = (v,w) - \int_s^t (A(r)U(r,s)v,w)\,dr.$$

Thus we obtain (3.10) and (3.11).
(b) is a direct consequence of (3.10). □

It is easy to prove the uniqueness of the evolution operator constructed above.

Lemma 3.8. *Let* $\{U(t,s)\}$ *be as in* Lemma 3.5. *Suppose that* $\{V(t,s)\}$ *is another family in* $B(X)$ *with the properties* (i), (ii) *and* (v). *Then* $U(t,s) \equiv V(t,s)$ *on* Δ.

In fact, we see from Lemma 3.7 (b) that for $v \in Y$,

$$(\partial/\partial r)V(t,r)U(r,s)v = 0 \quad \text{a.a. } r \in (s,t).$$

Hence we obtain $U(t,s)v = V(t,s)v$. Since Y is dense in X, the assertion follows.

Lemma 3.9. *Let* $\{A(t)\}$ *and* S *be as in* Theorem 1.1. *Assume that conditions* (I) *and* (III) *are satisfied, with the inclusion* $D(S) \subset D(A(t))$. *Let* $\{S_\varepsilon\}$ *be the Yosida approximation of* S. *Then*

$$|\operatorname{Re}(A(t)v, S_\varepsilon v)| \le \beta(t)(v, S_\varepsilon v), \quad v \in D(A(t)), \quad \text{a.a. } t \in (0,T).$$

In particular, if $D(S^{1/2}) \subset D(A(t))$ *(this is condition* (II)*), then*

$$|\operatorname{Re}(A(t)v, S_\varepsilon v)| \le \beta(t)\left\|S^{1/2}v\right\|^2, \quad v \in D(S^{1/2}), \quad \text{a.a. } t \in (0,T). \tag{3.12}$$

The conclusion follows from Lemma 2.4 (this fact is first noted in [**13**]).

Lemma 3.10. *Let* $\{U(t,s)\}$ *be as in* Lemma 3.5. *Let* $v \in Y$. *Then*

(a) $S^{1/2}U(t,s)v$ *is weakly continuous on* Δ.

(a$'$) $S^{1/4}U(t,s)v$ *is strongly continuous on* Δ.

(b) $S^{1/2}U(t,s)v \to S^{1/2}v$ *as* $(t,s) \to (t_0,t_0)$.

(c) *For* $t \in (0,T]$, $U(t,\cdot)v \in C([0,T];Y)$.

Proof. (a) Let $\{S_\varepsilon\}$ be the Yosida approximation of S. Then for $v \in Y$, $S_\varepsilon^{1/2}U(t,s)v$ is continuous on Δ. Noting that $(1+\varepsilon S)^{-1/2}w \to w$ $(\varepsilon \downarrow 0)$, we see by (3.5) that

$$S^{1/2}U(t,s)v = \operatorname*{w\text{-}lim}_{\varepsilon\downarrow 0} S_\varepsilon^{1/2}U(t,s)v,$$

where the convergence is uniform on Δ and hence the limit function is also weakly continuous on Δ.

(a$'$) is a direct consequence of Lemma 3.5 (a) and (3.5).

(b) Let $t_0 \in [0,T]$. Then it suffices by (a) to show that

$$\left\|S^{1/2}U(t,s)v\right\| \to \left\|S^{1/2}v\right\| \quad \text{as} \quad (t,s) \to (t_0,t_0).$$

We see again by (a) that

$$\left\|S^{1/2}v\right\| \le \liminf_{(t,s)\to(t_0,t_0)} \left\|S^{1/2}U(t,s)v\right\|.$$

On the other hand, it follows from (3.5) that

$$\limsup_{(t,s)\to(t_0,t_0)} \left\|S^{1/2}U(t,s)v\right\| \le \left\|S^{1/2}v\right\|.$$

(c) follows from (b) and (3.5). □

Now we are in a position to prove (iii) and $U(\cdot,\cdot) \in W^{1,1}(\Delta; B(Y,X))$ of Theorem 1.1.

Lemma 3.11. *Let* $\{A(t)\}$ *and* S *be as in* Theorem 1.1. *Assume that conditions* (I)–(IV) *are satisfied. Let* $\{U(\cdot,\cdot)\}$ *be as in* Lemma 3.5. *Then*

(a) *For* $v \in Y$ *and* $s \in [0,T]$, $U(\cdot,s)v \in C([s,T];Y)$.

(b) $U(\cdot,\cdot)$ *is strongly continuous on* Δ *to* $B(Y)$.

(c) *For* $v \in Y$, $U(\cdot,\cdot)v \in W^{1,1}(\Delta;X)$.

Proof. (a) Lemmas 3.5 (a) and 3.7 (b) yield that $U(\cdot,s)v \in W^{1,1}(s,T;X) \subset C([s,T];X)$. Thus it suffices to show that

$$S^{1/2}U(\cdot,s)v \in C([s,T];X). \tag{3.13}$$

Let $t_0 \in [s,T]$. Then we have

$$\left\|S^{1/2}U(t,s)v - S^{1/2}U(t_0,s)v\right\|^2 = \left\|S^{1/2}U(t,s)v\right\|^2 - \left\|S^{1/2}U(t_0,s)v\right\|^2 \\ - 2\operatorname{Re}\bigl(S^{1/2}U(t,s)v - S^{1/2}U(t_0,s)v, S^{1/2}U(t_0,s)v\bigr).$$

Since $S^{1/2}U(t,s)v$ is weakly continuous on Δ (see Lemma 3.10 (a)), we obtain (3.13) if we show that

$$\left\|S^{1/2}U(t,s)v\right\|^2 \longrightarrow \left\|S^{1/2}U(t_0,s)v\right\|^2 \quad \text{as} \quad t \longrightarrow t_0. \tag{3.14}$$

To this end we can use (3.2). Integrating (3.2) on $[t_0,t]$, we have

$$\left\|S_\varepsilon^{1/2}U_n(t,s)v\right\|^2 - \left\|S_\varepsilon^{1/2}U_n(t_0,s)v\right\|^2 = -2\int_{t_0}^t \operatorname{Re}\bigl(A_n(r)U_n(r,s)v, S_\varepsilon U_n(r,s)v\bigr)\,dr.$$

Letting $n \longrightarrow \infty$, we see from (3.4), (3.8) and Lemma 3.4 (c) that

$$\left\|S_\varepsilon^{1/2}U(t,s)v\right\|^2 - \left\|S_\varepsilon^{1/2}U(t_0,s)v\right\|^2 = -2\int_{t_0}^t \operatorname{Re}(A(r)U(r,s)v, S_\varepsilon U(r,s)v)\,dr.$$

It follows from (3.12) and (3.5) that

$$\left|\left\|S_\varepsilon^{1/2}U(t,s)v\right\|^2 - \left\|S_\varepsilon^{1/2}U(t_0,s)v\right\|^2\right| \le 2\left|\int_{t_0}^t \beta(r)\exp\Bigl[2\int_s^r \beta(\tau)\,d\tau\Bigr]\,dr\right| \|v\|_Y^2 \\ = \left|\exp\Bigl[2\int_s^t \beta(r)\,dr\Bigr] - \exp\Bigl[2\int_s^{t_0} \beta(r)\,dr\Bigr]\right| \|v\|_Y^2.$$

Noting that $(1+\varepsilon S)^{-1}w \longrightarrow w$ ($\varepsilon \downarrow 0$) for every $w \in X$, we have

$$\left|\left\|S^{1/2}U(t,s)v\right\|^2 - \left\|S^{1/2}U(t_0,s)v\right\|^2\right| \le \left|\exp\Bigl[2\int_s^t \beta(r)\,dr\Bigr] - \exp\Bigl[2\int_s^{t_0} \beta(r)\,dr\Bigr]\right| \|v\|_Y^2.$$

Thus we obtain (3.14).

(b) We follow the idea in Kato [**4**, Remark 5.4]. First let $t_0 = s_0$. Then the assertion follows from Lemma 3.10 (b). Next let $s_0 < t_0$. Set $a := 2^{-1}(s_0+t_0)$. Then $s < a < t$ for $(t,s) \in B\bigl((t_0,s_0), 2^{-1}(t_0-s_0)\bigr) \cap \Delta$. Thus we have

$$\|U(t,s)v - U(t_0,s_0)v\|_Y \\ \le \|U(t,a)\|_{B(Y)}\|U(a,s)v - U(a,s_0)v\|_Y + \|(U(t,a) - U(t_0,a))U(a,s_0)v\|_Y.$$

Therefore the assertion follows from (a), (3.5) and Lemma 3.10 (c).

(c) $U(\cdot,\cdot)v \in C(\Delta;X)$ is a direct consequence of (b). It follows from Lemma 3.5 (c) and 3.7 (b) that

$$\iint_\Delta \|(\partial/\partial t)U(t,s)v\|\,dtds = \iint_\Delta \|A(t)U(t,s)v\|\,dtds \\ \le \iint_\Delta \gamma(t)\exp\Bigl[\int_s^t \beta(r)\,dr\Bigr]\|v\|_Y\,dtds \\ \le T\|\gamma\|_{L^1(0,T)}\exp\Bigl[\int_0^T \beta(r)\,dr\Bigr]\|v\|_Y.$$

Similarly by Lemma 3.5 (a) and 3.6 (c) we have

$$\iint_\Delta \|(\partial/\partial s)U(t,s)v\|\ dtds \le T\,\|\gamma\|_{L^1(0,T)} \exp\Big[\int_0^T \alpha(r)\,dr\Big]\ \|v\|_Y\,.$$

Therefore the assertion follows. □

4. Inhomogeneous equations

In this section we prove Theorem 1.2. Let $A(t)$ and S be as in Theorem 1.1. First assume that condition $(\mathrm{I})_+, (\mathrm{II}), (\mathrm{III})_+$ and (IV) are satisfied. Let $\{U(t,s); (t,s) \in \Delta\}$ be the evolution operator with the properties stated in Lemmas 3.5–3.7. Then for $u_0 \in Y$,

$$(d/dt)U(t,0)u_0 + A(t)U(t,0)u_0 = 0 \quad \text{a.a. } t \in (0,T). \tag{4.1}$$

Let $f(\cdot) \in L^1(0,T;Y)$ and put

$$v(t) := \int_0^t U(t,s)f(s)\,ds. \tag{4.2}$$

Then clearly $v(\cdot) \in L^\infty(0,T;X)$. We want to show that

$$(d/dt)v(t) + A(t)v(t) = f(t) \quad \text{a.a. } t \in (0,T). \tag{4.3}$$

Lemma 4.1. *Let $v(\cdot)$ be as above and $t \in [0,T]$. Then*

(a) $v(\cdot) \in L^\infty(0,T;Y)$, *with* $\|v(t)\|_Y \le \exp\Big[\int_0^T \beta(r)dr\Big]\ \|f(\cdot)\|_{L^1(0,T;Y)}$.

(b) $S^{1/2}v(\cdot)$ *is weakly continuous on* $[0,T]$.

(c) $v(t) \in D(A(t))$ *and* $\|A(\cdot)v(\cdot)\|_{L^1(0,T;X)} \le \|\gamma\|_{L^1(0,T)}\ \|v(\cdot)\|_{L^\infty(0,T;Y)}$.

Proof. (a) Let $\{S_\varepsilon\}$ be the Yosida approximation of S. Then we have

$$S_\varepsilon^{1/2}v(t) = \int_0^t S_\varepsilon^{1/2}U(t,s)f(s)\,ds.$$

Since $\|S_\varepsilon^{1/2}w\| \le \|S^{1/2}w\| \le \|w\|_Y$, it follows from (3.5) that

$$\|S_\varepsilon^{1/2}v(t)\| \le \int_0^t \|U(t,s)\|_{B(Y)}\ \|f(s)\|_Y\ ds \le \exp\Big[\int_0^T \beta(r)dr\Big]\ \|f(\cdot)\|_{L^1(0,T;Y)}$$

Hence we see that $v(t) \in Y$ and

$$S^{1/2}v(t) = \operatorname*{w\text{-}lim}_{\varepsilon\downarrow 0} S_\varepsilon^{1/2}v(t), \quad t \in [0,T]. \tag{4.4}$$

Thus the assertion follows.

(b) The convergence in (4.4) is uniform on $[0,T]$ and therefore $S^{1/2}v(\cdot)$ is weakly continuous on $[0,T]$.

(c) follows from (a) and the condition (II). □

Next let $\{U_n(t,s)\}$ be as in Theorem 3.2 and put

$$v_n(t) := \int_0^t U_n(t,s)f(s)\,ds.$$

Then $v_n(\cdot) \in W^{1,1}(0,T;X)$ and

$$(d/dt)v_n(t) = -A_n(t)v_n(t) + f(t) \quad \text{a.a. } t \in (0,T). \tag{4.5}$$

Now we can prove (4.3).

Lemma 4.2. *Let $v(\cdot)$ be as above. Then*

(a) $v_n(\cdot) \to v(\cdot)$ *in* $C([0,T];X)$ *as* $n \to \infty$.

(b) $A(t)v(t) = \underset{n\to\infty}{\text{w-lim}} A_n(t)v_n(t) \quad \text{a.a. } t \in (0,T)$.

(c) $A(\cdot)v(\cdot)$ *is Bochner integrable on* $[0,T]$ *and*

$$v(t) = -\int_0^t A(s)v(s)\,ds + \int_0^t f(s)\,ds. \tag{4.6}$$

(d) $(d/dt)v(t) = -A(t)v(t) + f(t) \quad \text{a.a. } t \in (0,T)$.

Proof. (a) follows from (3.4).

(b) (a) and Lemma 4.1 (c) implies by Lemma 2.6 that $A(\cdot)v(\cdot)$ is the weak limit of $A_n(\cdot)v_n(\cdot)$ as $n \to \infty$.

(c) It follows from (b) that $A(\cdot)v(\cdot)$ is strongly measurable. Furthermore, by Lemma 4.1 (c) we have $A(\cdot)v(\cdot) \in L^1(0,T;X)$. Therefore $A(\cdot)v(\cdot)$ is Bochner integrable on $[0,T]$. On the other hand, we see from (4.5) that for each $w \in X$,

$$\big(v_n(t), w\big) = -\int_0^t \big(A_n(s)v_n(s), w\big)\,ds + \int_0^t \big(f(s), w\big)\,ds.$$

Letting $n \to \infty$, we have

$$\big(v(t), w\big) = -\int_0^t \big(A(s)v(s), w\big)\,ds + \int_0^t \big(f(s), w\big)\,ds.$$

Hence we obtain (4.6).

(d) Strong differentiability of $v(t)$ is a consequence of (4.6). □

The next lemma guarantees that the strong solution of (E) is expressed by the variation of constant formula.

Lemma 4.3. *Let $\{U(t,s)\}$ be the evolution operator with properties* (i), (ii) *and* (v). *Let $u(\cdot)$ be a strong solution of* (E) *with $u(0) = u_0 \in Y$. If $f \in L^1(0,T;X)$ then*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds. \tag{4.7}$$

In fact, it suffices to integrate the identity:

$$(\partial/\partial s)U(t,s)u(s) = U(t,s)f(s) \quad \text{a.a. } s \in (0,t).$$

Consequently, it follows from (4.1) and (4.3) that if $f(\cdot) \in L^1(0,T;Y)$ then $u(\cdot)$ given by (4.7) is a unique solution of (E) with $u(0) = u_0 \in Y$.

Now we are in a position to prove Theorem 1.2.

Lemma 4.4. *Let $\{A(t)\}$ and S be as in* Theorem 1.1. *Assume that conditions* (I)–(IV) *are satisfied. Let $\{U(t,s)\}$ be the evolution operator on X generated by $\{A(t)\}$. For $f(\cdot) \in L^1(0,T;Y)$ let $v(\cdot)$ be as in (4.2). Then*

$$v(\cdot) \in W^{1,1}(0,T;X) \cap C([0,T];Y).$$

Proof. It follows from Lemma 4.2 (d) that $v \in W^{1,1}(0,T;X)$. Hence it suffices to show that

$$v(\cdot) \in C([0,T];Y). \tag{4.8}$$

This is shown by the similar way as in Lemma 3.11 (a). Let $\{S_\varepsilon\}$ be the Yosida approximation of S. Then it follows from (4.5) that

$$\begin{aligned}(d/ds)\left\|S_\varepsilon^{1/2}v_n(s)\right\|^2 &= 2\operatorname{Re}\big((d/ds)v_n(s), S_\varepsilon v_n(s)\big)\\ &= 2\operatorname{Re}\big(-A_n(s)v_n(s) + f(s), S_\varepsilon v_n(s)\big) \quad \text{a.a. } s \in (0,T).\end{aligned}$$

Integrating this equality from $s = t_0$ to $s = t$, we have

$$\begin{aligned}&\left\|S_\varepsilon^{1/2}v_n(t)\right\|^2 - \left\|S_\varepsilon^{1/2}v_n(t_0)\right\|^2\\ &= -2\int_{t_0}^t \operatorname{Re}\big(A_n(s)v_n(s), S_\varepsilon v_n(s)\big)\,ds + 2\int_{t_0}^t \operatorname{Re}\big(f(s), S_\varepsilon v_n(s)\big)\,ds.\end{aligned}$$

Letting $n \to \infty$, we see from Lemma 4.2 (a) and (b) that

$$\begin{aligned}&\left\|S_\varepsilon^{1/2}v(t)\right\|^2 - \left\|S_\varepsilon^{1/2}v(t_0)\right\|^2\\ &= -2\int_{t_0}^t \operatorname{Re}\big(A(s)v(s), S_\varepsilon v(s)\big)\,ds + 2\int_{t_0}^t \operatorname{Re}\big(f(s), S_\varepsilon v(s)\big)\,ds.\end{aligned}$$

It follows from (3.12) and Lemma 4.1 (a) that

$$\begin{aligned}&\left|\left\|S_\varepsilon^{1/2}v(t)\right\|^2 - \left\|S_\varepsilon^{1/2}v(t_0)\right\|^2\right|\\ &\le 2\left|\int_{t_0}^t \beta(t)\left\|S^{1/2}v(s)\right\|^2 ds\right| + 2\left|\int_{t_0}^t \left\|S^{1/2}f(s)\right\| \cdot \left\|S^{1/2}v(s)\right\| ds\right|\\ &\le 2\|\beta\|_{L^1(t_0,t)}\|v(\cdot)\|^2_{L^\infty(0,T;Y)} + 2\|f(\cdot)\|_{L^1(t_0,t;Y)}\|v(\cdot)\|_{L^\infty(0,T;Y)}\,.\end{aligned}$$

Thus we have

$$\left\|S_\varepsilon^{1/2}v(t)\right\|^2 \to \left\|S_\varepsilon^{1/2}v(t_0)\right\|^2 \quad (t \to t_0). \tag{4.9}$$

By both Lemma 4.1 (b) and (4.9) we obtain (4.8). □

In view of Lemma 4.2 (d) and Lemma 3.11 this completes the proof of Theorem 1.2.

5. Preliminaries for applications

Put $\langle x \rangle := \left(1 + |x|^2\right)^{1/2}$. In this section we consider the selfadjointness of

$$S := (H_D + V)^2 + \langle x \rangle^2 I \quad \text{for} \quad u \in D(S) := \{u \in L^2(\mathbb{R}^3)^4; Su \in L^2(\mathbb{R}^3)^4\}. \tag{5.1}$$

Here H_D is the free Dirac operator

$$H_D := \alpha \cdot p + m\beta = \sum_{j=1}^{3} \alpha_j i^{-1} \frac{\partial}{\partial x_j} + m\beta,$$

acting in the Hilbert space $L^2(\mathbb{R}^3)^4$; $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = \alpha_4$ are the usual 4×4 Hermitian matrices satisfying the commutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \quad (j, k = 1, 2, 3, 4), \tag{5.2}$$

and m is a positive constant (cf. Fattorini [**2**]).

The potential V is an operator of multiplication with a 4×4 Hermitian matrix-valued, measurable function $V(x)$ defined on $\mathbb{R}^3$. It is assumed that

$$|V(x)| \le a|x|^{-1} + b, \tag{5.3}$$

where $|V(x)|$ denotes the operator norm of $V(x) : \mathbb{C}^4 \to \mathbb{C}^4$ and a, b are nonnegative constants with $a < 1/2$.

First, we consider the selfadjointness of $H_D + V$.

Theorem 5.1 (Kato-Rellich theorem). *Let A be a selfadjoint operator in a Hilbert space H and B a symmetric operator in H, with $D(A) \subset D(B)$. Assume that there exist two constants a_0, $b_0 \ge 0$ such that for all $u \in D(A)$,*

$$\|Bu\| \le a_0 \|u\| + b_0 \|Au\|.$$

If $b_0 < 1$ then $A + B$ is also selfadjoint on $D(A)$.

For a proof see [**7**, Theorem V.4.3].

Lemma 5.2. *Let H_D and V be as above. Then $H_D + V$ is selfadjoint on $H^1(\mathbb{R}^3)^4$.*

Proof. Let $u \in H^1(\mathbb{R}^3)^4$. H_D is selfadjoint and V is symmetric. It follows from (5.3) and the Hardy inequality that

$$\|Vu\| \le a\||x|^{-1}u\| + b\|u\| \le 2a\|\nabla u\| + b\|u\|.$$

On the other hand, we see from (5.2) that $\|H_D u\|^2 = \|\nabla u\|^2 + m^2\|u\|^2$. Therefore, V is H_D-bounded, with H_D-bound $2a < 1$. Now the assertion follows from Theorem 5.1. $\square$

The selfadjointness of $(H_D + V)^2$ is clear. Let us consider the selfadjointness of S. Clearly, S is symmetric. Thus we have only to consider the m-accretivity of S.

Lemma 5.3 ([10]). *Let A and B be linear m-accretive operators in a Hilbert space H. Let D be a linear manifold invariant under $(1+n^{-1}A)^{-1}$ for $n \in \mathbb{N}$. Assume that D is a core of B and there exist two constants a, $b \geq 0$ such that for all $u \in D_0 := (1+A)^{-1}D$,*

$$0 \leq \mathrm{Re}(Au, Bu) + a\,\|u\|^2 + b\,\|Au\|^2 .$$

If $b < 1$ then $A + B$ is also m-accretive in H.

Lemma 5.4. *Let H_D and V be as above. Then S is selfadjoint on $D(S)$.*

Proof. Let $u \in \mathcal{S}(\mathbb{R}^3)^4$, where $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space. Then we have

$$\begin{aligned}
\mathrm{Re}\big((H_D+V)^2u, \langle x\rangle^2 u\big) &= \mathrm{Re}\big((H_D+V)u, (H_D+V)(\langle x\rangle^2 u)\big) \\
&= \big\|\langle x\rangle(H_D+V)u\big\|^2 - 2\,\mathrm{Im}\big((H_D+V)u, \alpha\cdot xu\big) \\
&\geq \big\|\langle x\rangle(H_D+V)u\big\|^2 - 2\big\|\langle x\rangle(H_D+V)u\big\|\cdot\|u\| \\
&\geq -\|u\|^2.
\end{aligned}$$

The assertion follows from Theorem 5.3. □

6. Applications to the Dirac equation

Let H_D and V be as in Section 5. In this section we consider, as an application of Theorem 1.1, the Cauchy problem for the Dirac equation:

$$\text{(DE)} \qquad \begin{cases} i\dfrac{d}{dt}u = H(t)u + f(t) & \text{for } t \in (0,T), \\ u(0) = u_0 \end{cases}$$

in the Hilbert Space $X = L^2(\mathbb{R}^3)^4$, where $u_0 \in Y := H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$.

First we define $H(t)$ precisely. Let

$$\mathcal{H}(t) := H_D + V + q(t)I$$

with domain $D(\mathcal{H}(t)) = C_0^\infty(\mathbb{R}^3)^4$. $q(t)I$ is a maximal multiplication operator by $q(x,t)$, where $q(x,t) : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ is the time-dependent measurable real-valued potential. Furthermore, we impose $q(t)$ satisfying following conditions:

$$\text{(q1)} \qquad q(\cdot) \in L^1\big(0,T; \langle x\rangle L^\infty(\mathbb{R}^3)\big),$$

$$\text{(q2)} \qquad |\nabla q(\cdot)| \in L^1\big(0,T; L^\infty(\mathbb{R}^3)\big),$$

where $\langle x\rangle L^\infty(\mathbb{R}^3) := \{\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^3); \langle x\rangle^{-1}\varphi \in L^\infty(\mathbb{R}^3)\}$.

Since $\mathcal{H}(t)$ is symmetric, $\mathcal{H}(t)$ is closable. Then we take as $H(t)$ the closure $\widetilde{\mathcal{H}}(t)$ of $\mathcal{H}(t)$, i.e., $H(t) = \widetilde{\mathcal{H}}(t)$.

Let S be as in (5.1). Then S is selfadjoint on $D(S)$, with $S \geq 1$. Thus $Y = D(S^{1/2})$ is regarded as a Hilbert space, embedded continuously and densely in $L^2(\mathbb{R}^3)^4$, with inner product

$$(u,v)_{D(S^{1/2})} = (S^{1/2}u, S^{1/2}v), \quad u,\, v \in D(S^{1/2}).$$

Lemma 6.1. *Let S be as above. Then $D(S^{1/2}) = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$ and there exist positive constants c_1, c_2 such that*

$$c_1\|S^{1/2}u\|^2 \le \|u\|^2 + \|\nabla u\|^2 + \||x|u\|^2 \le c_2\|S^{1/2}u\|^2, \quad u \in D(S^{1/2}). \tag{6.1}$$

Proof. Let $u \in D(S)$. Then we have

$$\begin{aligned}\|S^{1/2}u\|^2 &= (Su, u)\\ &= \big((H_D + V)^2u + u + |x|^2u, u\big)\\ &= \|u\|^2 + \|(H_D + V)u\|^2 + \||x|u\|^2.\end{aligned}$$

On the other hand, there exist positive constants c'_1, c'_2 such that

$$c'_1(\|u\| + \|\nabla u\|) \le \|u\| + \|(H_D + V)u\| \le c'_2(\|u\| + \|\nabla u\|). \tag{6.2}$$

Since $D(S)$ is a core for $S^{1/2}$, (6.1) holds for $u \in D(S^{1/2}) = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$. □

Now we shall verify conditions (I)–(IV) of Theorem 1.1.

Lemma 6.2. *Let $A(t) = iH(t)$ and S be as above. Assume that* (q1), (q2) *are satisfied. Then for each $T > 0$*

(I) $\operatorname{Re}(A(t)v, v) = 0$, $v \in D(A(t))$, a.a. $t \in (0, T)$.

(II) $Y = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4 \subset D(A(t))$, a.a. $t \in (0, T)$.

(III) *There exists $\beta \in L^1(0, T)$, $\beta \ge 0$ such that*

$$|\operatorname{Re}(A(t)u, Su)| \le \beta(t)\|S^{1/2}u\|^2, \ u \in D(S), \ \text{a.a. } t \in (0, T).$$

(IV) $A(\cdot) \in L^1_*(0, T; B(H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4))$.

Proof. Noting that $\operatorname{Re}(A(t)u, u) = -\operatorname{Im}(H(t)u, u)$, the assertion follows from symmetry of $H(t)$. Therefore, it is sufficient to show that there exist $\beta, \gamma \in L^1(0, T)$ such that

$$\|H(t)u\| \le \gamma(t)\|S^{1/2}u\|, \ u \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4, \quad \text{a.a. } t \in (0, T). \tag{6.3}$$

$$|\operatorname{Im}(H(t)u, Su)| \le \beta(t)\|S^{1/2}u\|^2, \ u \in D(S), \quad \text{a.a. } t \in (0, T). \tag{6.4}$$

First, we verify (6.3). It follows from condition (q1) that

$$\begin{aligned}\|H(t)u\| &\le \|(H_D + V)u\| + \|q(t)u\|\\ &\le \|(H_D + V)u\| + \gamma_q(t)\|\langle x\rangle u\|,\end{aligned}$$

where $\gamma_q \in L^1(0, T)$ depends on q. Thus we obtain (6.3).

Next, we verify (6.4). By integration by parts we have

$$\begin{aligned}\operatorname{Im}(H(t), Su) &= \operatorname{Im}\big((H_D + V)u, |x|^2u\big) + \operatorname{Im}\big(q(t)u, (H_D + V)^2u\big)\\ &= \operatorname{Re}\big((\alpha \cdot x)u, u\big) - \operatorname{Re}\big((\alpha \cdot \nabla q(t))u, (H_D + V)u\big).\end{aligned}$$

Hence it follows from the Cauchy-Schwarz inequality and condition (q2) that

$$\begin{aligned}\left|\mathrm{Im}(H(t), Su)\right| &\le \| |x|u\| \cdot \|u\| + \left\| |\nabla q(t)|u \right\| \cdot \left\|(H_D + V)u\right\| \\ &\le \| |x|u\| \cdot \|u\| + \beta_q(t)\|u\| \cdot \left\|(H_D + V)u\right\|,\end{aligned}$$

where $\beta_q \in L^1(0,T)$ depends on q. Therefore we obtain (6.4). □

Assume further that

$$\text{(f1)} \qquad f \in L^1\big(0,T; H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4\big).$$

Then we can apply Theorems 1.1 and 1.2 to conclude that the Dirac equation (DE) admits a unique solution $u \in W^{1,1}(0,T; L^2(\mathbb{R}^3)^4) \cap C(0,T; H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4)$.

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