

Shape Optimization in Multi-Phase Stefan Problem

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1. Formulation of the optimization problem

Let us consider the enthalpy formulation of Stefan problem described as follows:

$$SP(\Omega) \begin{cases} u_t - \Delta \beta(u) = f & \text{in } Q(\Omega) := (0, T) \times \Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ \beta(u) = g & \text{on } \Sigma(\Omega) := (0, T) \times \partial\Omega, \end{cases}$$

where $\hat{\Omega}$ is a fixed smooth bounded domain in R^N ($N \geq 2$), and Ω is a smooth subdomain of $\hat{\Omega}$, $0 < T < \infty$, $\hat{Q} := (0, T) \times \hat{\Omega}$ and $\hat{\Sigma} := (0, T) \times \partial\hat{\Omega}$; $\beta : R \rightarrow R$ is a nondecreasing function on R such that

$$(1.1) \quad \begin{cases} \beta(0) = 0, |\beta(r)| \geq C_0|r| - C'_0 & \text{for all } r \in R, \\ |\beta(r) - \beta(r')| \leq L_0|r - r'| & \text{for all } r, r' \in R, \end{cases}$$

where $C_0 > 0$, $C'_0 \geq 0$ and $L_0 > 0$ are constants. Here we suppose that $f \in L^2(\hat{Q})$, $g \in W^{2,2}(0, T; L^2(\hat{\Omega})) \cap L^2(0, T; H^2(\hat{\Omega}))$ and $u_0 \in L^2(\hat{\Omega})$. In this paper, u represents the enthalpy and $\beta(u)$ the temperature.

Now we give the weak formulation of $SP(\Omega)$.

DEFINITION 1.1. A function $u : [0, T] \rightarrow L^2(\Omega)$ is a weak solution of $SP(\Omega)$, if the following three conditions (w1) – (w3) are satisfied:

- (w1) $u \in C_w([0, T]; L^2(\Omega))$, $u(0) = u_0$;
- (w2) $\beta(u) \in L^2(0, T; H^1(\Omega))$ and $\beta(u) - g \in L^2(0, T; H_0^1(\Omega))$;
- (w3) $-\int_{Q(\Omega)} u \eta_t dx dt + \int_{Q(\Omega)} \nabla \beta(u) \nabla \eta dx dt = \int_{Q(\Omega)} f \eta dx dt$
for all $\eta \in L^2(0, T; H_0^1(\Omega))$ with $\eta_t \in L^2(Q(\Omega))$ and $\eta(0, \cdot) = \eta(T, \cdot) = 0$.

REMARK 1.1. (1) In (w3) of Definition 1.1, it is enough to take as test function η

any smooth function of the form ρz , with $\rho \in \mathcal{D}(0, T)(= \{\rho \in C^\infty(R); \text{supp } \rho \subset (0, T)\})$ and $z \in H_0^1(\Omega)$.

(2) We denote by $C_w([0, T]; L^2(\Omega))$ the space of all weakly continuous functions from $[0, T]$ to $L^2(\Omega)$ and by $\langle \cdot, \cdot \rangle_\Omega$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Now we introduce the notion of convergence of closed convex sets in a Banach space X , which is due to Mosco [13]. Let $\{K_n\}$ be a sequence of closed convex sets in X and K be a closed convex set in X . Then we say " $K_n \rightarrow K$ in X as $n \rightarrow \infty$ (in the sense of Mosco)" if the following two conditions (M1) and (M2) are satisfied:

(M1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in K_{n_k}$, and $z_k \rightarrow z$ weakly in X as $k \rightarrow \infty$, then $z \in K$.

(M2) For any $z \in K$ there is a sequence $\{z_n\} \subset X$ such that

$$z_n \in K_n, n = 1, 2, \dots, \text{ and } z_n \rightarrow z \text{ in } X \text{ as } n \rightarrow \infty.$$

We denote by χ_Ω the characteristic function of Ω in $\hat{\Omega}$ for any subset Ω of $\hat{\Omega}$. We put

$$O := \{\Omega \subset \hat{\Omega}; \Omega \text{ is a smooth subdomain of } \hat{\Omega}\}$$

and for each $\Omega \in O$ denote by $V(\Omega)$ the set

$$\{z \in H_0^1(\hat{\Omega}); z = 0 \text{ a.e. on } \hat{\Omega} - \Omega\}.$$

Clearly $V(\Omega)$ is a closed linear subspace of $H_0^1(\hat{\Omega})$.

We consider the shape optimization problem for any non-empty subset O_c of O which is compact in the following sense:

$$(C) \left\{ \begin{array}{l} \text{For any sequence } \{\Omega_n\} \subset O_c \text{ there is a subsequence } \{\Omega_{n_k}\} \text{ of } \{\Omega_n\} \text{ with } \Omega \in O_c \\ \text{such that } \chi_{\Omega_{n_k}} \rightarrow \chi_\Omega \text{ in } L^1(\hat{\Omega}) \text{ as } k \rightarrow \infty \text{ and } V(\Omega_{n_k}) \rightarrow V(\Omega) \text{ in } H_0^1(\hat{\Omega}) \\ \text{as } k \rightarrow \infty \text{ (in the sense of Mosco).} \end{array} \right.$$

We give below typical examples of O_c , which are very important in the application of our main results

EXAMPLE 1.1. (1) Let $\hat{\Omega}$ and O be the same as stated before. Let Θ be the class of

all C^1 -diffeomorphisms from $\overline{\hat{\Omega}}$ onto itself. Here we give Θ the topology induced from $C^1(\overline{\hat{\Omega}})$. Let Ω' be a smooth subdomain of $\hat{\Omega}$ with $\overline{\Omega'} \subset \hat{\Omega}$. For a given a non-empty compact subset Θ_c of Θ , we put

$$(1.2) \quad O_c = \{\theta(\Omega'); \theta \in \Theta_c\}.$$

Then this subset O_c of O satisfies condition (C).

Let $\{\Omega_n = \theta_n(\Omega')\}$ be any sequence in O_c . Then, by the compactness of Θ_c , there is a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\theta_{n_k} \rightarrow \theta$ in $C^1(\overline{\hat{\Omega}})$ as $k \rightarrow \infty$ for some $\theta \in \Theta_c$. We see easily that $\chi_{\Omega_{n_k}} \rightarrow \chi_\Omega$, with $\Omega = \theta(\Omega')$, in $L^1(\hat{\Omega})$ as $k \rightarrow \infty$. Moreover, $V(\Omega_{n_k}) \rightarrow V(\Omega)$ in $H_0^1(\Omega)$ as $k \rightarrow \infty$ (in the sense of Mosco). In fact, if $z_{k'} \rightarrow z$ weakly in $H_0^1(\hat{\Omega})$ as $k' \rightarrow \infty$ for a subsequence $\{n_{k'}\}$ and $z_{k'} \in V(\Omega_{n_{k'}})$, then $\tilde{z}_{k'}(x) = z_{k'}(\theta_{n_{k'}} \circ \theta^{-1}(x)) \in V(\Omega)$ and $\tilde{z}_{k'} \rightarrow z(\theta \circ \theta^{-1}) = z$ weakly in $H_0^1(\hat{\Omega})$. So we see that $z \in V(\Omega)$. Also, let $z \in V(\Omega)$ and put $z_k(x) := z(\theta \circ \theta_{n_k}^{-1}(x)) \in V(\Omega_{n_k})$. Then, clearly, we have $z_k \rightarrow z$ in $H_0^1(\hat{\Omega})$.

EXAMPLE 1.2. Let $\hat{\Omega} := \{x; |x| < 2\} \subset \mathbb{R}^3$, $\Omega_a := \{x; a < |x| < 1\}$ for any $0 < a \leq \frac{1}{2}$ and $\Omega := \{x; |x| < 1\}$. Here we put $O_c := \{\Omega_a; 0 < a \leq \frac{1}{2}\} \cup \{\Omega\}$. Then, we see that this subset O_c of O satisfies condition (C).

In fact, by [13; Lemma 1.8], the 2-capacity of any singleton is zero. Then, by [13], we see that $V(\Omega_a) \rightarrow V(\Omega)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a \rightarrow 0$. In the other hand, by the same argument as in Example 1.1, we obtain that $V(\Omega_{a'}) \rightarrow V(\Omega_a)$ in $H_0^1(\hat{\Omega})$ in the sense of Mosco as $a' \rightarrow a$. Hence O_c satisfies condition (C). $\diamond$

In the case of Example 1.1, problems $SP(\Omega)$ can be reformulated as degenerate parabolic equations on the fixed domain Ω' by using the variable transformation $y = \theta^{-1}(x)$. However, in the case of Example 1.2, the situation is quite different, because there is no C^1 -diffeomorphism between domains Ω_a and Ω .

Based on an abstract result of [1] about the solvability of $SP(\Omega)$, we consider a shape optimization problem. For a given non-empty subset O_c of O , our optimization problem,

denoted by $P(O_c)$, is formulated as follows:

$$P(O_c) \quad \text{Find } \Omega_* \in O_c \text{ such that } J(\Omega_*) = \inf_{\Omega \in O_c} J(\Omega),$$

where

$$(1.3) \quad J(\Omega) = \frac{1}{2} \int_{Q(\Omega)} |\beta(u_\Omega) - \beta_d|^2 dxdt + \frac{1}{2} \int_{\hat{Q}-Q(\Omega)} |g|^2 dxdt \quad \text{for } \Omega \in O,$$

u_Ω is the weak solution of $SP(\Omega)$, and β_d is a given function in $L^2(\hat{Q})$.

In real problem, the driving variables are f, g and Ω . But, in this paper, we are interested in the effect of the domain Ω for the shape optimization. So, we fix the functions f and g , and take Ω as the driving variable.

The main results are stated in the following theorems. To prove the existence of solutions to $P(O_c)$, an important part is to show the continuous dependence of weak solution $u = u_\Omega$ to $SP(\Omega)$ upon $\Omega \in O$.

THEOREM 1.1. *Let $\{\Omega_n\} \subset O$ and $\Omega \in O$ such that $V(\Omega_n) \rightarrow V(\Omega)$ in $H_0^1(\hat{\Omega})$ as $n \rightarrow \infty$ (in the sense of Mosco) and $\chi_{\Omega_n} \rightarrow \chi_\Omega$ in $L^1(\hat{\Omega})$ as $n \rightarrow \infty$. Also, denote by u_n and u the weak solutions of $SP(\Omega_n)$ and $SP(\Omega)$, respectively. Then, as $n \rightarrow \infty$,*

$$(1.4) \quad (u_n(t), z)_{\Omega_n} \rightarrow (u(t), z)_\Omega \quad \text{for any } z \in L^2(\hat{\Omega})$$

and

$$(1.5) \quad \tilde{\beta}(u_n) \rightarrow \tilde{\beta}(u) \quad \text{in } L^2(\hat{Q}).$$

Here we denote by $(\cdot, \cdot)_{\Omega'}$ the inner product in $L^2(\Omega')$ and put

$$\tilde{\beta}(u_{\Omega'}) = \begin{cases} \beta(u_{\Omega'}) & \text{in } Q(\Omega'), \\ g & \text{in } \hat{Q} - Q(\Omega'), \end{cases}$$

for any $\Omega' \in O$.

The next theorem is concerned with the existence of a solution to $P(O_c)$.

THEOREM 1.2. *Problem $P(O_c)$ has at least one optimal solution Ω_* .*

We shall prove Theorems 1.1 and 1.2 in section 3.

2. Uniform estimates for the weak solutions to $SP(\Omega)$

In this section, we obtain some results from [1] on the existence, uniqueness and uniform estimates for weak solutions to $SP(\Omega)$. We use the following notations.

For simplicity, we denote by H the space $L^2(\hat{\Omega})$ and by X the Sobolev space $H_0^1(\hat{\Omega})$. Moreover, $|\cdot|_H$ stands for the norm in H and $(\cdot, \cdot)$ the inner product in H . For each $\Omega \in O$, we define a bilinear form $a_\Omega(\cdot, \cdot)$ on $H^1(\Omega)$ by

$$a_\Omega(u, v) := \int_{\Omega} \nabla u \nabla v dx \quad \text{for all } u, v \in H^1(\Omega),$$

and denote by F_Ω the duality mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ which is given by the formula

$$\langle F_\Omega v, z \rangle := a_\Omega(v, z) \quad \text{for all } v, z \in H_0^1(\Omega),$$

where $\langle \cdot, \cdot \rangle_\Omega$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$. In particular, we put $a(\cdot, \cdot) := a_{\hat{\Omega}}(\cdot, \cdot)$.

According to the abstract result of [1; Theorem 2.1], problem $SP(\Omega)$ has a unique weak solution u such that $u \in W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and $\beta(u) - g \in L^2(0, T; H_0^1(\Omega))$ for any $\Omega \in O$. In fact, the weak solution u is obtained as a unique solution of the following evolution problem in $H^{-1}(\Omega)$:

$$(2.1) \quad \begin{cases} u'(t) + F_\Omega(\beta(u(t)) - g(t)) = f(t) + \Delta g(t) & \text{for a.e. } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

We give some uniform estimates for weak solutions of $SP(\Omega)$ with respect to $\Omega \in O$.

LEMMA 2.1 *There exists a positive constant M_1 independent of Ω such that*

$$(2.2) \quad \|u_\Omega\|_{L^\infty(0, T; L^2(\Omega))} \leq M_1, \quad \|\beta(u_\Omega)\|_{L^2(0, T; H^1(\Omega))} \leq M_1$$

$$(2.3) \quad \|t^{1/2} \frac{d}{dt} \beta(u_\Omega)\|_{L^2(0, T; L^2(\Omega))} \leq M_1, \quad \|t^{1/2} \beta(u_\Omega)\|_{L^\infty(0, T; H^1(\Omega))} \leq M_1$$

for all $\Omega \in O$, where u_Ω is the weak solution of $SP(\Omega)$.

Proof. As was seen in [1], problem $SP(\Omega)$ is able to be approximated by non-degenerated problem $SP(\Omega)^\varepsilon$, $\varepsilon \in (0, 1]$:

$$SP(\Omega)^\varepsilon \begin{cases} u_t - \Delta \beta^\varepsilon(u) = f & \text{in } Q(\Omega), \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ \beta^\varepsilon(u) = g & \text{on } \Sigma(\Omega), \end{cases}$$

where $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$, $r \in R$.

In fact, this problem has one and only one weak solution $u^\varepsilon \in C([0, T]; L^2(\Omega))$ such that $t^{1/2} \frac{d}{dt} \beta^\varepsilon(u^\varepsilon) \in L^2(Q(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \in L^2(0, T; H^1(\Omega))$. Moreover, we see that $u^\varepsilon \rightarrow u_\Omega$ in $C_w([0, T]; L^2(\Omega))$ and $\beta^\varepsilon(u^\varepsilon) \rightarrow \beta(u_\Omega)$ weakly in $L^2(0, T; H^1(\Omega))$ as $\varepsilon \rightarrow 0$. After some calculations, we obtain that there is a positive constant C' independent of ε and Ω such that

$$(2.4) \quad \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla(\beta^\varepsilon(u^\varepsilon(t)))\|_{L^2(\Omega)}^2 dt \leq C'.$$

Moreover, multiply both sides of $u_t - \Delta \beta^\varepsilon(u^\varepsilon) = f$ by $t \frac{d}{dt}(\beta^\varepsilon(u^\varepsilon) - g)$ and integrate over $Q(\Omega)$. Then, by (2.4), we have

$$(2.5) \quad \begin{aligned} \|t^{1/2} \beta^\varepsilon(u^\varepsilon)\|_{L^\infty(0, T; H^1(\Omega))} &\leq C'', \quad \|t^{1/2} \frac{d}{dt} \beta^\varepsilon(u^\varepsilon)\|_{L^2(0, T; L^2(\Omega))} \leq C'', \\ &\text{for any } \varepsilon \in (0, 1] \text{ and } \Omega \in O, \end{aligned}$$

where C'' is a constant independent of $\varepsilon \in (0, 1]$ and $\Omega \in O$. Therefore, letting $\varepsilon \rightarrow 0$, we see that (2.2) and (2.3) hold. $\diamond$

3. Proofs of Theorems 1.1 and 1.2

First we prove Theorem 1.1.

Proof of THEOREM 1.1. Let consider the function $u_g \in L^\infty(0, T; H)$ such that $g(t, x) = \beta(u_g(t, x))$ in $\hat{Q}$. Here, we put

$$\tilde{u}_n = \begin{cases} u_n & \text{in } Q_n := Q(\Omega), \\ u_g & \text{in } \hat{Q} - Q_n. \end{cases}$$

Then, we see that $\tilde{u}_n \in L^\infty(0, T; H)$. Moreover, we put $v_n := \beta(\tilde{u}_n)$ in $\hat{Q}$. By using Lemma 2.1, there exist a subsequence $\{n_k\}$ of $\{n\}$, $v \in L^2(0, T; H^1(\hat{\Omega}))$ and $\tilde{u} \in L^\infty(0, T; H)$ such that

$$(3.1) \quad \tilde{u}_{n_k} \rightarrow \tilde{u} \quad \text{weakly}^* \text{ in } L^\infty(0, T; H)$$

and

$$(3.2) \quad \begin{cases} v_{n_k} \rightarrow v & \text{weakly in } L^2(0, T; H^1(\hat{\Omega})), \\ v_{n_k}(t) \rightarrow v(t) & \text{weakly in } H^1(\hat{\Omega}) \text{ for all } t \in (0, T]. \end{cases}$$

By using Ascoli-Arzelà's theorem and Lemma 2.1, we easily verify that

$$v_{n_k} \rightarrow v \text{ in } L^2(0, T; H) \text{ as } k \rightarrow \infty.$$

Since $v_{n_k} = \beta(\tilde{u}_{n_k})$ in $\hat{Q}$, from (3.1) and (3.2) we show that $v = \beta(\tilde{u})$ and that $\beta(\tilde{u}(t)) - g(t) \in V(\Omega)$ for any $t \in (0, T]$.

Next, let z be any function in $V(\Omega)$ and ρ be any function in $\mathcal{D}(0, T)$. By the assumptions of Theorem 1.1, there exists a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \rightarrow z$ in X . Then by the definition of solution to $SP(\Omega)$ we have

$$-\int_0^T (u_{n_k}(t), z_{n_k})_{\Omega_{n_k}} \rho'(t) dt + \int_0^T a_{\Omega_{n_k}}(v_{n_k}(t), z_{n_k}) \rho(t) dt = \int_0^T (f(t), z_{n_k})_{\Omega_{n_k}} \rho(t) dt.$$

Letting $k \rightarrow \infty$, by $z_{n_k} = 0$ a.e. on $\hat{\Omega} - \Omega_{n_k}$ we obtain

$$-\int_0^T (\tilde{u}(t), z) \rho'(t) dt + \int_0^T a(v(t), z) \rho(t) dt = \int_0^T (f(t), z) \rho(t) dt.$$

This shows that $u = \tilde{u}|_{Q(\Omega)}$ is the solution of $SP(\Omega)$. $\diamond$

Proof of THEOREM 1.2. Since $J(\Omega) \geq 0$, there exists a minimizing sequence $\{\Omega_n\}$ in O_c such that

$$J(\Omega_n) \rightarrow J_* := \inf\{J(\Omega); \Omega \in O_c\}$$

Then, by the compactness of O_c , there are a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ and $\Omega_* \in O_c$ such that $V(\Omega_{n_k}) \rightarrow V(\Omega_*)$ in X (in the sense of Mosco) for some $\Omega_* \in O_c$ and $\chi_{\Omega_{n_k}} \rightarrow \chi_{\Omega_*}$ in $L^1(\hat{\Omega})$ as $k \rightarrow \infty$. Now, denote by u_k the weak solution of $SP(\Omega_{n_k})$ and by u_* the weak solution of $SP(\Omega_*)$. Then put

$$v_k := \begin{cases} \beta(u_k) & \text{in } Q_k = Q(\Omega_{n_k}), \\ g & \text{in } \hat{Q} - Q_k, \end{cases}$$

and

$$v := \begin{cases} \beta(u_*) & \text{in } Q = Q(\Omega_*), \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

From Theorem 1.1, it follows that $v_k \rightarrow v$ in $L^2(0, T; H)$ as $k \rightarrow \infty$. Then we see that

$$J(\Omega_{n_k}) \rightarrow J(\Omega_*).$$

Therefore $J(\Omega_*) = J_*$. Hence Ω_* is a solution of $P(O_c)$. $\diamond$

4. Approximations for $SP(\Omega)$ and $P(O_c)$

In this section, from some numerical points of view, we discuss approximations of $SP(\Omega)$ and $P(O_c)$ by smooth problems. At first, we introduce the approximation β^ε and χ_Ω^ν for β and χ_Ω , respectively.

Let $\{\beta^\varepsilon\} = \{\beta^\varepsilon; 0 < \varepsilon \leq 1\}$ be a family of (smooth) functions $\beta^\varepsilon : R \rightarrow R$ such that

$$(\beta) \begin{cases} |\beta^\varepsilon(r) - \beta(r)| \leq \varepsilon(|r| + 1) & \text{for all } r \in R; \\ \beta^\varepsilon(0) = 0, |\beta^\varepsilon(r) - \beta^\varepsilon(r')| \leq \tilde{L}_0 |r - r'| & \text{for all } r, r' \in R, \\ \frac{d}{dr} \beta^\varepsilon(r) \geq \varepsilon & \text{for a.e. } r \in R, \end{cases}$$

where $\tilde{L}_0 > 0$ is a constant independent of ε .

Next, let $\{\chi_\Omega^\nu\} = \{\chi_\Omega^\nu; 0 < \nu \leq 1, \Omega \in O_c\}$ be a family of smooth functions on $\hat{\Omega}$ and suppose that the following two conditions ($\chi 1$) and ($\chi 2$) hold :

$$(\chi 1) \quad 0 \leq \chi_\Omega \leq \chi_\Omega^\nu \leq 1 \text{ in } \hat{\Omega} \text{ and } \text{supp } (\chi_\Omega^\nu) \subset \{x \in \hat{\Omega}; \text{dist}(x, \Omega) \leq \nu\}$$

for any $\nu \in (0, 1]$ and $\Omega \in O_c$.

$$(\chi 2) \quad \text{For each } \nu \in (0, 1], \{\chi_\Omega^\nu; \Omega \in O_c\} \text{ is compact in } L^1(\hat{\Omega}).$$

We give below typical examples of approximations β^ε and χ_Ω^ν for β and χ_Ω , respectively, which satisfy the conditions mentioned above.

EXAMPLE 4.1. (1) We define $\beta^\varepsilon : R \rightarrow R$ by $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$ for any $r \in R$. Then, the family of $\{\beta^\varepsilon\}$ satisfies the condition (β) for $\tilde{L}_0 = L_0 + 1$ where L_0 is the constant of (1.1).

(2) Let $\hat{\Omega}$, Ω' and O_c be the same as in Example 1.1. Now, for each $\nu \in (0, 1]$ and $\Omega \in O_c$, we denote by $\Omega(\frac{\nu}{2})$ the set $\{x \in \hat{\Omega}; \text{dist}(x, \Omega) \leq \frac{\nu}{2}\}$. Let χ_Ω^ν be the regularization of $\chi_{\Omega(\frac{\nu}{2})}$ by means of usual mollifiers on $\hat{\Omega}$. Clearly, we see that $(\chi 1)$ holds. Also, we obtain that $(\chi 2)$ holds. Because we can prove that

$$(4.1) \quad \text{if } \Omega_n = \theta_n(\Omega'), \theta_n \rightarrow \theta \text{ in } C^1(\bar{\hat{\Omega}}) \text{ and } \Omega = \theta(\Omega'), \text{ then } \chi_{\Omega_n} \rightarrow \chi_\Omega \text{ in } L^1(\hat{\Omega}).$$

Now, we define the approximate problem $SP(\Omega)^{\varepsilon\nu\mu}$, $\varepsilon, \nu, \mu \in (0, 1]$, by using the penalty method for $SP(\Omega)$:

$$SP(\Omega)^{\varepsilon\nu\mu} \begin{cases} u_t - \Delta \beta^\varepsilon(u) = f - \frac{1 - \chi_\Omega^\nu}{\mu} (\beta^\varepsilon(u) - g) & \text{in } \hat{Q}, \\ u(0, \cdot) = u_0 & \text{in } \hat{\Omega}, \\ \beta^\varepsilon(u) = g & \text{on } \hat{\Sigma}. \end{cases}$$

Here we give the weak formulation of $SP(\Omega)^{\varepsilon\nu\mu}$.

DEFINITION 4.1. A function $u : [0, T] \rightarrow H$ is a solution of $SP(\Omega)^{\varepsilon\nu\mu}$, if the following three conditions (aw1) – (aw3) are satisfied:

$$(aw1) \quad u \in C([0, T]; H) \cap W_{loc}^{1,2}((0, T]; H) \cap L^2(0, T; H^1(\hat{\Omega})), \quad u(0) = u_0 \text{ in } \hat{\Omega};$$

$$(aw2) \quad \beta^\varepsilon(u(t)) - g(t) \in X \text{ for a.e. } t \in [0, T];$$

$$(aw3) \quad \langle u'(t), z \rangle_{\hat{\Omega}} + a(\beta^\varepsilon(u(t)), z) = (f(t) - \frac{1 - \chi_\Omega^\nu}{\mu} (\beta^\varepsilon(u(t)) - g(t)), z) \\ \text{for any } z \in X, \text{ a.e. } t \in [0, T].$$

According to the abstract result in [9; Chapter 2] (or [10]), problem $SP(\Omega)^{\varepsilon\nu\mu}$ has a unique solution u .

Our approximate optimization problem $P(O_c)^{\varepsilon\nu\mu}$, associated with $SP(\Omega)^{\varepsilon\nu\mu}$, is formu-

lated as follows:

$$P(O_c)^{\varepsilon\nu\mu} \quad \text{Find } \Omega_*^{\varepsilon\nu\mu} \in O_c \text{ such that } J^{\varepsilon\nu\mu}(\Omega_*^{\varepsilon\nu\mu}) = \inf_{\Omega \in O_c} J^{\varepsilon\nu\mu}(\Omega),$$

where

$$J^{\varepsilon\nu\mu}(\Omega) = \frac{1}{2} \int_{\hat{Q}} \chi_{\Omega}^{\nu} |\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu}) - \beta_d|^2 dxdt + \frac{1}{2} \int_{\hat{Q}} (1 - \chi_{\Omega}^{\nu}) |g|^2 dxdt,$$

$u_{\Omega}^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Next, we give the convergence results in the following theorem.

THEOREM 4.1. *We have the following statements (1) and (2):*

(1) *For each $\varepsilon, \nu, \mu \in (0, 1]$, $P(O_c)^{\varepsilon\nu\mu}$ has at least one solution.*

(2) *Let $\{\varepsilon_n\}, \{\nu_n\}, \{\mu_n\}$ be null sequences and let $\{\Omega_n\} \subset O_c$ and $\Omega \in O_c$ such that $V(\Omega_n) \rightarrow V(\Omega)$ in X as $n \rightarrow \infty$ (in the sense of Mosco), $\chi_{\Omega_n}^{\nu_n} \rightarrow \chi_{\Omega}$ in $L^1(\hat{\Omega})$ as $n \rightarrow \infty$. Denote by u_n the solution of $SP(\Omega_n)^{\varepsilon_n\nu_n\mu_n}$. Then as $n \rightarrow \infty$,*

$$\begin{cases} \chi_{\Omega_n} u_n \rightarrow \chi_{\Omega} u & \text{weakly}^* \text{ in } L^{\infty}(0, T; H), \\ \beta^{\varepsilon_n}(u_n) \rightarrow v & \text{in } L^2(0, T; H) \text{ and weakly in } L^2(0, T; H^1(\hat{\Omega})), \end{cases}$$

Moreover u is the weak solution of $SP(\Omega)$ and

$$v = \begin{cases} \beta(u) & \text{in } Q = (0, T) \times \Omega, \\ g & \text{in } \hat{Q} - Q. \end{cases}$$

In particular, if Ω_n is a solution of $P(O_c)^{\varepsilon\nu\mu}$ with $\varepsilon = \varepsilon_n, \nu = \nu_n$ and $\mu = \mu_n$ for $n = 1, 2, \dots$, then Ω is a solution of $P(O_c)$.

In this theorem, $\{\varepsilon_n\}$, $\{\nu_n\}$, and $\{\mu_n\}$ are chosen independently. This is very convenient for numerical computation. Moreover, we show that $P(O_c)^{\varepsilon\nu\mu}$ converges to $P(c)$ in some sense.

5. Energy estimates for $SP(\Omega)^{\varepsilon\nu\mu}$

For the proof of Theorem 4.1, we prepare some lemmas on energy estimates for solutions of $SP(\Omega)^{\varepsilon\nu\mu}$ with respect to $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$.

LEMMA 5.1. *There is a positive constant M_2 such that*

$$(5.1) \quad |u_{\Omega}^{\varepsilon\nu\mu}|_{L^{\infty}(0,T;H)} \leq M_2, |\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu})|_{L^2(0,T;H^1(\widehat{\Omega}))} \leq M_2$$

and

$$(5.2) \quad \frac{1}{\mu} \int_{\widehat{\Omega}} (1 - \chi_{\Omega}^{\nu}) |\beta^{\varepsilon}(u_{\Omega}^{\varepsilon\nu\mu}) - g|^2 dx dt \leq M_2$$

for all $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$, where $u_{\Omega}^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Proof. For $0 < \nu, \mu \leq 1, \Omega \in O, 0 \leq t \leq T$, we introduce a proper lower semi-continuous convex function $\varphi_{\Omega}^{\nu\mu}$ on H as follows:

$$(5.3) \quad \varphi_{\Omega}^{\nu\mu}(t, z) = \begin{cases} \frac{1}{2} |\nabla z|_H^2 + \frac{1}{2\mu} \int_{\widehat{\Omega}} (1 - \chi_{\Omega}^{\nu}) |z - g(t)|^2 dx & \text{for } z - g(t) \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

We easily see that the subdifferential $\partial\varphi_{\Omega}^{\nu\mu}(t, \cdot)$ in H is singlevalued in H and

$$(5.4) \quad z^* = \partial\varphi_{\Omega}^{\nu\mu}(t, z) \Leftrightarrow \begin{cases} z - g(t) \in X, z^* \in H, \\ z^* = -\Delta z + \frac{1 - \chi_{\Omega}^{\nu}}{\mu} (z - g(t)) \in H. \end{cases}$$

By using (5.4), we can show that $SP(\Omega)^{\varepsilon\nu\mu}$ can be reformulated by the following evolution problem in H :

$$(5.5) \quad \begin{cases} u'(t) + \partial\varphi_{\Omega}^{\nu\mu}(t, \beta^{\varepsilon}(u(t))) = f(t) & \text{in } H \text{ for a.e. } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

For simplicity, we write u for $u_{\Omega}^{\varepsilon\nu\mu}$, χ for χ_{Ω}^{ν} and $\varphi(t, \cdot)$ for $\varphi_{\Omega}^{\nu\mu}(t, \cdot)$. Multiplying $u'(t) + \partial\varphi(t, \beta^{\varepsilon}(u(t))) = f(t)$ by $\beta^{\varepsilon}(u(t)) - g(t)$, by using (5.4), we obtain

$$\begin{aligned} & (u'(t), \beta^{\varepsilon}(u(t)) - g(t)) + a(\beta^{\varepsilon}(u(t)), \beta^{\varepsilon}(u(t)) - g(t)) \\ & + \frac{1}{\mu} \int_{\widehat{\Omega}} (1 - \chi) |\beta^{\varepsilon}(u(t)) - g(t)|^2 dx \\ & = (f(t), \beta^{\varepsilon}(u(t)) - g(t)). \end{aligned}$$

After some calculations, we obtain the following inequality:

$$(5.6) \quad \begin{aligned} & \frac{d}{dt} \left\{ \int_{\widehat{\Omega}} \widehat{\beta^{\varepsilon}}(u(t)) dx - (g(t), u(t)) \right\} \\ & + R_1 \left\{ |\nabla(\beta^{\varepsilon}(u(t)) - g(t))|_H^2 + \frac{1}{\mu} \int_{\widehat{\Omega}} (1 - \chi) |\beta^{\varepsilon}(u(t)) - g(t)|^2 dx \right\} \\ & \leq R_2 \left\{ \int_{\widehat{\Omega}} \widehat{\beta^{\varepsilon}}(u(t)) dt - (g(t), u(t)) \right\} \\ & + R_3 (1 + |g(t)|_{H^1(\widehat{\Omega})}^2 + |g'(t)|_H^2 + |f(t)|_H^2) \end{aligned}$$

where $R_i, i = 1, 2, 3$, are positive constants independent of ε, ν, μ and Ω . By using Gronwall's inequality and (5.6), we show (5.1) and (5.2) for a positive constant M_2 independent of $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$. $\diamond$

LEMMA 5.2. *There is a positive constant M_3 such that*

$$(5.7) \quad |t^{1/2} \beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})|_{L^\infty(0,T;H^1(\hat{\Omega}))} \leq M_3, |t^{1/2} \frac{d}{dt} \beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})|_{L^2(0,T;H)} \leq M_3,$$

and

$$(5.8) \quad \sup_{t \in (0,T]} \frac{t}{\mu} \int_{\hat{\Omega}} (1 - \chi_\Omega^\nu) |\beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu}(t)) - g(t)|^2 dx \leq M_3,$$

for all $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$, where $u_\Omega^{\varepsilon\nu\mu}$ is the solution of $SP(\Omega)^{\varepsilon\nu\mu}$.

Proof. Simply write u for $u_\Omega^{\varepsilon\nu\mu}$ and $\tilde{\beta}$ for $\beta^\varepsilon(u_\Omega^{\varepsilon\nu\mu})$. Let us consider the convex function $\psi := \psi_\Omega^{\nu\mu}$ on H given by

$$\psi_\Omega^{\nu\mu}(z) = \begin{cases} \frac{1}{2} \|\nabla z\|_H^2 + \frac{1}{2\mu} \int_{\hat{\Omega}} (1 - \chi_\Omega^\nu) |z|^2 dx & \text{for } z \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, it is easy to see that ψ is proper lower semicontinuous and convex on H , and the subdifferential $\partial\psi$ is singlevalued in H . Besides,

$$z^* = \partial\psi(z) \Leftrightarrow \begin{cases} z \in X, z^* \in H, \\ z^* = -\Delta z + \frac{1 - \chi_\Omega^\nu}{\mu} z \in H. \end{cases}$$

Moreover, by the standard argument of convex analysis, we have

$$(5.9) \quad \frac{d}{dt} \psi(z(t)) = (\partial\psi(z(t)), z'(t)) \text{ for } z \in W^{1,2}(0, T; H).$$

Then, by using (5.4) and (5.5), we see that

$$\begin{aligned} & (u'(t), \tilde{\beta}'(t) - g'(t)) + (-\Delta(\tilde{\beta}(t) - g(t)) + \frac{1 - \chi_\Omega^\nu}{\mu}(\tilde{\beta}(t) - g(t)), \tilde{\beta}'(t) - g'(t)) \\ & = (f(t) + \Delta g(t), \tilde{\beta}'(t) - g'(t)). \end{aligned}$$

Then, by (5.9), we show that

$$\begin{aligned}
 (5.10) \quad & \frac{t}{2\tilde{L}_0} |\tilde{\beta}'(t)|_H^2 \\
 & + \frac{d}{dt} \left\{ \frac{t}{2} |\nabla(\tilde{\beta}(t) - g(t))|_H^2 - t(u(t), g'(t)) + \frac{t}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^{\nu}) |\tilde{\beta}(t) - g(t)|^2 dx \right\} \\
 & \leq T |f(t) + \Delta g(t)|_H \{ |g'(t)|_H + \frac{\tilde{L}_0}{2} |f(t) + \Delta g(t)|_H \} + T |u(t)|_H \cdot |g''(t)|_H \\
 & + \frac{1}{2} |\nabla(\tilde{\beta}(t) - g(t))|_H^2 - (u(t), g'(t)) + \frac{1}{2\mu} \int_{\Omega} (1 - \chi_{\Omega}^{\nu}) |\tilde{\beta}(t) - g(t)|^2 dx.
 \end{aligned}$$

Here, integrating (5.10) over $[0, t]$ and using Lemma 5.1, we derive the estimates (5.7) and (5.8) for some positive constant M_3 independent of $\varepsilon, \nu, \mu \in (0, 1]$ and $\Omega \in O_c$. $\diamond$

6. Proof of Theorem 4.1.

Now we prove Theorem 4.1.

Proof of (1) of THEOREM 4.1. Fix $\varepsilon, \nu, \mu \in (0, 1]$ and put $I_* = \inf\{J^{\varepsilon\nu\mu}(\Omega); \Omega \in O_c\} \geq 0$. Then, there exists a minimizing sequence $\{\Omega_n\}$ in O_c such that

$$J^{\varepsilon\nu\mu}(\Omega_n) \rightarrow I_* \text{ (as } n \rightarrow \infty \text{)}.$$

By (χ_2) , there is a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ such that $V(\Omega_{n_k}) \rightarrow V(\Omega)$ in X (in the sense of Mosco) and $\chi_k := \chi_{\Omega_{n_k}}^{\nu} \rightarrow \chi_{\Omega}^{\nu} =: \chi$ in $L^1(\hat{\Omega})$ for some $\Omega \in O_c$. In a similar way to that of the proof of Theorem 1.1, we can prove that the solution $u_k := u_{\Omega_{n_k}}^{\varepsilon\nu\mu}$ converges to the weak solution $u := u_{\Omega}^{\varepsilon\nu\mu}$ of $SP(\Omega)^{\varepsilon\nu\mu}$ in the sense that

$$\begin{cases} u_k \rightarrow u & \text{in } L^2(0, T; H) \\ \beta^{\varepsilon}(u_k) \rightarrow \beta^{\varepsilon}(u) & \text{in } L^2(0, T; H) \end{cases}$$

Therefore

$$I_* = \lim_{k \rightarrow \infty} J^{\varepsilon\nu\mu}(\Omega_k) = J^{\varepsilon\nu\mu}(\Omega),$$

and we see that Ω is a solution of $P(O_c)^{\varepsilon\nu\mu}$. $\diamond$

Proof of (2) of Theorem 4.1. By Lemma 5.1 and Lemma 5.2, we may assume that

$$(6.1) \quad u_n \rightarrow \tilde{u} \text{ weakly* in } L^{\infty}([0, T]; H),$$

and

$$(6.2) \quad \begin{cases} \tilde{\beta}_n := \beta^{\varepsilon_n}(u_n) \rightarrow \beta(\tilde{u}) =: \tilde{\beta} \text{ in } C_{loc}((0, T]; H) \text{ and weakly in } L^2(0, T; H^1(\hat{\Omega})), \\ \tilde{\beta}_n(t) \rightarrow \tilde{\beta}(t) \text{ weakly in } H^1(\hat{\Omega}) \text{ for any } t \in (0, T]. \end{cases}$$

In fact, (6.1) and (6.2) are obtained in a similar way to the proof of Theorem 1.2. Moreover, by using (5.8) of Lemma 5.2 and (6.2), we have

$$\begin{cases} \chi_{\Omega_n} u_n \rightarrow \chi_{\Omega} u \text{ weakly* in } L^\infty(0, T; H), \\ \tilde{\beta}_n \rightarrow \tilde{\beta} \text{ in } L^2(0, T; H), \\ \int_{\hat{\Omega}} (1 - \chi_{\Omega_n}^{\nu_n}) |\tilde{\beta}_n(t) - g(t)|^2 dx \rightarrow 0 = \int_{\hat{\Omega}} (1 - \chi_{\Omega}) |\tilde{\beta}(t) - g(t)|^2 dx \\ \text{for any } t \in (0, T], \end{cases}$$

so that

$$(6.3) \quad \tilde{\beta}(t) - g(t) \in V(\Omega) \quad \text{for any } t \in (0, T].$$

Next, let ρ be any function in $\mathcal{D}(0, T)$. By assumption, for any $z \in V(\Omega)$, there is a sequence $\{z_n\}$ such that $z_n \in V(\Omega_n)$ and $z_n \rightarrow z$ in X . From (5.5) it follows that

$$\begin{aligned} & - \int_0^T (u_n(t), z_n) \rho(t) dt + \int_0^T a(\tilde{\beta}_n(t), z_n) \rho(t) dt + \frac{1}{\mu_n} \int_0^T ((1 - \chi_{\Omega_n}^{\nu_n})(\tilde{\beta}_n - g)(t), z_n) \rho(t) dt \\ & = \int_0^T (f(t), z_n) \rho(t) dt. \end{aligned}$$

Since $(1 - \chi_{\Omega_n}^{\nu_n})z_n = 0$ a.e. on $\hat{\Omega}$, as $n \rightarrow \infty$, we get that

$$\int_0^T \langle \tilde{u}'(t), z \rho(t) \rangle_{\hat{\Omega}} dt + \int_0^T a(\tilde{\beta}(t), z) \rho(t) dt = \int_0^T (f(t), z) \rho(t) dt.$$

Therefore $\tilde{u}$ is the weak solution of $SP(\Omega)$.

In particular, let Ω_n be a solution of $P(O_c)^{\varepsilon_n \nu_n \mu_n}$ for each n . Just as above

$$J^{\varepsilon_n \nu_n \mu_n}(\Omega_n) \rightarrow J(\Omega)$$

and

$$J^{\varepsilon_n \nu_n \mu_n}(\Omega') \rightarrow J(\Omega') \quad \text{for any } \Omega' \in O_c.$$

Therefore, for any $\Omega' \in O_c$,

$$J(\Omega') = \lim_{n \rightarrow \infty} J^{\varepsilon_n \nu_n \mu_n}(\Omega') \geq \lim_{n \rightarrow \infty} J^{\varepsilon_n \nu_n \mu_n}(\Omega_n) = J(\Omega).$$

This shows that Ω is a solution of $P(O_c)$. $\diamond$

For the detailed proofs of all results stated in this note, see the forthcoming paper [17].

References

- [1] A.Damlamian, Some results on the multi-phase Stefan problem, Comm. P.D.E. 2. (1977), 1017 - 1044.
- [2] A.Damlamian and N.Kenmochi, Asymptotic behavior of solutions to a multi-phase Stefan problems. Japan J. Appl. Math., 3(1986), 15 - 36.
- [3] A.Freidman, The Stefan problem in several space variables, Trans. Amer. Math. Soc. 133(1968), 51 - 87.
- [4] N.Fujii, Existence of an optimal domain optimization problem, Lecture Notes in Control and Information Sciences 113, System Modelling and Optimization, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 251 - 258.
- [5] Y.Goto, N.Fujii and Y.Muramatsu, Second Order Necessary Optimality Conditions for Domain Optimization Problem with a Neumann Problem, Lecture Notes in Control and Information Sciences 113, System Modelling and Optimization, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 259 - 268.
- [6] J.Haslinger and P.Neittaanmäki, Finite Element Approximation for Optimal Shape Design: Theory and Applications, John Willy & Sons Ltd, Chichester-New York-Brisbane-Toronto-Singapore, 1988.
- [7] J.Haslinger, P.Neittaanmäki and D.Tiba, On state constrained optimal shape design problems, Optimal Control of Partial Differential Equations II, ISNM 78, Birkhäuser, Basel-Boston-Stuttgart, 1987, 109 - 122.

- [8] S.L. Kamenomostskaja, On Stefan's problem, *Mat. Sb.* 53(1961), 489 - 514.
- [9] N.Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, *Bull. Fac. Education, Chiba Univ.* 30(1981), 1 - 87.
- [10] N.Kenmochi and I.Pawlow, A class of nonlinear elliptic-parabolic equations with time-dependent constraints, *Nonlinear Anal.* 10(1986), 1181 - 1202.
- [11] M.Koda, Sensitivity Analysis of A Descriptor Distributed Parameter System and Its Application to Shape Optimization, *Lecture Notes in Control and Information Sciences* 113, System Modelling and Optimization, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 241 - 250.
- [12] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Transl. Math. Monogr. 23, Amer. Math. Soc., Providence R.I., 1968.
- [13] U.Mosco, Convergence of convex sets and of solutions of variational inequalities, *Advances Math.* 3(1969), 510 - 585.
- [14] P.Neittaanmäki, J.Sokolowski and J.P.Zolesio, *Optimization of the Domain in Elliptic Variational Inequalities*, *App. Math. Optim.* 18: 58 - 98 (1988) Springer.
- [15] I.Pawlow, Analysis and control in free boundary problems, *Bull. Fac. Education, Chiba Univ.* 36(1988), 9 - 67.
- [16] C.Saguez, *Contrôle Optimal de Systemes a frontière libre*, Thèse, Univ. Technologique, Compiègne, 1980.
- [17] A.Kadoya and N.Kenmochi, Optimal Shape Design in Multi-Phase Stefan Problems, *Tech. Report Math. Sci.*, Vol.6, No.8, Chiba Univ., 1990.